千人計劃

RECRUITMENT
PROGRAM OF GLOBAL EXPERTS

FQXi
FOUNDATIONAL QUESTIONS INSTITUTE

CATEGORICAL PURIFICATION

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Celebrating 10 years of CQM

@Jericho Tavern,

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PROLOGUE

PURIFICATION IN HILBERT SPACE QUANTUM MECHANICS

For every mixed state of system A, say $\rho = \sum_{k=1}^{r} p_k \, |\alpha_k\rangle\langle\alpha_k|$

there exists a system B and a pure state (the "purification" of ρ)

$$|\Psi\rangle = \sum_{k=1}^{r} \sqrt{p_k} \, |\alpha_k\rangle|\beta_k\rangle$$

such that $\rho = \mathrm{Tr}_B\big[\, |\Psi\rangle\langle\Psi| \,\big]$

Once ρ is given, the purification is essentially fixed.

FROM THEOREMS TO AXIOMS

Purification = GNS construction

- fascinating structure at the basis of dilation theorems (Stinespring/Kraus, Naimark)

- ubiquitous in the derivation of quantum protocols/ quantum features

Idea: turn purification into an axiom

OPERATIONAL-PROBABILISTIC THEORIES (CDP 2009)

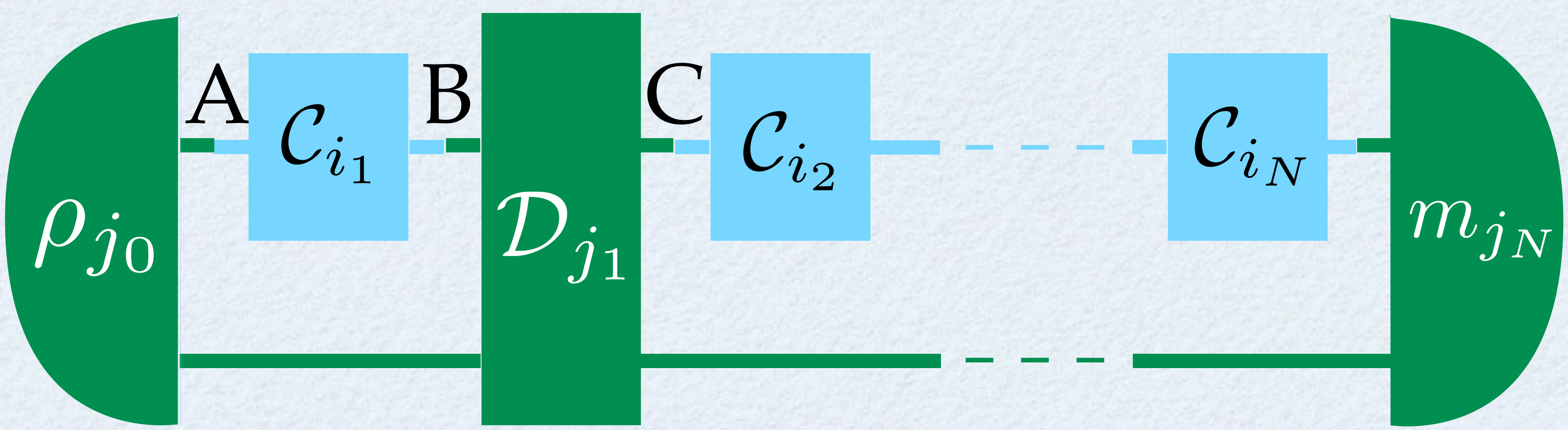
Operational-probabilistic theory

=

operational structure

+

probabilistic structure



$$= p(j_0, j_1, \ldots, j_N)$$

Processes embedded in a category of ordered vector spaces and positive maps.

PURE/MIXED STATES IN OPERATIONAL-PROBABILISTIC THEORIES

Mixed process: process obtained by coarse-graining over some outcomes of a given test

ignorance about which process is actually taking place in the lab

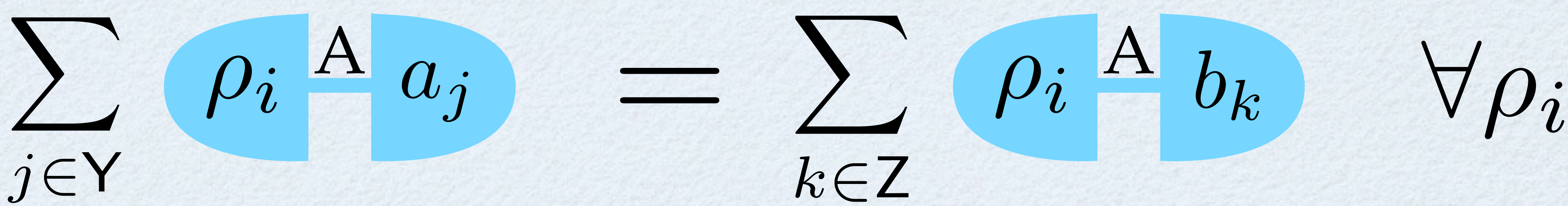
Pure process : process that cannot be obtained by a (non-trivial) coarse-graining

maximal knowledge about what is happening

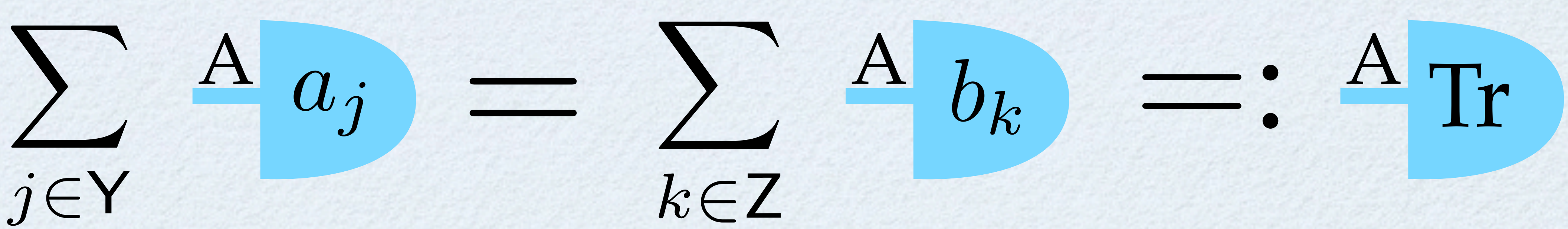
CAUSALITY

The choice of tests performed in the future cannot influence the probabilities of outcomes of tests performed in the present.

