Elementary Category Theory In Pictures

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Proofs in Elementary Category Theory

Commuting Diagrams

$$
\begin{array}{ccc}
T'F(X) & \xrightarrow{T'F(h)} & T'F(Y) \\
\downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} \\
FT(X) & \xrightarrow{FT(h)} & FT(Y) \\
\downarrow{\scriptstyle F(a)} & & \downarrow{\scriptstyle F(a')} \\
F(X) & \xrightarrow[F(h)]{} & F(Y)
\end{array}
$$

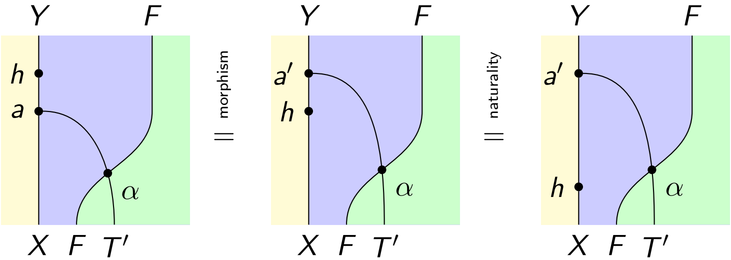
Proofs in Elementary Category Theory

Fokkinga and Meertens: Equational Proof

$$
\begin{aligned}
& F(h) \circ F(a) \circ \alpha_X \\
= & \quad \{ \text{ functoriality } \} \\
& F(h \circ a) \circ \alpha_X \\
= & \quad \{ \ h \text{ is a } T\text{-algebra homomorphism } \} \\
& F(a' \circ T(h)) \circ \alpha_X \\
= & \quad \{ \text{ functoriality } \} \\
& F(a') \circ FT(h) \circ \alpha_X \\
= & \quad \{ \text{ naturality } \} \\
& F(a') \circ \alpha_Y \circ T'F(h)
\end{aligned}
$$

Proofs in Elementary Category Theory

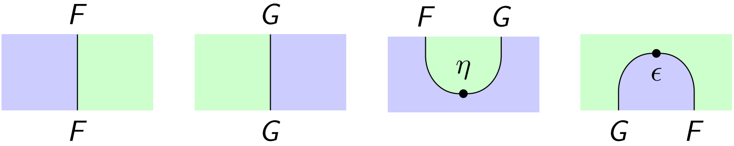
String Diagrams



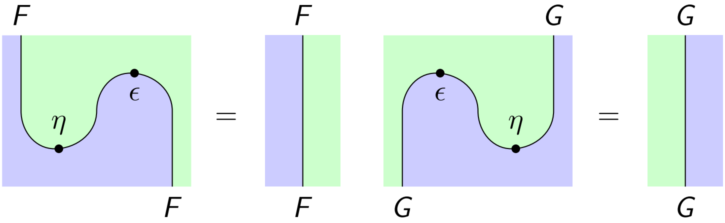
Conventional Material

Adjunctions and Monads

An adjunction consists of functors and natural transformations:



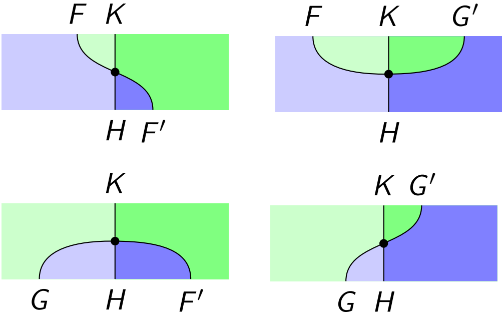
satisfying the following “snake equations”



Conventional Material

Adjunctions and Monads

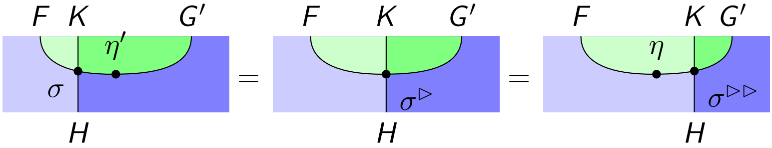
An exercise in wire bending from MacLane, given $F \dashv G$ and $F' \dashv G'$:



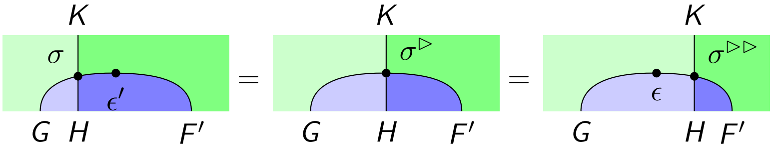
Conventional Material

Adjunctions and Monads

We can slide natural transformations around unit bends:



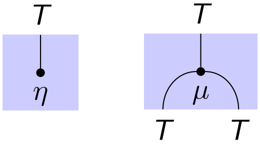
Similarly we can slide natural transformations around counit bends:



