Functorial backpropagation and symmetric lenses

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Lenses are a well-established structure for modelling bidirectional transformations, such as the interactions between a database and a view of it. Lenses may be symmetric or asymmetric, and may be composed, forming the morphisms of a monoidal category. More recently, the notion of a learner has been proposed: these provide a compositional way of modelling supervised learning algorithms, and again form the morphisms of a monoidal category. In this paper, we show that the two concepts are tightly linked: there is a faithful, identity-on-objects symmetric monoidal functor embedding the category of learners into a category of symmetric lenses.

1 Background

In informal discussions at the Leiden meeting on Applied Category Theory in 2018 the authors sketched their ideas for a relationship between *learners*, a compositional treatment of certain supervised learning algorithms which will shortly appear in LICS [FST19], and certain generalised symmetric lenses. We have now worked out the details of the correspondence, and some of their implications, and a paper about them will be presented at the Eighth Workshop on Bidirectional Transformations in Philadelphia in June, [CK19].

Because that proposed application of category theory raised considerable interest at ACT2018, and because it formed part of a group of applications of lenses in unexpected areas that is in itself unusual and potentially productive, we would like the opportunity to present the now finalised correspondence at ACT2019.

The paper that provides all of the details can be found on the arXiv at

https://arxiv.org/abs/1903.03671

and the following brief summary is lightly edited from the paper's introduction and conclusion sections to give the ACT2019 Program Committee an overview of its content.

2 Overview

The paper presents surprising links between two apparently disparate areas: a compositional treatment of supervised learning algorithms, called *learners*, and a mathematical treatment of bidirectional transformations, called *lenses*. Until this work there had been no known non-trivial relationships between the two areas, and, naively at least, there seemed to be little reason to expect them to be closely related mathematically.

But lenses and learners are indeed mathematically closely related. The main result that we present is the existence of a faithful, identity-on-objects, symmetric monoidal functor, from the category whose arrows are learners to a category whose arrows are symmetric lenses. The symmetric lenses in the image of that functor have particularly simple structure: as spans of asymmetric lenses, their left legs are the long studied and easily understood lenses known as *constant complement lenses*. Roughly speaking, this means that a supervised learning algorithm may be understood as a special type of symmetric lens.

The right legs of the symmetric lenses in the image of the functor are also simple, but in a different way. The right legs are bare asymmetric lenses — lenses which have, as all lenses do, a Put and Get, but which have no axioms restricting the way the Put and Get interact. Such simple lenses were defined in the very first papers on lenses, and were the ones blessed with the name "lens", but the study of such bare lenses has so far been somewhat neglected by the bidirectional transformation community, and certainly by us, because of a lack of examples or applications requiring them. This paper provides some compelling examples and has led us to study them seriously. Such bare lenses have been more heavily used in the Haskell community, and this paper is, we hope, the beginning of a category theoretic treatment of some of the mathematical properties of those bare lenses.

The symmetric lenses that correspond to learners also have links with recent work on some quite sophisticated notions of lenses called *amendment lenses*, or in a slightly simplified form, *aa-lenses*. These aa-lenses do have extra axioms — the ones we have looked at most are called Stable PutGet (or SPG) aa-lenses — and it turns out that the extra structure provided by amendment lenses can be added in a canonical way to the lenses that we study here, and in a way that allows them to recover the amendment-lens versions of GetPut and PutGet axioms.

3 Outline

The paper exposites both asymmetric and symmetric lenses, as well as learners, providing details of how each of them compose. Because of the bare lenses that we are using in the paper, we need to extend somewhat the usual definition of composition of symmetric lenses which is generally defined only when the lenses each satisfy the PutGet axiom. We show that, with a slightly complicated construction, the usual definition of composition can be extended to symmetric lenses in which one leg satisfies PutGet even if the other leg satisfies no axioms at all. The composition of learners can seem, to some readers, to be slightly complicated too, so we include string diagrams to illustrate that composition, and many of the other constructions in the paper as well. After establishing this background we present in detail the theorem already alluded to, and illustrate the precise and remarkably parallel relationship between learners and lenses. And then we discuss some of the observations that follow from this work, and conclude with some speculations on possible directions for further studies flowing from the results.

4 Conclusion

To summarise: in the paper we describe a faithful, identity-on-objects, symmetric monoidal functor from a category which captures the notion of composable supervised learning algorithms to a suitable category of lenses. To do this, we presented a slight generalisation of the usual notion of symmetric lens, in which we require a very weak form of well behavedness: a span of asymmetric lenses in which the left leg satisfies the PutGet law. Despite the general definition, these symmetric lenses still compose, and indeed equivalence classes of them form the morphisms of a symmetric monoidal category. Our main theorem describes the aforementioned close, functorial relationship between the category of learners, as defined in [FST19], with this category of lenses. In this theorem, we witness a surprising yet highly robust link between two previously unrelated fields. We believe this to be a rich connection deserving of further exploration.

References

[CK19] Cheney, J. and Ko, H.-S. eds. (2019) Proceedings of the Eighth International Workshop on Bidirectional Transformations (Bx 2019), Philadelphia, PA, USA, June 4, 2019, to be published at `http://ceur-ws.org`

[FST19] Fong, B., Spivak, D. I. and Tuyéras, R. (2019) Backprop as functor: a compositional perspective on supervised learning. To appear in *Proceedings of the 34th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2019*. Preprint available as arXiv:1711.10455.