An effect-theoretic reconstruction of quantum theory

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This is an extended abstract for the paper [30].

An often used model for quantum theory is to associate to every physical system a C*-algebra. From a physical point of view it is unclear why operator algebras would form a good description of nature. For this reason, many have ventured to find an explanation for why operator algebras should represent physical systems [16, 26, 25, 8, 11, 17, 36, 31, 14, 5, 27, 28, 12, 20, 13, 2]. These results are known as *reconstructions of quantum theory*. They start with a general physical framework that ideally can represent any physical theory and then postulate a series of requirements, or axioms, that should be intuitive and reasonable to be true. From these requirements they then derive (part of) the content of quantum theory.

In this paper, we use principles from *effectus theory*, a categorical logical framework generalising classical and quantum logic [9], to reconstruct the category of finite-dimensional C*-algebras. Like Generalised Probabilistic Theories [7] (GPTs) and Categorical Quantum Mechanics [1] (CQM), effectus theory is an abstract framework to reason about physical theories. Whereas CQM focuses on tensor products and compact closure, and GPTs take convex structure as given, an effectus does not require either. Instead an effectus requires that coproducts exist, and that the effects of a system form an *effect algebra*, a generalisation of the space of effects of a quantum system. The tensor product in CQM allows one to consider conjunctive composites of systems: $A \otimes B$ represents system A *and* B. The coproducts $A + B$ of effectus theory however represent disjunctive composites: A *or* B.

Due to this different focus and the lack of conventional convex or monoidal structure, effectus theory has developed new ideas for reasoning about physical systems. In particular, a new notion of *pure* maps came forth from effectus theory [32, 35]. A map in an effectus is pure when it is a composition of a *compression* and a *filter*. These maps satisfy particular universal properties and respectively correspond to 'forgetting' and 'measuring' the validity of an effect.

We define a *pure effect theory* (PET) to be an effectus where the pure maps form a dagger-category and filters and compressions are adjoint. A PET has a very rich structure, but it is too abstract to retrieve the full structure of operator algebras. To close this gap we introduce the notion of an *operational* effect theory, where the scalars are required to be real numbers, which will force the systems to be described by vector spaces, and we further assume that the spaces are finite-dimensional, along with a few other technical conditions. An operational effect theory is very similar to a GPT.

Our main result is the following, which relates operational PETs to *Euclidean Jordan algebras* (EJAs), generalised models of quantum theory that also include real and quaternionic Hilbert spaces and C*-algebras [24, 23]:

Theorem 1. Let $\mathbb{E}$ be an operational PET. Then there exists a functor into the opposite category of Euclidean Jordan algebras and sub-unital positive maps $F : \mathbb{E} \to \mathbf{EJA}_{\text{psu}}^{\text{op}}$ such that $\text{Eff}(F(A)) \cong \text{Eff}(A) := \text{Hom}(A, I)$. Furthermore, this functor is faithful if and only if $\mathbb{E}$ satisfies *local tomography.*

EJAs appear in many reconstructions as an intermediate step along the way to deriving the full structure of quantum theory. It is a well-known fact that there is no well-behaved monoidal structure on Euclidean Jordan algebras [6, 4, 15]. By assuming additional monoidal structure on an operational PET, we are therefore able to retrieve the standard C*-algebraic structure:

Theorem 2. Let $\mathbb{E}$ be a monoidal operational PET. Then the functor of Theorem 1 restricts either to the category of real C*-algebras or to the category of complex C*-algebras. Furthermore, this functor is faithful if and only $\mathbb{E}$ satisfies tomography.

The real and complex case can be further distinguished by requiring the PET to satisfy *local tomography* [18]: that $f = g$ if and only if $q \circ f = q \circ g$ for all effects q.

Theorem 2 shows that the ideas arising from the categorical language of effectus theory suffice to reconstruct the operator algebra structure of quantum theory. Conversely, a monoidal PET can be seen as a categorical generalisation of the category of C*-algebras.