For every pair of measurements $\{a_j\}_{j\in\mathsf{Y}}$ and $\{b_k\}_{k\in\mathsf{Z}}$



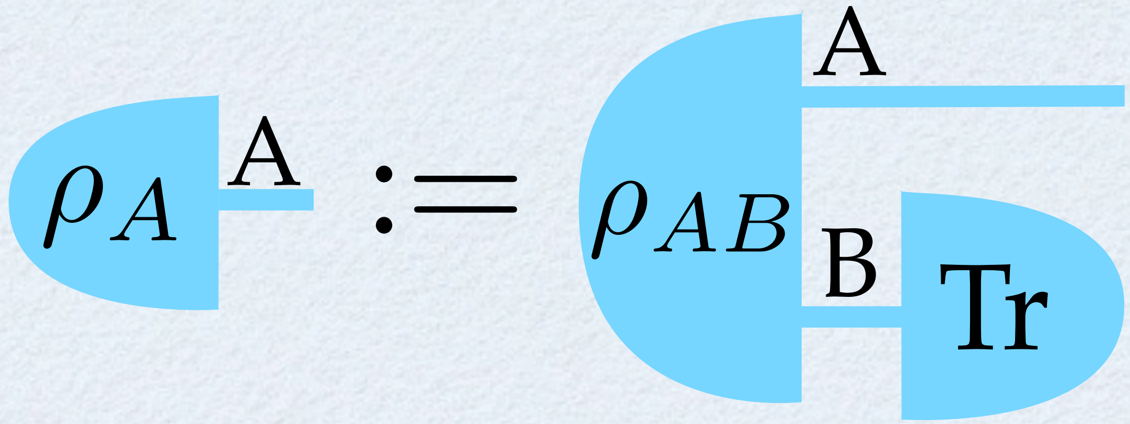
Equivalent condition: there exists a unique deterministic effect



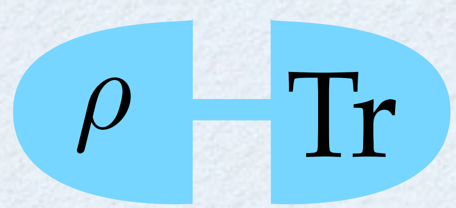
MARGINAL STATES

Uniqueness of the deterministic effect → only one way to discard a system

→ marginal states are uniquely defined



e.g. in QT:

