# Completeness of Graphical Languages for Mixed States Quantum Mechanics

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Graphical languages that speak of quantum information can be formalised through the notion of symmetric monoidal categories. Hence, it has a nice graphical representation using string diagrams [34]. Qubits are represented by wires, and morphisms by graphical elements where some wires go in, and some others go out, just as in quantum circuits (which is actually a particular case of symmetric monoidal category), and where these graphical elements can be composed either in sequence (usual composition) or in parallel (tensor product). They usually come with an additional structure, a contravariant functor called dagger.

Examples of graphical languages for quantum mechanics and quantum computing are the quantum circuits and the ZX-Calculus [8]. Some variants of the ZX-calculus have been introduced more recently like the ZW-calculus [21] and the ZH-calculus [5]. All these languages are defined using generators (elementary gates) and come with an interpretation functor which associates to any diagram a pure quantum evolution, i.e. a morphism in the category of Hilbert spaces. Given a graphical language, there are generally several ways to represent a quantum evolution, thus a graphical language is also equipped with an equational theory which allows to transform a diagram into another equivalent diagram. A fundamental property is the completeness of the language: given two diagrams representing the same quantum evolution, one can be turned into the other using only the transformation rules in the theory.

The languages considered have usually been built so as to be able to rep-

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resent any pure quantum evolution, i.e. any perfectly isolated quantum system which hence does not interact with the environment. In this case, the language is called universal for pure quantum mechanics. Finite presentations for the quantum circuits were shown to be complete for some restrictions – namely Clifford [35], one-qubit Clifford+T [31], two-qubit Clifford+T [36], CNot-dihedral [1] –, however none of these restrictions is universal, nor approximately universal. Regarding the ZX-calculus, completeness results exist for non-universal restrictions of the ZX-Calculus [2, 3, 12, 20], but also for the many-qubit Clifford+T ZX-Calculus [26], which was the first completeness result for an approximately universal fragment of the language. Then complete theories have been introduced for the universal ZX-Calculus [23, 27, 28, 37] and ZW-Calculus [22, 23]. The completeness of the graphical languages for pure quantum mechanics is one of the main achievements of the categorical approach to quantum mechanics, and is the cornerstone for the application of this formalism in many areas of quantum information processing. The ZX-Calculus already proved to be useful for quantum information processing [10] (e.g. measurement-based quantum computing [14, 19, 24], quantum codes [13, 6, 16, 18], circuit optimisation [17], etc.). Moreover the ZX-calculus can be concretely used through two softwares: Quantomatic [30] and PyZX [29].

The existence of complete graphical languages beyond pure quantum mechanics for more general, not necessarily pure, quantum evolutions is an open question that we address in the present paper.

While pure quantum evolutions correspond to linear maps over Hilbert spaces, probability distributions over quantum states as well as some quantum evolutions like discarding a quantum system can be represented, following the von Neumann approach, by means of density matrices and completely positive maps. Selinger introduced a construction called CPM to turn a category for pure quantum mechanics into a category for density matrices and completely positive maps [33]. Another approach to relate pure quantum mechanics to the general one is the notion of environment structure [7, 9, 11]. The notion of *purification* is central in the definition of environment structure. The CPM-construction and the environment structure approaches have been proved to be equivalent [9].

In terms of graphical languages, the environment structure approach cannot be used in a straightforward way to extend a graphical language beyond pure quantum mechanics. Roughly speaking the environment structure approach provides second order axioms which cannot be easily handled by a equational theory on diagrams. Regarding the CPM-construction, the main property which has been exploited in [10] is that CPM(**C**) is essentially a subcategory of **C**, thus one can use a graphical language which has been designed for **C** in order to represent morphisms in CPM(**C**): Given a complete graphical language for **C**, we can use a subset of the pure diagrams to represent the evolutions in CPM(**C**). The main caveat of this approach is that this subset is not necessarily closed under the equational theory on pure diagrams, and as a consequence does not provide a complete graphical language for CPM(**C**).