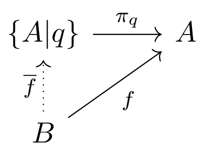
The results in more depth

We will give here some more concrete details to enlighten the theorems stated above. We refer to the main paper [30], in particular Section 3, for the formal definitions, examples, and context.

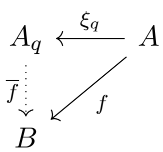
First of all, an *effect theory* is simply a category with a designated *trivial* object I that represent the empty physical system. The maps $\omega : I \to A$ are the *states* of the physical theory, with the maps $q : A \to I$ being the *effects*, i.e. the measurements. The only structure we require on an effect theory is that the set of effects $\mathrm{Eff}(A) := \{q : A \to I\}$ is an *effect algebra* and that all morphisms preserve this effect algebra structure. An effect algebra is a set with a designated 0 and 1 element with a partial addition operation $+$ and a negation operation $(\cdot)^\perp$. These operations model the structure on the real unit interval $[0, 1]$ (or more generally, the unit interval of any ordered vector space) where $x + y$ is defined when $x + y \leq 1$ and negation is simply $x^\perp = 1 - x$. We also consider *monoidal* effect theories, where the trivial object is also the monoidal unit and the effect algebra structure is preserved by the tensor product.

There are two important types of maps we consider on an effect theory: *compressions* and *filters*, which are 'dual' in a certain sense. Compressions and filters are defined with respect to an effect $q : A \to I$, and represent respectively forgetting that the effect q, when viewed as a predicate, holds and post-selecting, i.e. forcing, q to hold.

Formally, a compression for q is a map $\pi_q : \{A|q\} \to A$ such that $1 \circ \pi_q = q \circ \pi_q$, which is *final* with respect to this property, i.e. for all $f : B \to A$ satisfying $1 \circ f = q \circ f$ there is a unique $\overline{f} : B \to \{A|q\}$ making the following diagram commute:



Similarly, a filter for q is a map $\xi_q : A \to A_q$ such that $1 \circ \xi_q \leq q$ which is *initial* with respect to this property: for every map $f : A \to B$ satisfying $1 \circ f \leq q$ there is a unique $\overline{f} : A_q \to B$ making the following diagram commute:



In effectus theory compressions come from a functor that is right-adjoint to the 'truth' functor, while filters are left-adjoint to 'falsity' [10, 9].

Following [32, 33, 34, 35] we call a map f in an effect theory *pure* when there is a filter ξ and compression π such that $f = \pi \circ \xi$. For C*-algebras the pure maps correspond to isomorphisms and Krauss rank 1 operators.

We need one final concept, which is that of *sharpness* of an effect. In a C*-algebra, the sharp effects are precisely the projections. In an effect theory, we define it using the *image* of a map. The image $\operatorname{im} f$ of a map $f : A \to B$ (when it exists) is an effect $q : B \to I$ that is the *smallest* effect satisfying $q \circ f = 1 \circ f$.

For π_q a compression for q, we always have $\operatorname{im} \pi_q \leq q$. We call q *sharp* when equality holds here, or equivalently when q is the image of any map in the effect theory.

Now we can give the main definition of the paper, that of a pure effect theory.

Definition 3. A (monoidal) **Pure Effect Theory** (PET) is a (monoidal) effect theory satisfying the following properties.

1. All effects have filters and compressions.
2. The pure maps form a (monoidal) dagger-category.
3. All maps have images.
4. The negation of a sharp effect is sharp (if q is sharp, then $q^\perp$ is sharp)
5. Filters of sharp effects are adjoint to its compressions (if π_q is a compression for sharp q, then $\pi_q^\dagger$ is a filter for q, and vice versa).
6. Compressions of sharp effects are isometries ($\pi_q^\dagger \circ \pi_q = \mathrm{id}$ for sharp q).

