Hypernormalisation, linear exponential monads and the Giry tricocycloid (extended abstract)

Richard Garner

Centre of Australian Category Theory, Macquarie University, Australia

Background A basic construction in probability theory is that of normalising a sub-probability distribution of weight $\leqslant 1$ to a probability distribution of weight 1. The simplest case is that of finitely supported, discrete probability sub-distributions on a set A, i.e., finitely supported functions $\omega \colon A \to [0,1]$ with $\omega(A) := \sum_{a \in A} \omega(a) \leqslant 1$. If $\omega(A) \neq 0$, then the *normalisation* $\bar{\omega}$ of ω is defined by $\bar{\omega}(a) = \omega(a)/\omega(A)$. This is, of course, a probability distribution, i.e., $\bar{\omega}(A) = 1$. But if $\omega(A) = 0$, then we cannot normalise ω; so normalisation is only a partial operation. In [2], Jacobs introduces *hypernormalisation* which, among other things, addresses this defect.

Hypernormalisation is a *total* function

$$\mathcal{N} \colon \mathcal{D}(A_1 + \cdots + A_n) \to \mathcal{D}(\mathcal{D}A_1 + \cdots + \mathcal{D}A_n)$$

where $\mathcal{D}(X)$ will denote the set of finitely supported probability distributions on X. To define $\mathcal{N}$ at $\omega \in \mathcal{D}(A_1 + \cdots + A_n)$, we first restrict ω along the n coproduct injections to get sub-distributions ω_i on A_i; we then select the *non-zero* sub-distributions among these, say $\omega_{i_1}, \ldots, \omega_{i_m}$; finally, we define $\mathcal{N}(\omega)$ to take the value $\omega_{i_k}(A_{i_k})$ at the element $\overline{\omega_{i_k}}$ in the $\mathcal{D}A_{i_k}$-summand of $\mathcal{D}A_1 + \cdots + \mathcal{D}A_n$, and to be zero elsewhere. So $\mathcal{N}(\omega)$ "normalises the non-zero distributions among $\omega_1, \ldots, \omega_n$ and records the weights".

In [1], I establish links between hypernormalisation, and structures arising in monoidal category theory, linear logic and quantum algebra—as I will now explain.

Convex coproducts The assignation $X \mapsto \mathcal{D}X$ underlies the *finite Giry monad* $\mathbb{D}$ on the category of sets, whose algebras are *convex spaces*. A (abstract) convex space is a set A with with a "convex combination" operation $(0,1) \times A \times A \to A$, which we write as $r, a, b \mapsto r(a,b)$ or $r, a, b \mapsto r \cdot a + r^* \cdot b$, where $r^* := 1 - r$. The axioms are that $r(a,a) = a$, $r(a,b) = r^*(b,a)$ and $r(s(a,b),c) = (rs)(a, \frac{r \cdot s^*}{(rs)^*}(b,c))$ for $a,b,c \in A$ and $r, s \in (0,1)$.

The first recasting of hypernormalisation is in terms of coproducts in the category $\mathbf{Conv}$ of convex spaces. These are unusually simple; the binary coproduct is:

$$A \star B = A + (0,1) \times A \times B + B \qquad (1)$$

with a suitable convex structure. The outer summands give the coproduct inclusions $\iota_1 \colon A \to A \star B \leftarrow B \colon \iota_2$, and the middle summand gives elements of the form $r \cdot a + r^* \cdot b$.

Richard Garner: richard.garner@mq.edu.au, Extended abstract of the arXiv preprint [1].

Now the free functor $\mathbf{Set} \to \mathbf{Conv}$ sends a set A to $\mathcal{D}A$ with the convex structure induced pointwise from $[0,1]$. Being a left adjoint, F preserves coproducts, and so we have an isomorphism

$$\varphi\colon \mathcal{D}(A+B) \xrightarrow{\cong} \mathcal{D}A \star \mathcal{D}B$$

of convex spaces. Working through the definitions, we see that φ is *very close* to being (binary) hypernormalisation:

$$\varphi(\omega) = \begin{cases} \iota_1(\omega|_A) & \text{if } \omega(A) = 1; \\ \iota_2(\omega|_B) & \text{if } \omega(B) = 1; \\ \omega(A) \cdot \overline{\omega|_A} + \omega(B) \cdot \overline{\omega|_B} & \text{otherwise.} \end{cases}$$

Recapturing $\mathcal{N}$ Nice as it is, this map φ is not quite hypernormalisation. How do we close the gap? Since hypernormalisation $\mathcal{D}(A+B) \to \mathcal{D}(\mathcal{D}A+\mathcal{D}B)$ fails to be a map of convex spaces, we must for this go outside the category $\mathbf{Conv}$ of convex spaces, and we do so in a seemingly simple-minded manner, by passing to the category $\mathbf{Conv}_{\mathrm{arb}}$ of convex spaces and *arbitrary* maps.

The key point is that the coproduct monoidal structure $(\star, 0)$ on $\mathbf{Conv}$ *extends* to a monoidal structure on $\mathbf{Conv}_{\mathrm{arb}}$. On objects this is (necessarily) defined as before; while the tensor of maps in $\mathbf{Conv}_{\mathrm{arb}}$ is given by $f \star g = f + ((0,1) \times f \times g) + g$, i.e., exactly the same formula as in $\mathbf{Conv}$.