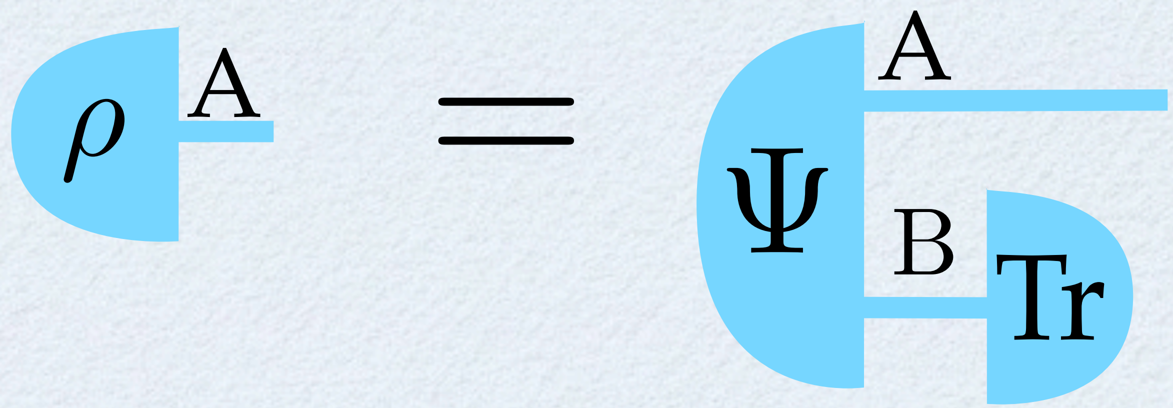


$= \mathrm{Tr}[\rho]$

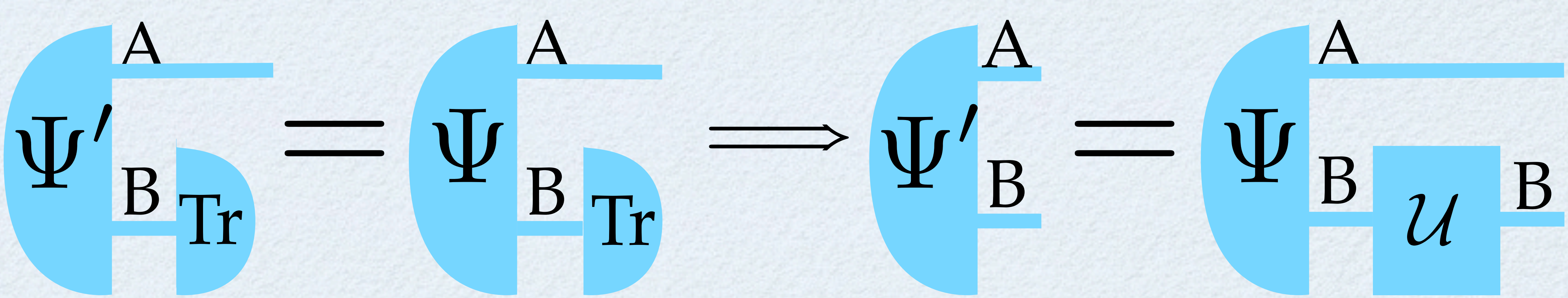
$$\rho_A := \mathrm{Tr}_A[\rho_{AB}]$$

THE PURIFICATION AXIOM

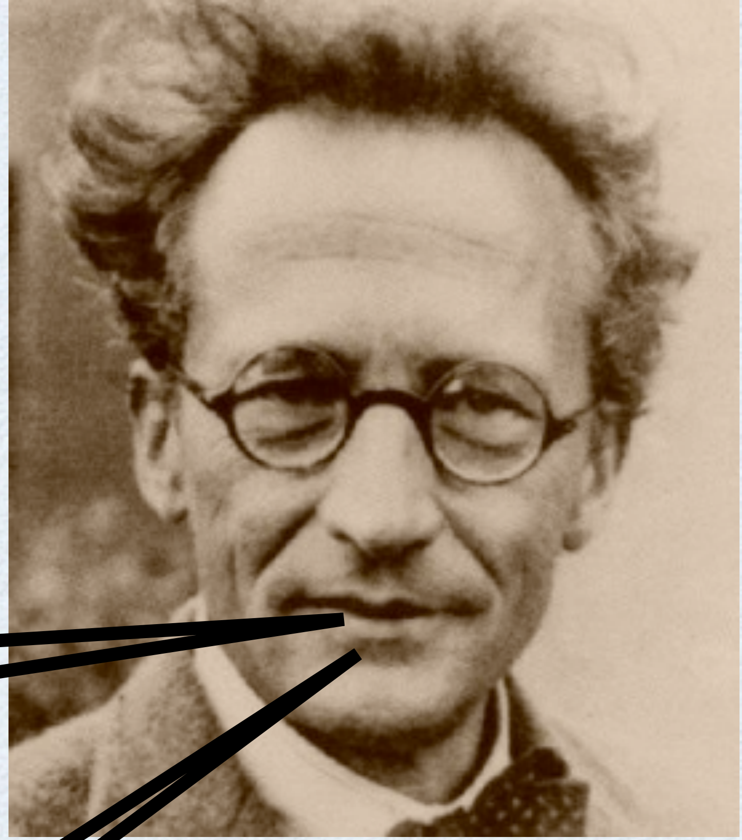
- Existence: For every state ρ of A
  there is a system B and a pure state Ψ of $A \otimes B$
  such that



- Uniqueness: two purifications of the same state are equivalent
  up to a reversible transformation



SCHROEDINGER AND PURIFICATION



The best possible knowledge of a whole does not necessarily imply the best possible knowledge of its parts.

I would not call that *one* but rather *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought.

AXIOMATIZATION OF QT (CDP 2010)

- Causality
- Fine-Grained Composition
- Perfect Distinguishability
- Ideal Compression
- Local tomography

$+$

PURIFICATION

$=$

Quantum Theory
(in finite dimensions)

AXIOMATIZATION OF QUANTUM PROTOCOLS

- Entanglement
- No Cloning
- No Information Without Disturbance
- Teleportation
- Steering
- Existence of perfectly-correlating states
- Ancilla-assisted process tomography
- Reversible simulation of irreversible processes
- No Bit Commitment
- Principle of Delayed Measurement
- No Programming Theorem
- Error correction balance
- Structure of no-signalling channels

...

COOL, BUT...

- do we really need the probabilistic structure?

cf. Schumacher's and Westmoreland's quantum theory on finite fields

- and what is so special about pure states?

(they play a prominent role
in most of the known axiomatization schemes)

THE CQM COUSINS OF PURIFICATION

- Selinger's CPM construction: it constructs a category of "mixed processes" starting from a category of "pure processes" (dagger compact $\longrightarrow$ dagger compact)

- Coecke, Coecke-Perdrix environment structure: it axiomatizes CPM by adding a "partial trace" to a dagger compact category of "pure processes".

- Coecke-Lal purification: environment structure + uniqueness of purification up to isometries.

STILL...

- do we really need the dagger compact structure?
  (perhaps not)

- what *are* the pure processes?

Are they pure because they have some special property?
Or being “pure” is just a name that indicates membership to a distinguished---but otherwise arbitrary---class of processes?

NOT JUST QUANTUM FOUNDATIONS

The foundation of the notion of pure state/pure process is related to two rather fundamental questions:

- What is "maximal knowledge"?
- How can one acquire an integral piece of information?

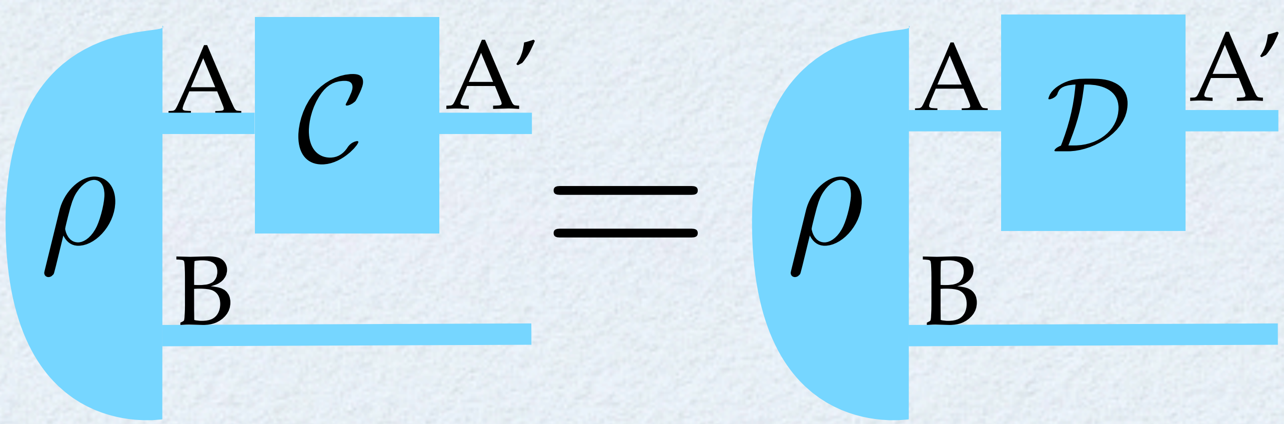
DE-CONVEXIFICATION
OF
PURE STATES

THE FRAMEWORK:
CAUSAL DETERMINISTIC CATEGORIES

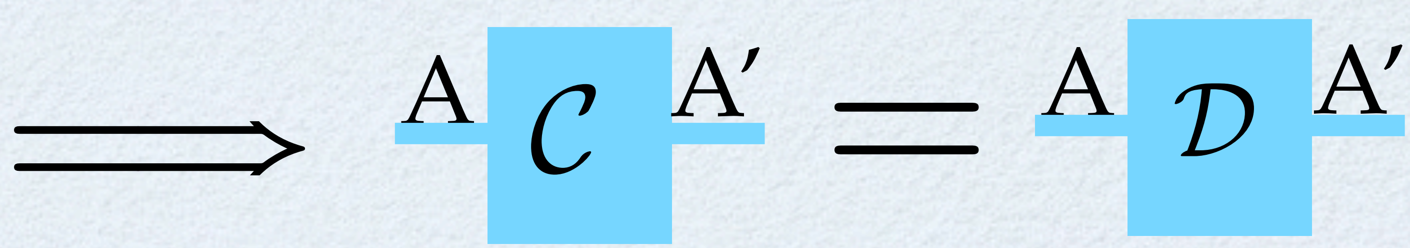
Consider a process category Det with the following features:

-the monoidal unit is terminal (Coecke-Lal)

-states separate processes



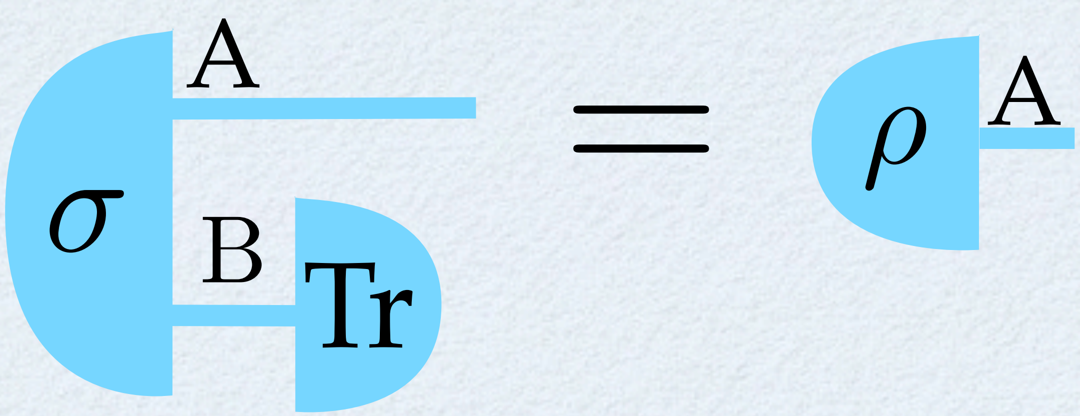
$$\forall B, \forall \rho : I \to A \otimes B$$



CONTEXTS

Think of a state as a piece of information.
What are the contexts that are compatible with that piece of information?

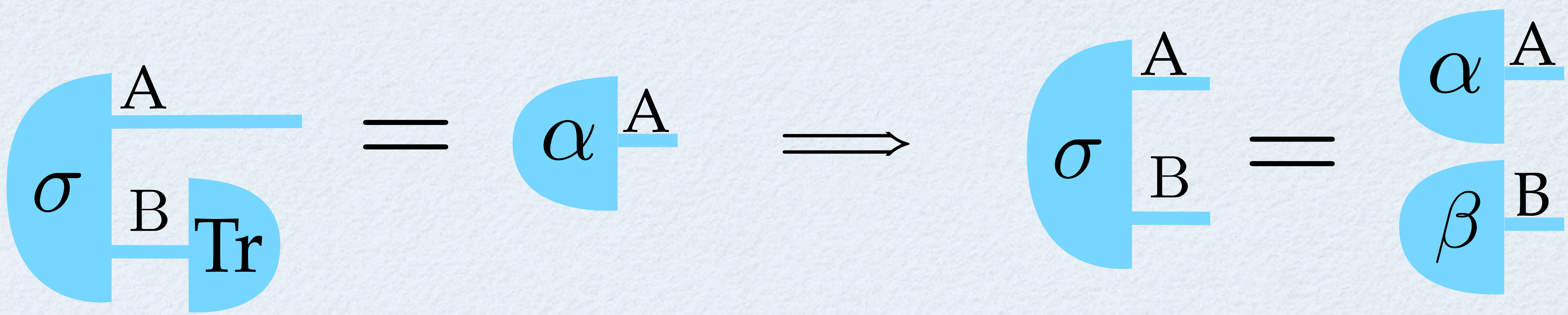
Definition: σ is an extension of ρ iff



CATEGORICAL DEFINITION OF PURE STATE

Definition: a state is pure iff it only has trivial extensions:

α is pure iff



Informally,

pure state = piece of information that is independent of the context.

PROPERTIES

- Pure states form a monoid:

$$\alpha : I \to A \text{ pure}\,,\, \beta : I \to B \text{ pure}$$

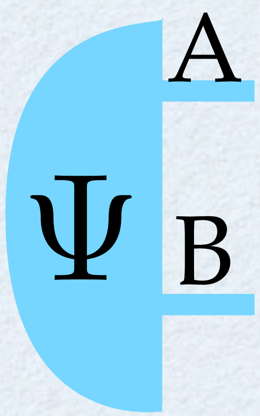
$$\Longrightarrow \alpha \otimes \beta : I \to A \otimes B \text{ pure}$$

- Reversible transformations (i.e. isomorphisms) preserve pure states

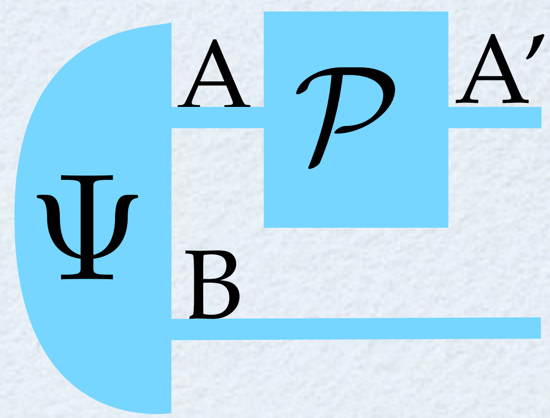
$$\alpha : I \to A \text{ pure, } \mathcal{U} : A \to B \text{ iso} \qquad \Longrightarrow \alpha\,;\mathcal{U} \text{ pure}$$