We can relate some of these properties to perhaps more familiar notions. An effect theory has images and filters and compressions for sharp effects if and only if it has all *kernels* and *cokernels*, while points 5 and 6 above ensure that the subcategory of pure maps is a *dagger kernel category* [19].

While interesting results about PETs can be proven [30, 35], in order to state our main result we need to bring it closer to the concrete physical world by requiring some *operational* structure modelled after *operational probabilistic theories* [29]. Without going into details, the properties we require of an operational effect theory are that its *scalars* equal the real unit interval, which makes its effect spaces embed into real vector spaces, that these effect spaces are finite-dimensional and *order-separated* by the states of the object, and that the set of states of an object is *closed* when viewed as a set of functionals on the effect space.

Theorem 1 states that the effect spaces of an operational pure effect theory are isomorphic to the set of effects of a *Euclidean Jordan algebra* (EJA) in a functorial way. A *Jordan algebra* is a real vector space with a commutative unital operation $*$ that satisfies a weak form of associativity called the *Jordan identity*: $(a * a) * (b * a) = ((a * a) * b) * a$. It is called Euclidean when it is also a finite-dimensional Hilbert space with the inner product satisfying $\langle a * b, c\rangle = \langle b, a * c\rangle$. EJAs are of interest because they can equivalently be described as *homogeneous self-dual positive cones*, and because many of the properties and structure present in C*-algebras extend to them.

Finally, Theorem 2 states that if an operational pure effect theory also has monoidal structure, then the effect spaces must be isomorphic to a subset of the effect spaces of EJAs, namely those of real and complex C*-algebras. With this theorem we see that the abstract category theory inspired properties of Definition 3 suffice, in a finite-dimensional convex setting, to recover the familiar operator algebraic structure of quantum theory. Conversely, without assuming the operational structure, we can view a monoidal pure effect theory as a type of category that models the structure of an operator algebra.

References

[1] Samson Abramsky and Bob Coecke. Categorical quantum mechanics. *Handbook of quantum logic and quantum structures: quantum logic*, pages 261–324, 2009.

[2] Erik M Alfsen and Frederic W Shultz. *Geometry of state spaces of operator algebras.* Springer Science & Business Media, 2012.

[3] Erik M Alfsen and Frederik W Shultz. *State spaces of operator algebras: basic theory, orientations, and C*-products.* Springer Science & Business Media, 2012.

[4] Howard Barnum, Matthew Graydon, and Alexander Wilce. Composites and categories of Euclidean Jordan algebras. *arXiv preprint arXiv:1606.09331*, 2016.

[5] Howard Barnum, Markus P Müller, and Cozmin Ududec. Higher-order interference and single-system postulates characterizing quantum theory. *New Journal of Physics*, 16(12):123029, 2014.

[6] Howard Barnum and Alexander Wilce. Local tomography and the Jordan structure of quantum theory. *Foundations of Physics*, 44(2):192–212, 2014.

[7] Jonathan Barrett. Information processing in generalized probabilistic theories. *Physical Review A*, 75(3):032304, 2007.

[8] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. Informational derivation of quantum theory. *Physical Review A*, 84(1):012311, 2011.

[9] Kenta Cho, Bart Jacobs, Bas Westerbaan, and Abraham Westerbaan. An introduction to effectus theory. *arXiv preprint arXiv:1512.05813*, 2015.

[10] Kenta Cho, Bart Jacobs, Bas Westerbaan, and Bram Westerbaan. Quotient-comprehension chains. In Chris Heunen, Peter Selinger, and Jamie Vicary, editors, Proceedings of the 12th International Workshop on *Quantum Physics and Logic,* Oxford, U.K., July 15-17, 2015, volume 195 of *Electronic Proceedings in Theoretical Computer Science*, pages 136–147. Open Publishing Association, 2015.

[11] Borivoje Dakic and Caslav Brukner. Quantum theory and beyond: is entanglement special? *arXiv preprint arXiv:0911.0695*, 2009.