**Our contributions.** In [25] was shown that the category **CPTPM** of completely positive trace-preserving maps is the universal monoidal category with a terminal unit and a functor from the category of isometries. We build upon this result by introducing a new construction, the *discard construction*, which transforms any †-symmetric monoidal category into a symmetric monoidal category equipped with a discard map. Roughly speaking this construction consists in making any isometry causal. Indeed, in quantum mechanics, the isometries (linear maps $U$ such $U^\dagger \circ U = I$) are known to be causal, i.e. applying $U$ and then discarding the subsystem on which it has been applied is equivalent to discarding the subsystem straightaway. Specifically, the discard construction proceeds as follows: first the discard is added to the subcategory of isometries, making the unit of the tensor a terminal object in this sub-category , as pointed out in [25]. Then the discard construction is obtained as the pushout of the resulting category and the initial one.

We show that the discard construction does not always produce an environment structure for the original category, and thus is not equivalent to the CPM construction. We show that a necessary and sufficient condition for the two constructions to be equivalent is that the initial category has enough isometries. We show that most of the categories usually used in the context of the categorical quantum mechanics, like **FHilb** and **Stab**, do have enough isometries, however **Clifford+T** does not.

Finally, we show that the discard construction provides a simple recipe to extend graphical languages beyond pure quantum mechanics. We provide an extension for several graphical languages that we prove to be complete for general quantum operations.

## References

[1] Matthew Amy, Jianxin Chen, and Neil J. Ross. A finite presentation of cnot-dihedral operators. In Bob Coecke and Aleks Kissinger, editors, *Proceedings 14th International Conference on Quantum Physics and Logic, Nijmegen, The Netherlands, 3-7 July 2017*, volume 266 of *Electronic Proceedings in Theoretical Computer Science*, pages 84–97. Open Publishing Association, 2018.

[2] Miriam Backens. The ZX-calculus is complete for stabilizer quantum mechanics. *New Journal of Physics*, 16(9):093021, sep 2014.

[3] Miriam Backens. The ZX-calculus is complete for the single-qubit Clifford+T group. *Electronic Proceedings in Theoretical Computer Science*, 172:293–303, dec 2014.

[4] Miriam Backens and Ali Nabi Duman. A complete graphical calculus for Spekkens' toy bit theory. *Foundations of Physics*, pages 1–34, 2014.

[5] Miriam Backens and Aleks Kissinger. Zh: A complete graphical calculus for quantum computations involving classical non-linearity. In Peter Selinger and Giulio Chiribella, editors, *Proceedings of the 15th International Conference on Quantum Physics and Logic, Halifax, Canada, 3-7th June 2018*, volume 287 of *Electronic Proceedings in Theoretical Computer Science*, pages 23–42. Open Publishing Association, 2019.

[6] Nicholas Chancellor, Aleks Kissinger, Joschka Roffe, Stefan Zohren, and Dominic Horsman. Graphical structures for design and verification of quantum error correction. last revised Jan. 2018, 2016.

[7] Bob Coecke. Axiomatic description of mixed states from Selinger's CPM-construction. *Electronic Notes in Theoretical Computer Science*, 210:3 – 13, 2008. Proceedings of the 4th International Workshop on Quantum Programming Languages (QPL 2006).

[8] Bob Coecke and Ross Duncan. Interacting quantum observables: Categorical algebra and diagrammatics. *New Journal of Physics*, 13(4):043016, apr 2011.

[9] Bob Coecke and Chris Heunen. Pictures of complete positivity in arbitrary dimension. *Information and Computation*, 250:50–58, 2016.

[10] Bob Coecke and Aleks Kissinger. *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*. Cambridge University Press, 2017.

[11] Bob Coecke and Simon Perdrix. Environment and Classical Channels in Categorical Quantum Mechanics. *Logical Methods in Computer Science*, Volume 8, Issue 4, November 2012.

[12] Bob Coecke and Quanlong Wang. ZX-rules for 2-qubit Clifford+T quantum circuits, 2018.

[13] Niel de Beaudrap and Dominic Horsman. The ZX-calculus is a language for surface code lattice surgery. *CoRR*, abs/1704.08670, 2017.

[14] Ross Duncan. A graphical approach to measurement-based quantum computing. In *Quantum Physics and Linguistics*, pages 50–89. Oxford University Press, feb 2013.

[15] Ross Duncan and Kevin Dunne. Interacting Frobenius algebras are Hopf. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS 2016, pages 535–544, New York, NY, USA, 2016. ACM.