THE CATEGORY OF PURITY-PRESERVING PROCESSES

Definition: a process $\mathcal{P}$ is purity-preserving iff



$\in \mathsf{PurSt}(A \otimes B) \quad \Longrightarrow$



$\in \mathsf{PurSt}(A' \otimes B)$

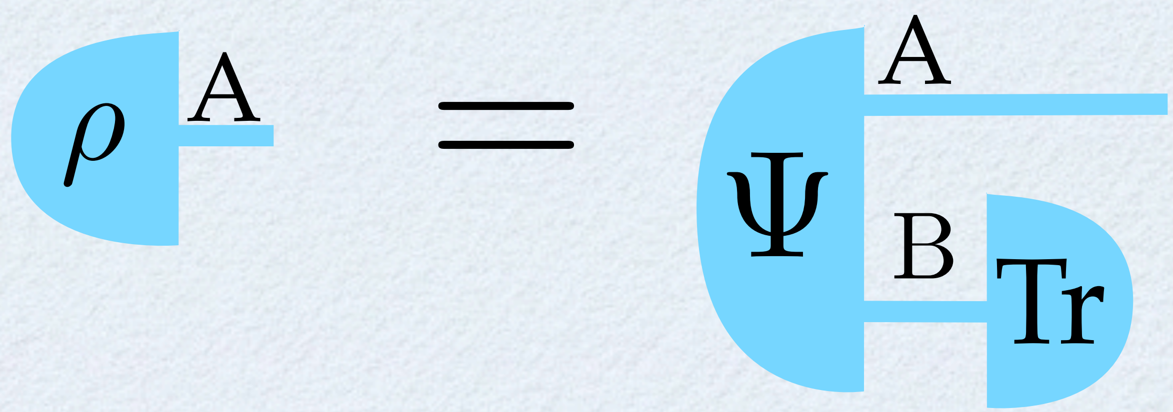
Property:

- purity-preserving processes form a symmetric monoidal subcategory of Det , containing the monoid of pure states

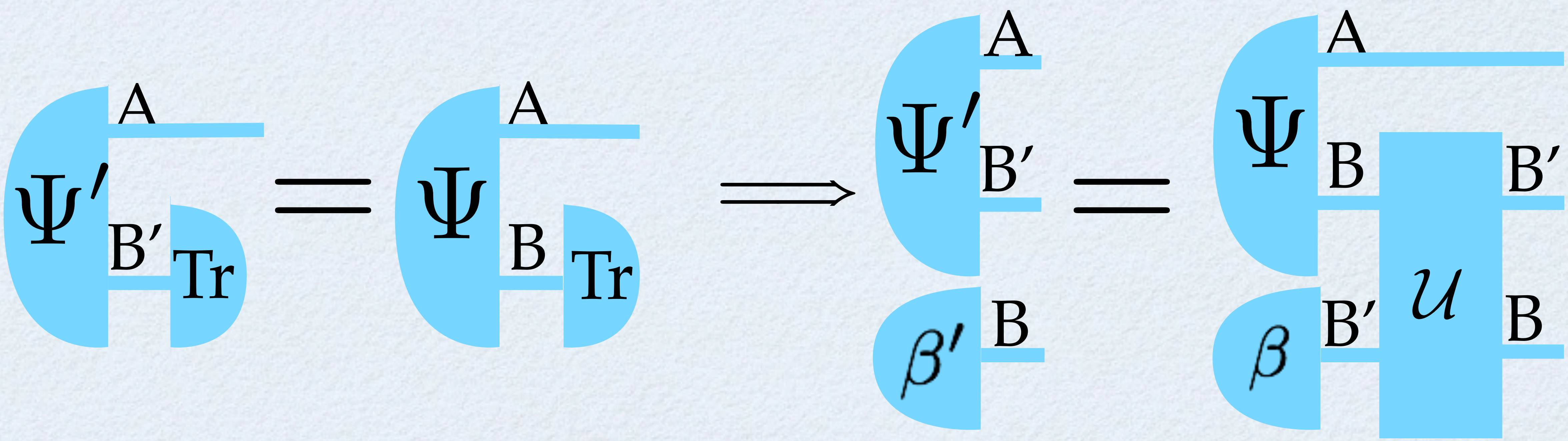
PURIFICATION

CATEGORICAL PURIFICATION

- Existence: For every state ρ of A
  there is a system B and a pure state Ψ of $A \otimes B$
  such that



- Uniqueness: all purifications of the same state are equivalent
  up to isos on the context



GOOD, BUT...

Does the Categorical Purification Axiom give all the features it gave in the convex world?

like, e. g. entanglement? or no-cloning?

mhm... wait!

We don't know yet if our category contains mixed states!

In fact, classical deterministic computation satisfies Purification, and has no entanglement nor a no-cloning theorem

MIXED AND FAITHFUL STATES

MIXED, MIXED, AND MORE MIXED...

Definition: a state is mixed if it is not pure ;-)

Good, but when is a state "more mixed" than another?

In the convex world, one can say that ρ is "more mixed" than σ

iff

$$\rho = p\,\sigma + (1-p)\,\tau$$

for some p>0 and some state τ

However, the above expression is not "legal" in our language...

EXTENSIONS OF MIXED STATES

In the convex world, if ρ is "more mixed" than σ

then, for every extension of σ, say $\sigma' \in \mathsf{St}(A \otimes B)$

there exists an extension of ρ, say ρ'

that is "more mixed" than σ'

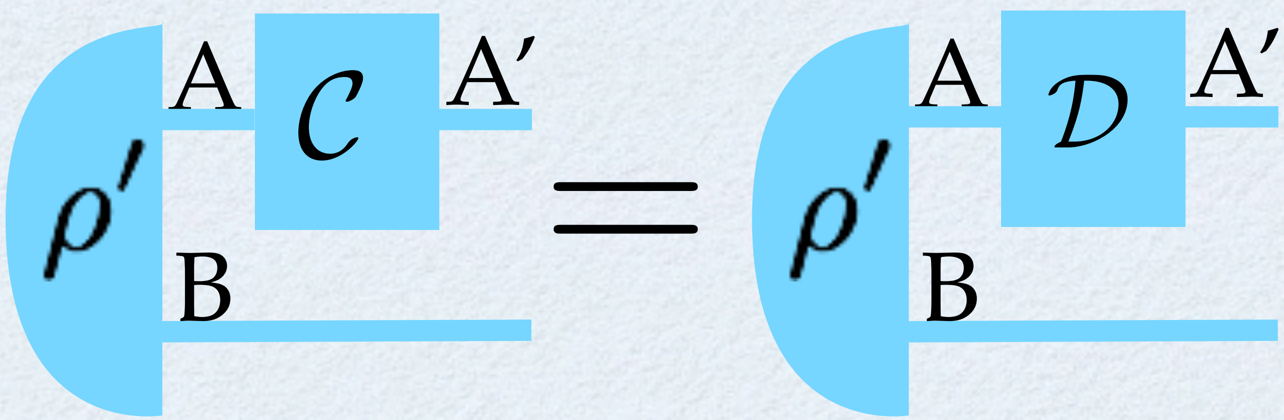
e. g. take $\rho' = p\sigma' + (1-p)\,\tau \otimes \beta$ for abitrary β