[12] Daniel I Fivel. Derivation of the rules of quantum mechanics from information-theoretic axioms. *Foundations of Physics*, 42(2):291–318, 2012.

[13] Philip Goyal. From information geometry to quantum theory. *New Journal of Physics*, 12(2):023012, 2010.

[14] Jack Gunson. On the algebraic structure of quantum mechanics. *Communications in mathematical physics*, 6(4):262–285, 1967.

[15] Harald Hanche-Olsen. JB-algebras with tensor products are C*-algebras. In *Operator Algebras and their Connections with Topology and Ergodic Theory*, pages 223–229. Springer, 1985.

[16] Lucien Hardy. Quantum theory from five reasonable axioms. *ArXiv preprint arXiv:quant-ph/0101012*, 2001.

[17] Lucien Hardy. Reconstructing quantum theory. In *Quantum Theory: Informational Foundations and Foils*, pages 223–248. Springer, 2016.

[18] Lucien Hardy and William K Wootters. Limited holism and real-vector-space quantum theory. *Foundations of Physics*, 42(3):454–473, 2012.

[19] Chris Heunen and Bart Jacobs. Quantum logic in dagger kernel categories. *Order*, 27(2):177–212, 2010.

[20] Philipp Andres Höhn. Toolbox for reconstructing quantum theory from rules on information acquisition. *Quantum*, 1:38, 2017.

[21] Bart Jacobs. Probabilities, distribution monads, and convex categories. *Theoretical Computer Science*, 412(28):3323–3336, 2011.

[22] Bart Jacobs, Jorik Mandemaker, and Robert Furber. The expectation monad in quantum foundations. *Information and Computation*, 250:87–114, 2016.

[23] P. Jordan, J. v. Neumann, and E. Wigner. On an algebraic generalization of the quantum mechanical formalism. *Annals of Mathematics*, 35(1):29–64, 1934.

[24] Pascual Jordan. *Über verallgemeinerungsmöglichkeiten des formalismus der quantenmechanik*. Weidmann, 1933.

[25] Ll Masanes, Markus P Mueller, David Perez-Garcia, and Remigiusz Augusiak. Entanglement and the three-dimensionality of the Bloch ball. *Journal of Mathematical Physics*, 55(12):122203, 2014.

[26] Lluís Masanes and Markus P Müller. A derivation of quantum theory from physical requirements. *New Journal of Physics*, 13(6):063001, 2011.

[27] John H Selby, Carlo Maria Scandolo, and Bob Coecke. Reconstructing quantum theory from diagrammatic postulates. *arXiv preprint arXiv:1802.00367*, 2018.

[28] Sean Tull. A categorical reconstruction of quantum theory. *arXiv preprint arXiv:1804.02265*, 2016.

[29] Sean Tull. Operational theories of physics as categories. *arXiv preprint arXiv:1602.06284*, 2016.

[30] John van de Wetering. An effect-theoretic reconstruction of quantum theory. *arxiv preprint arXiv:1801.05798*, 2018.

[31] John van de Wetering. Sequential Product Spaces are Jordan Algebras. *arxiv preprint arXiv:1803.11139*, 2018.

[32] Abraham Westerbaan and Bas Westerbaan. A universal property for sequential measurement. *Journal of Mathematical Physics*, 57(9):092203, 2016.

[33] Abraham Westerbaan, Bas Westerbaan, and John van de Wetering. Pure maps between euclidean jordan algebras. *arxiv preprint arXiv:1805.11496*, 2018.

[34] Abraham A. Westerbaan. *The Category of Von Neumann Algebras*. PhD thesis, Radboud Universiteit Nijmegen, 2018.

[35] Bas Westerbaan. *Dagger and dilations in the category of von Neumann algebras*. PhD thesis, Radboud Universiteit Nijmegen, 2018.

[36] Alexander Wilce. A royal road to quantum theory (or thereabouts). *Entropy*, 20(4):227, 2018.