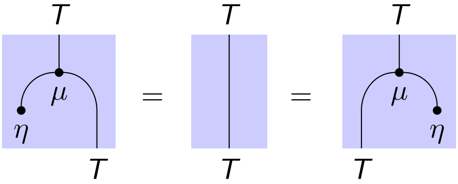
Conventional Material

Adjunctions and Monads

An endofunctor $T : \mathcal{C} \to \mathcal{C}$ with **unit** and **multiplication**:



With **unit axioms**:



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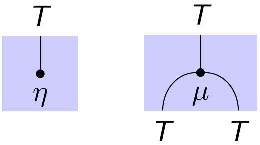
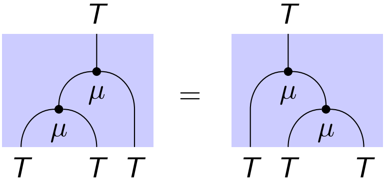


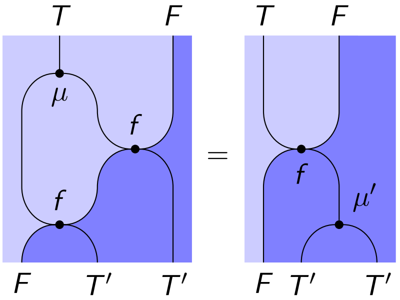
and **associativity axioms**:



Conventional Material

Distributive Laws - Artwork Matters

Given monads $(T : \mathcal{C} \to \mathcal{C}, \eta, \mu)$ and $(S : \mathcal{D} \to \mathcal{D}, \eta, \mu)$:



Conventional Material

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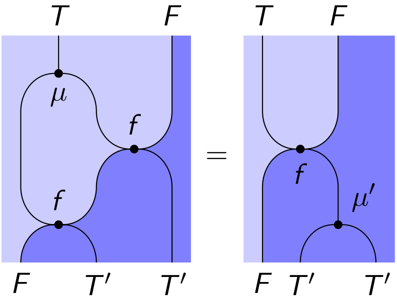
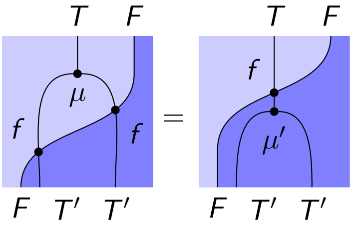
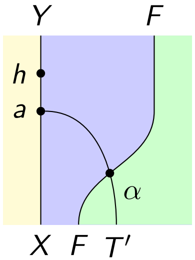




Unconventional Material

Objects and Morphisms

"Ordinary" objects are functors $1 \to \mathcal{C}$ and "ordinary" morphisms are natural transformations between them:

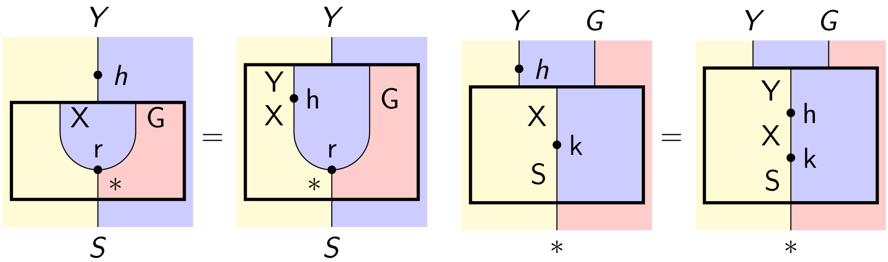


Unconventional Material

Representables

$$G \cong \mathcal{C}(S, -)$$

We represent the isomorphism as boxes satisfying "push and pop" axioms:



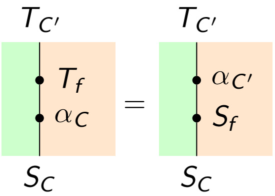
Unconventional Material

Bifunctors

Bifunctors and naturality in multiple variables:

$$[\mathcal{C} \times \mathcal{D}, \mathcal{E}] \cong [\mathcal{C}, [\mathcal{D}, \mathcal{E}]]$$

Naturality and functoriality equations:



Unconventional Material

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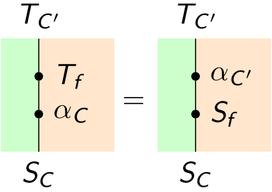
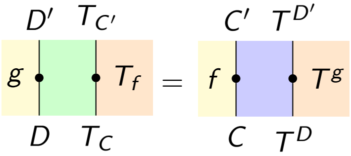
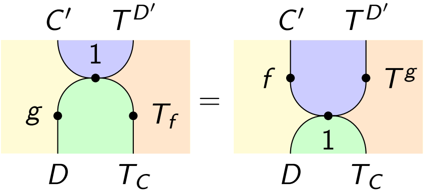


Witnessing identity 2-cells:

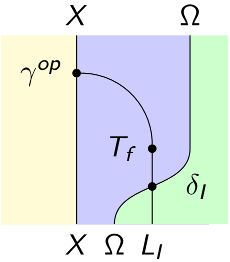




Concrete Application

Coalgebras and coalgebraic logic commonly use:

- "Ordinary morphisms"
- Transformations of signature functors
- Monads
- Distributive laws
- Logical connections



$= \{ \text{ Bifunctors } \}$