

Elementary Category Theory In Pictures

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Proofs in Elementary Category Theory

Commuting Diagrams

$$\begin{array}{ccc} T'F(X) & \xrightarrow{T'F(h)} & T'F(Y) \\ \alpha_X \downarrow & & \downarrow \alpha_Y \\ FT(X) & \xrightarrow{FT(h)} & FT(Y) \\ F(a) \downarrow & & \downarrow F(a') \\ F(X) & \xrightarrow{F(h)} & F(Y) \end{array}$$

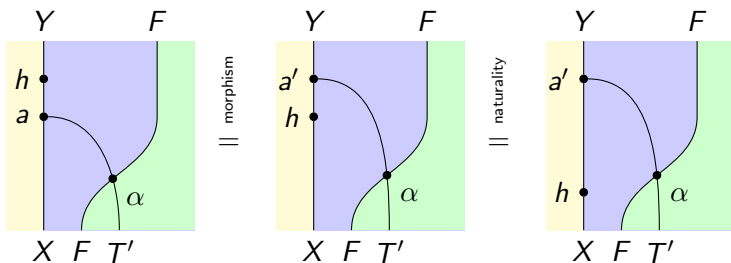
Proofs in Elementary Category Theory

Fokkinga and Meertens: Equational Proof

$$\begin{aligned} & F(h) \circ F(a) \circ \alpha_X \\ = & \{ \text{functoriality} \} \\ & F(h \circ a) \circ \alpha_X \\ = & \{ h \text{ is a } T\text{-algebra homomorphism} \} \\ & F(a' \circ T(h)) \circ \alpha_X \\ = & \{ \text{functoriality} \} \\ & F(a') \circ FT(h) \circ \alpha_X \\ = & \{ \text{naturality} \} \\ & F(a') \circ \alpha_Y \circ T'F(h) \end{aligned}$$

Proofs in Elementary Category Theory

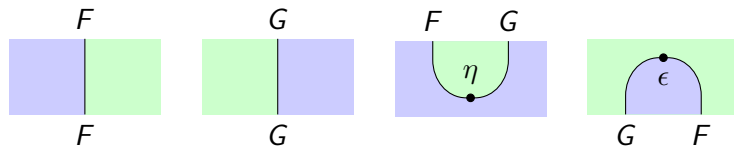
String Diagrams



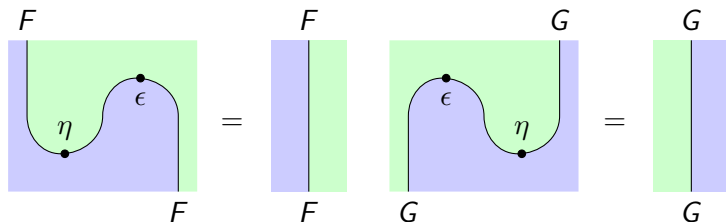
Conventional Material

Adjunctions and Monads

An adjunction consists of functors and natural transformations:



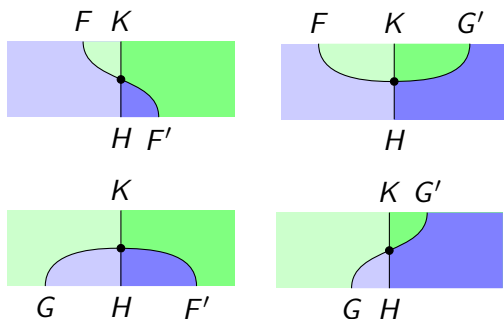
satisfying the following “snake equations”



Conventional Material

Adjunctions and Monads

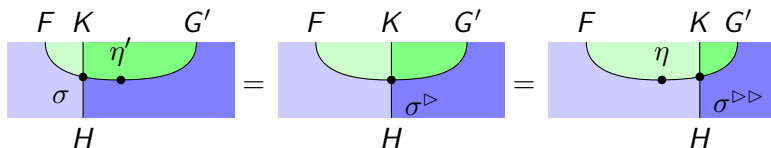
An exercise in wire bending from MacLane, given $F \dashv G$ and $F' \dashv G'$:



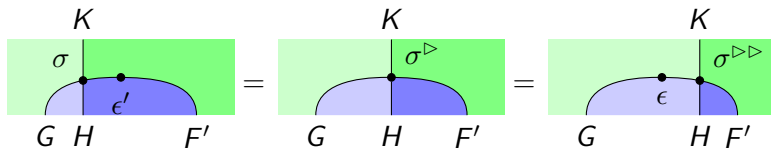
Conventional Material

Adjunctions and Monads

We can slide natural transformations around unit bends:



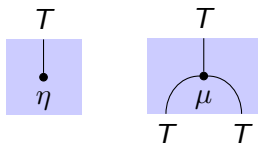
Similarly we can slide natural transformations around counit bends:



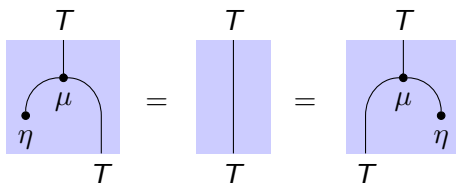
Conventional Material

Adjunctions and Monads

An endofunctor $T : \mathcal{C} \rightarrow \mathcal{C}$ with **unit** and **multiplication**:



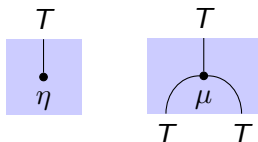
With **unit axioms**:



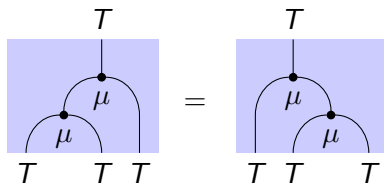
Conventional Material

Adjunctions and Monads

An endofunctor $T : \mathcal{C} \rightarrow \mathcal{C}$ with **unit** and **multiplication**:



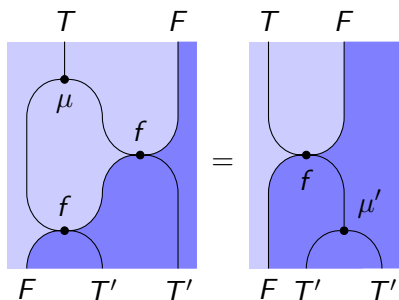
and **associativity** axioms:



Conventional Material

Distributive Laws - Artwork Matters

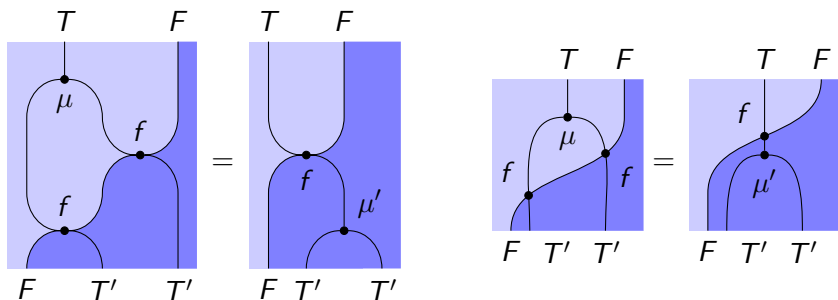
Given monads $(T : \mathcal{C} \rightarrow \mathcal{C}, \eta, \mu)$ and $(S : \mathcal{D} \rightarrow \mathcal{D}, \eta, \mu)$:



Conventional Material

Distributive Laws - Artwork Matters

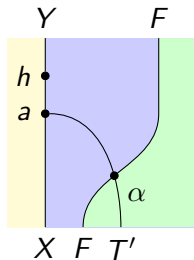
Given monads $(T : \mathcal{C} \rightarrow \mathcal{C}, \eta, \mu)$ and $(S : \mathcal{D} \rightarrow \mathcal{D}, \eta, \mu)$:



Unconventional Material

Objects and Morphisms

“Ordinary” objects are functors $1 \rightarrow \mathcal{C}$ and “ordinary” morphisms are natural transformations between them:

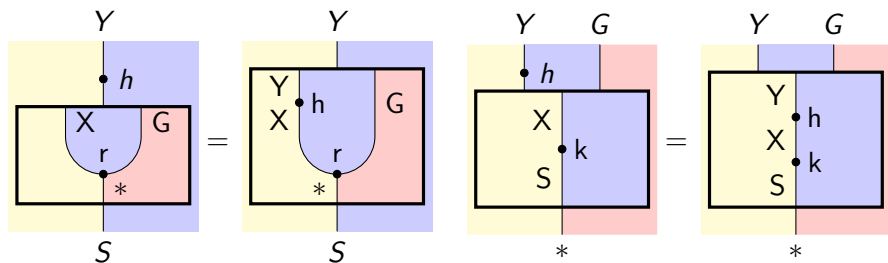


Unconventional Material

Representables

$$G \cong \mathcal{C}(S, -)$$

We represent the isomorphism as boxes satisfying “push and pop” axioms:



Unconventional Material

Bifunctors

Bifunctors and naturality in multiple variables:

$$[\mathcal{C} \times \mathcal{D}, \mathcal{E}] \cong [\mathcal{C}, [\mathcal{D}, \mathcal{E}]]$$

Naturality and functoriality equations:

$$\begin{array}{c} T_{C'} \\ \bullet T_f \\ \bullet \alpha_C \\ S_C \end{array} = \begin{array}{c} T_{C'} \\ \bullet \alpha_{C'} \\ \bullet S_f \\ S_C \end{array}$$

Unconventional Material

Bifunctors

Bifunctors and naturality in multiple variables:

$$[\mathcal{C} \times \mathcal{D}, \mathcal{E}] \cong [\mathcal{C}, [\mathcal{D}, \mathcal{E}]]$$

Witnessing identity 2-cells:

Naturality and functoriality equations:

A diagram illustrating the naturality equation. On the left, a vertical line is divided into a green section on the left and an orange section on the right. The green section has a dot labeled T_f and the orange section has a dot labeled α_C . Below the green section is the label S_C . On the right, a similar vertical line is shown, but the orange section has a dot labeled S_f and the green section has a dot labeled $\alpha_{C'}$. Below the green section is the label S_C . An equals sign is placed between the two diagrams.

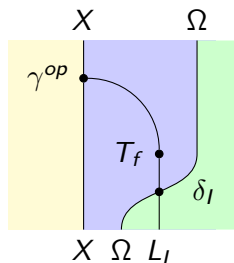
A diagram illustrating a witnessing identity 2-cell. On the left, a vertical line is divided into three colored sections: yellow on the left, green in the middle, and orange on the right. The yellow section has a dot labeled g , the green section has a dot labeled T_f , and the orange section has a dot labeled T_f . Above the green section is the label D' and above the orange section is the label $T_{C'}$. Below the green section is the label D and below the orange section is the label T_C . On the right, a similar vertical line is shown, but the middle section is blue. The yellow section has a dot labeled f , the blue section has a dot labeled T^g , and the orange section has a dot labeled T^g . Above the blue section is the label C' and above the orange section is the label $T^{D'}$. Below the blue section is the label C and below the orange section is the label T^D . An equals sign is placed between the two diagrams.

A diagram illustrating a witnessing identity 2-cell. On the left, a vertical line is divided into three colored sections: yellow on the left, green in the middle, and orange on the right. The yellow section has a dot labeled g , the green section has a dot labeled T_f , and the orange section has a dot labeled T_f . Above the green section is a blue semi-circle labeled 1 . Below the green section is the label D and below the orange section is the label T_C . On the right, a similar vertical line is shown, but the middle section is blue. The yellow section has a dot labeled f , the blue section has a dot labeled T^g , and the orange section has a dot labeled T^g . Above the blue section is a blue semi-circle labeled 1 . Below the blue section is the label D and below the orange section is the label T_C . An equals sign is placed between the two diagrams.

Concrete Application

Coalgebras and coalgebraic logic commonly use:

- ▶ “Ordinary morphisms”
- ▶ Transformations of signature functors
- ▶ Monads
- ▶ Distributive laws
- ▶ Logical connections



= { Bifunctors }