Idea: leverage on this property at the categorical level

CATEGORICAL DEFINITION OF "MORE MIXED"

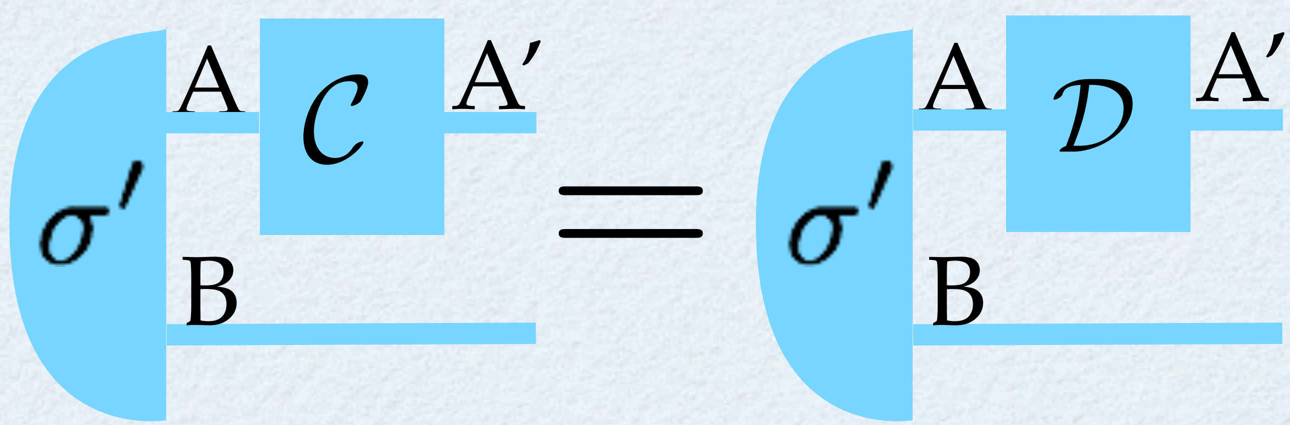
Definition: ρ is sufficient for σ , denoted $\rho \succeq \sigma$

iff



for every extension of ρ

$\Longrightarrow$

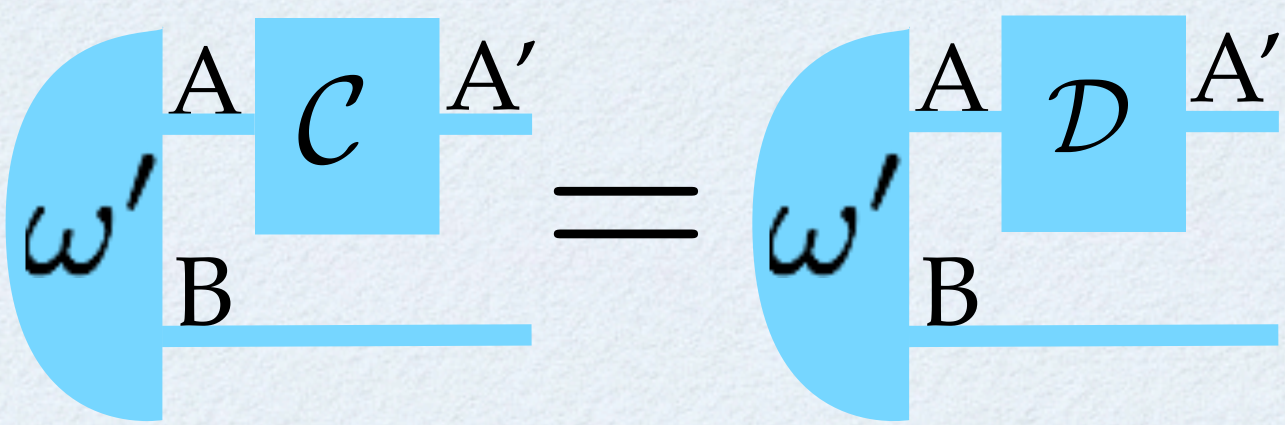


for every extension of σ

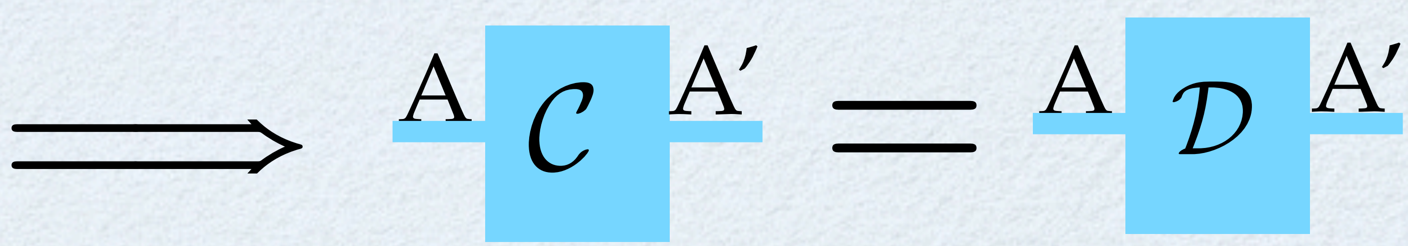
FAITHFUL STATES

Definition: $\omega \in \mathsf{St}(A)$ is faithful iff $\omega \succeq \rho\,, \quad \forall \rho \in \mathsf{St}(A)$