Using this tensor, we obtain for any convex spaces A and B a map in $\mathbf{Conv}_{\mathrm{arb}}$:

$$A \star B \xrightarrow{\eta_A \star \eta_B} \mathcal{D}A \star \mathcal{D}B \xrightarrow{\varphi^{-1}} \mathcal{D}(A+B)$$

where $\eta_X\colon X \to \mathcal{D}(X)$, the unit of the finite Giry monad, sends $x \in X$ to the Dirac distribution at x. Working through the definitions, the displayed composite sends elements $\iota_1(a)$ and $\iota_2(b)$ of $A \star B$ to the Dirac distributions on $A+B$ concentrated at a, respectively b; while an element $r \cdot a + r^* \cdot b$ of $A \star B$ is sent to the two-point distribution with weight r at a and weight r^* at b. Combined with our description of φ, this shows that $\mathcal{N}$ is the composite:

$$\begin{array}{ccc} \mathcal{D}(A+B) & \xrightarrow{\mathcal{N}} & \mathcal{D}(\mathcal{D}A+\mathcal{D}B) \\ \varphi \downarrow & & \uparrow \varphi^{-1} \\ \mathcal{D}A \star \mathcal{D}B & \xrightarrow{\eta_{\mathcal{D}A} \star \eta_{\mathcal{D}B}} & \mathcal{D}\mathcal{D}A \star \mathcal{D}\mathcal{D}B \end{array} \qquad (2)$$

Linear exponential monads This re-derivation of hypernormalisation leaves one question unanswered: *why* should there be an extension of the coproduct monoidal structure on $\mathbf{Conv}$ to $\mathbf{Conv}_{\mathrm{arb}}$? A moment's thought shows the fundamental reason to be that the underlying set of $A \star B$ depends only on the underlying sets of A and B, and not on their convex space structure.

This suggests that the symmetric monoidal structure on $\mathbf{Conv}$ could be a *lifting* of one on $\mathbf{Set}$; i.e., that $\mathbf{Set}$ could have a symmetric monoidal structure $(\star, 0)$ making $U\colon (\mathbf{Conv}, \star) \to (\mathbf{Set}, \star)$ strict symmetric monoidal. Were this so, then we could re-find the monoidal structure on $\mathbf{Conv}_{\mathrm{arb}}$ by factorising U as (bijective on objects, fully faithful) in the category of symmetric monoidal categories.

In fact, this *is* what happens; we describe the relevant monoidal structure on $\mathbf{Set}$—the *Giry monoidal structure*—below. However, first we note that this monoidal structure's lifting to $\mathbf{Conv}$ is really structure on the monad $\mathbb{D}$: it says that it is a linear exponential monad.

A *linear exponential monad* $\mathbb{T}$ on a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ is a monad for which $(\otimes, I)$ lifts to $\mathbb{T}$-**Alg**, and there becomes finite coproduct. Such monads interpret the connective ? ("why not?") of linear logic. In fact, they also interpret abstract hypernormalisation.

Indeed, if $\mathcal{C}$ has finite sums, then we get invertible maps ("Seely isomorphisms") $\varphi\colon T(A+B) \to TA \otimes TB$ from the fact that $TA \otimes TB$ is a *coproduct* of free $\mathbb{T}$-algebras TA and TB. Mimicking (2), we get hypernormalisation maps $\mathcal{N}\colon T(A+B) \to T(TA+TB)$ by taking $\mathcal{N} = \varphi^{-1} \circ (\eta_{TA} \otimes \eta_{TB}) \circ \varphi$.

These generalise precisely the maps $\mathcal{N}$ of the motivating case, and I show in [1] that many pleasant algebraic properties of that case carry over to the general one.

The Giry tricocycloid We now construct the Giry monoidal structure on **Set**. Remarkably, a construction from quantum algebra provides just what is needed.

An *abelian tricocycloid* [4] in a symmetric monoidal category $\mathcal{C}$ comprises an object H; an isomorphism $v\colon H \otimes H \to H \otimes H$ satisfying $(v \otimes 1)(1 \otimes \sigma)(v \otimes 1) = (1 \otimes v)(v \otimes 1)(1 \otimes v)$; and an involution $\gamma\colon H \to H$ satisfying $(1 \otimes \gamma)v(1 \otimes \gamma) = v(\gamma \otimes 1)v$. If $\mathcal{C}$ has finite coproducts distributing over $\otimes$, then (H, v, γ) induces a symmetric monoidal structure on $\mathcal{C}$, with unit 0 and binary tensor

$$A \star B = A + H \otimes A \otimes B + B\ . \qquad (3)$$

The maps v and γ appear in the associativity and symmetry constraints respectively.

Comparing (1) with (3) suggests instantiating this in **Set** with $H = (0,1)$. Indeed, defining v by $v(r,s) = (rs, \frac{rs^*}{(rs)^*})$—the terms appearing the third convex space axiom—and γ by $\gamma(r) = r^*$ yields an abelian tricocycloid, whose induced monoidal structure is the Giry one.

Other examples In [1] I examine the force of hypernormalisation for a range of linear exponential monads. In particular, I consider the *expectation monad* [3] on **Set**, involving involves finitely *additive* rather than finitely *supported* measures. This is linear exponential for the Giry monoidal structure; in fact, I conjecture that the expectation monad is *terminal* among such linear exponential monads.

References

[1] Garner, R. Hypernormalisation, linear exponential monads and the Giry tricocycloid. arXiv:1811.02710, 2018.

[2] Jacobs, B. Hyper normalisation and conditioning for discrete probability distributions. *Logical Methods in Computer Science 13* (2017), Paper No. 17, 29.

[3] Jacobs, B., Mandemaker, J., and Furber, R. The expectation monad in quantum foundations. *Information and Computation 250* (2016), 87–114.

[4] Street, R. Fusion operators and cocycloids in monoidal categories. *Applied Categorical Structures 6* (1998), 177–191.