[16] Ross Duncan and Liam Garvie. Verifying the smallest interesting colour code with quantomatic. In Bob Coecke and Aleks Kissinger, editors, *Proceedings 14th International Conference on Quantum Physics and Logic, Nijmegen, The Netherlands, 3-7 July 2017*, volume 266 of *Electronic Proceedings in Theoretical Computer Science*, pages 147–163. Open Publishing Association, 2018.

[17] Ross Duncan, Aleks Kissinger, Simon Perdrix, and John van de Wetering. Graph-theoretic simplification of quantum circuits with the ZX-calculus, 2019.

[18] Ross Duncan and Maxime Lucas. Verifying the Steane code with Quantomatic. *Electronic Proceedings in Theoretical Computer Science*, 171:33–49, dec 2014.

[19] Ross Duncan and Simon Perdrix. Rewriting measurement-based quantum computations with generalised flow. *Lecture Notes in Computer Science*, 6199:285–296, 2010.

[20] Ross Duncan and Simon Perdrix. Pivoting makes the ZX-calculus complete for real stabilizers. In *QPL 2013*, Electronic Proceedings in Theoretical Computer Science, pages 50–62, 2013.

[21] Amar Hadzihasanovic. A diagrammatic axiomatisation for qubit entanglement. In *2015 30th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 573–584, July 2015.

[22] Amar Hadzihasanovic. *The Algebra of Entanglement and the Geometry of Composition*. PhD thesis, University of Oxford, 2017.

[23] Amar Hadzihasanovic, Kang Feng Ng, and Quanlong Wang. Two complete axiomatisations of pure-state qubit quantum computing. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '18, pages 502–511, New York, NY, USA, 2018. ACM.

[24] Clare Horsman. Quantum picturalism for topological cluster-state computing. *New Journal of Physics*, 13(9):095011, sep 2011.

[25] Mathieu Huot and Sam Staton. Universal properties in quantum theory. In Peter Selinger and Giulio Chiribella, editors, *Proceedings of the 15th International Conference on Quantum Physics and Logic, Halifax, Canada, 3-7th June 2018*, volume 287 of *Electronic Proceedings in Theoretical Computer Science*, pages 213–223. Open Publishing Association, 2019.

[26] Emmanuel Jeandel, Simon Perdrix, and Renaud Vilmart. A complete axiomatisation of the ZX-calculus for Clifford+T quantum mechanics. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '18, pages 559–568, New York, NY, USA, 2018. ACM.

[27] Emmanuel Jeandel, Simon Perdrix, and Renaud Vilmart. Diagrammatic reasoning beyond Clifford+T quantum mechanics. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '18, pages 569–578, New York, NY, USA, 2018. ACM.

[28] Emmanuel Jeandel, Simon Perdrix, and Renaud Vilmart. A generic normal form for ZX-diagrams and application to the rational angle completeness, 2018.

[29] A. Kissinger and John van de Wetering. Pyzx, 2018.

[30] Aleks Kissinger and Vladimir Zamdzhiev. Quantomatic: A proof assistant for diagrammatic reasoning. In Amy P. Felty and Aart Middeldorp, editors, *Automated Deduction - CADE-25*, pages 326–336, Cham, 2015. Springer International Publishing.

[31] Ken Matsumoto and Kazuyuki Amano. Representation of Quantum Circuits with Clifford and $\pi/8$ Gates, June 2008.

[32] Peter Selinger. Towards a quantum programming language. *Mathematical. Structures in Comp. Sci.*, 14(4):527–586, August 2004.

[33] Peter Selinger. Dagger compact closed categories and completely positive maps. *Electronic Notes in Theoretical Computer Science*, 170:139–163, mar 2007.

[34] Peter Selinger. A survey of graphical languages for monoidal categories. In *New structures for physics*, pages 289–355. Springer, 2010.

[35] Peter Selinger. Generators and Relations for n-qubit Clifford Operators. *Logical Methods in Computer Science*, Volume 11, Issue 2, June 2015.

[36] Peter Selinger and Xiaoning Bian. Relations for Clifford+T operators on two qubits, 2015.

[37] Renaud Vilmart. A near-optimal axiomatisation of ZX-calculus for pure qubit quantum mechanics, 2018.