In other words:



for every extension of ω

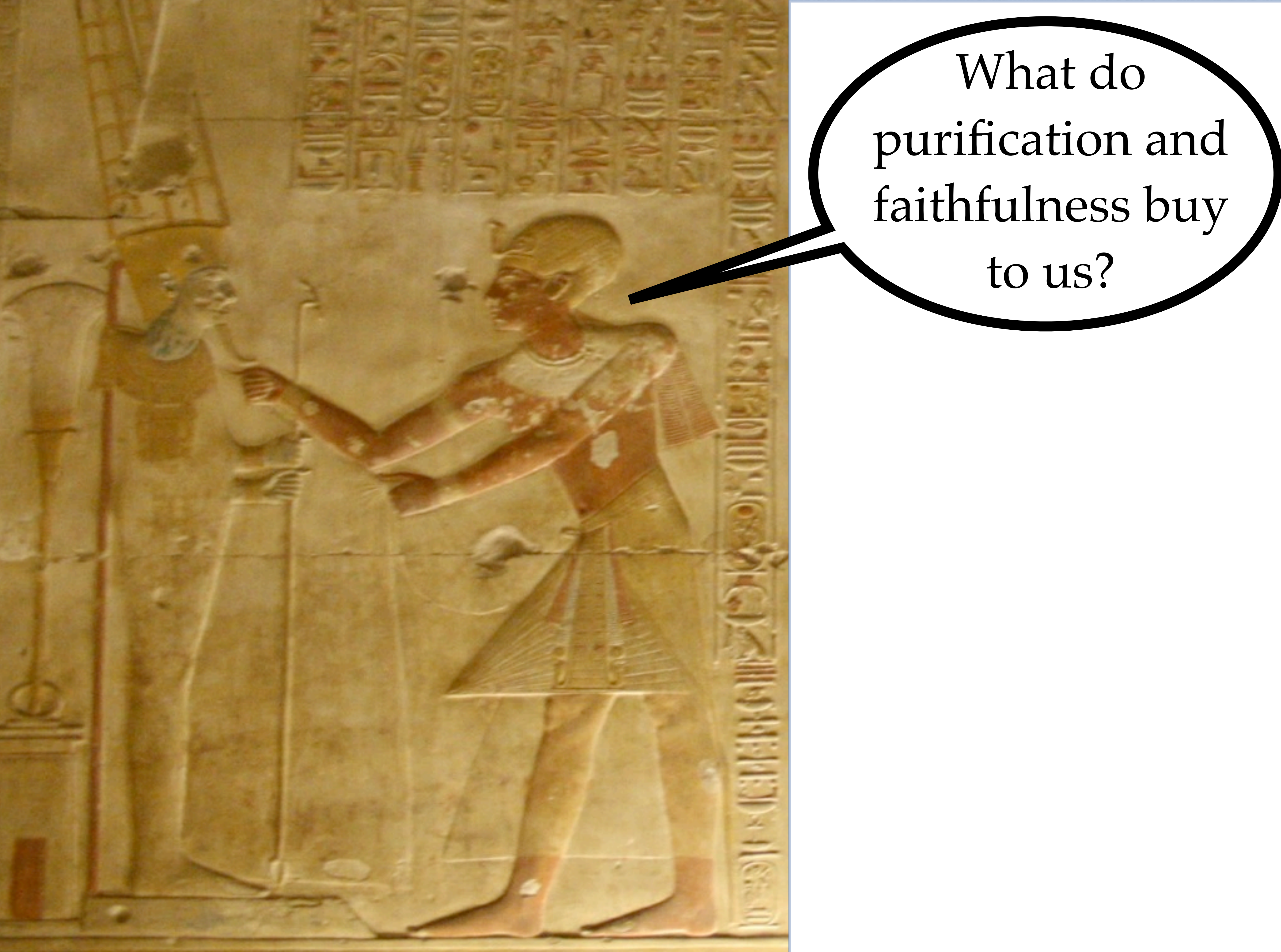


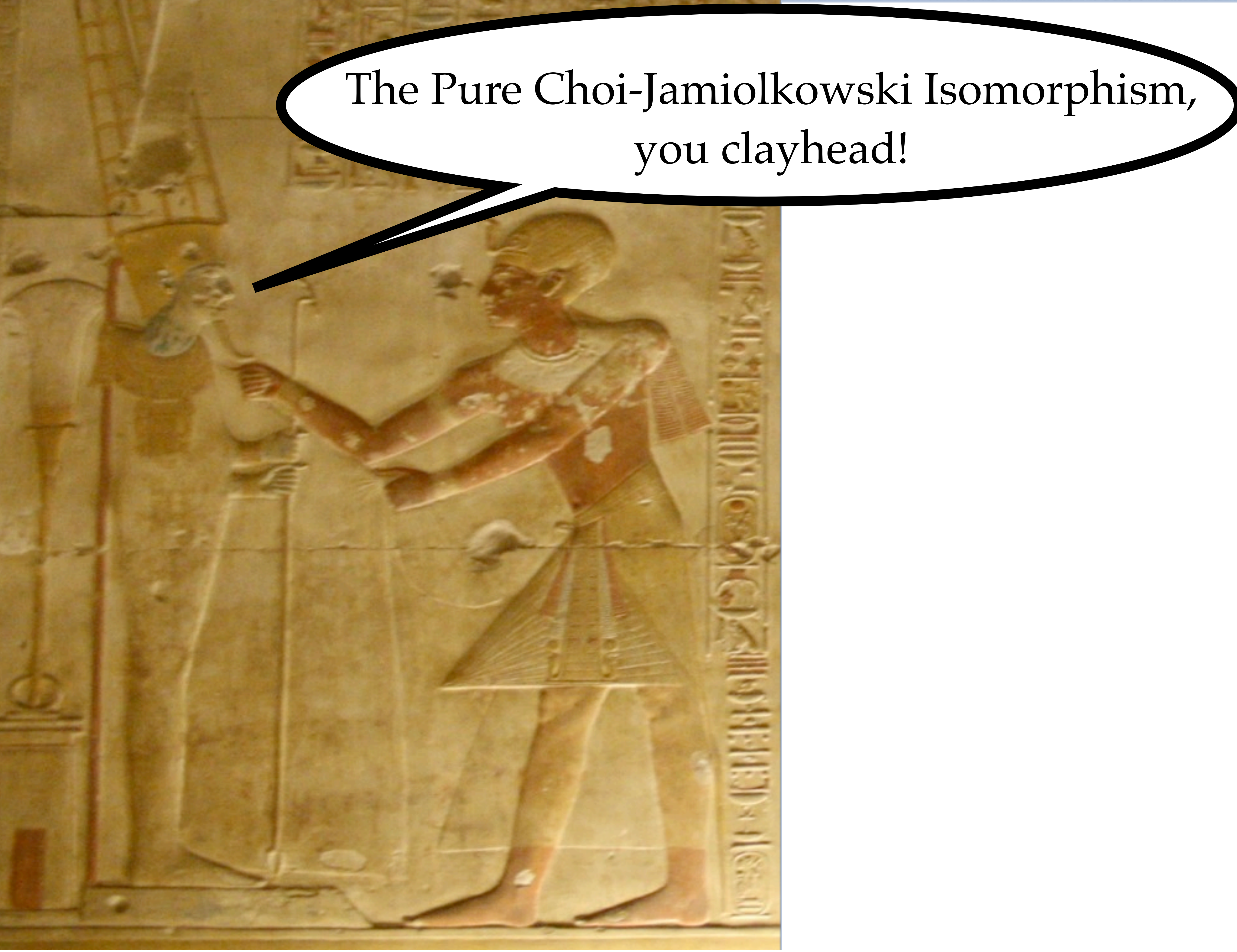
FAITHFULNESS AXIOM

Axiom: for every system type A,
the set of states contains at least one faithful state ω .

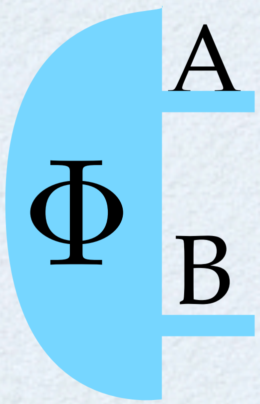
Satisfied by all convex operational-probabilistic theories:
-quantum theory on complex and real fields,
-classical probability theory
-...

but also by non-probabilistic theories
-Schumacher-Westmoreland quantum theory on finite fields
-...

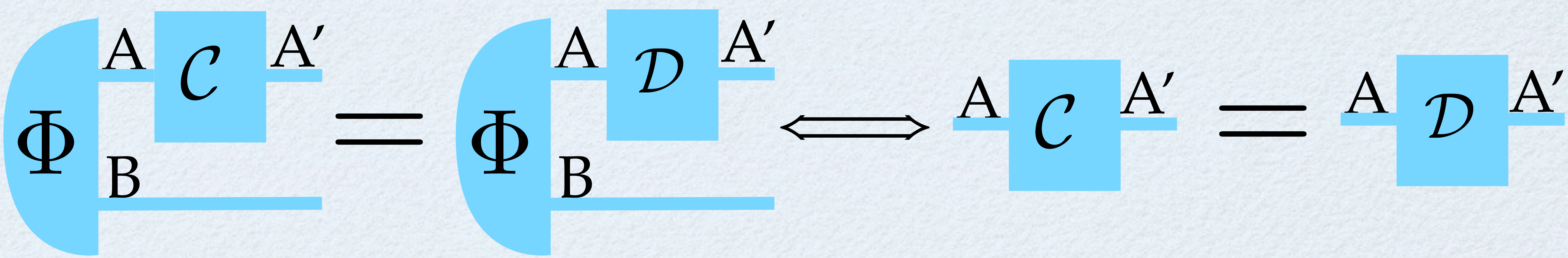

What do purification and faithfulness buy to us?


The Pure Choi-Jamiolkowski Isomorphism,
you clayhead!

PURE STATE-TRANSFORMATION ISOMORPHISM

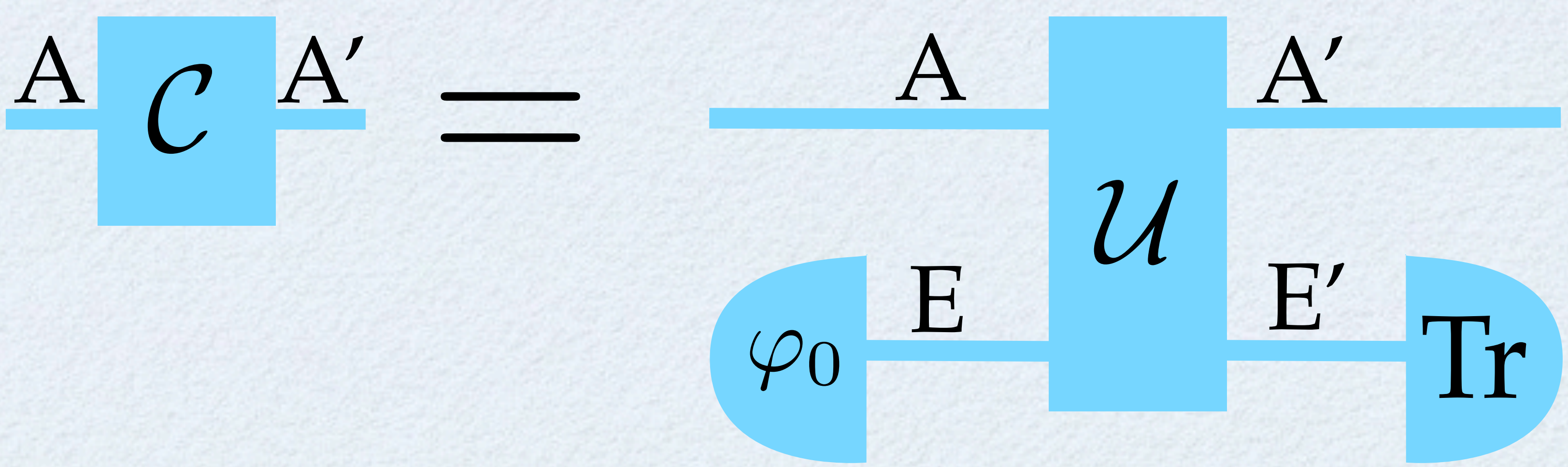


:= purification of the faithful state



PURE AND REVERSIBLE PROCESS SIMULATION

Theorem: For every process
there exist environments E and E′
a pure state of E ,
and a reversible process from AE to BE′ such that



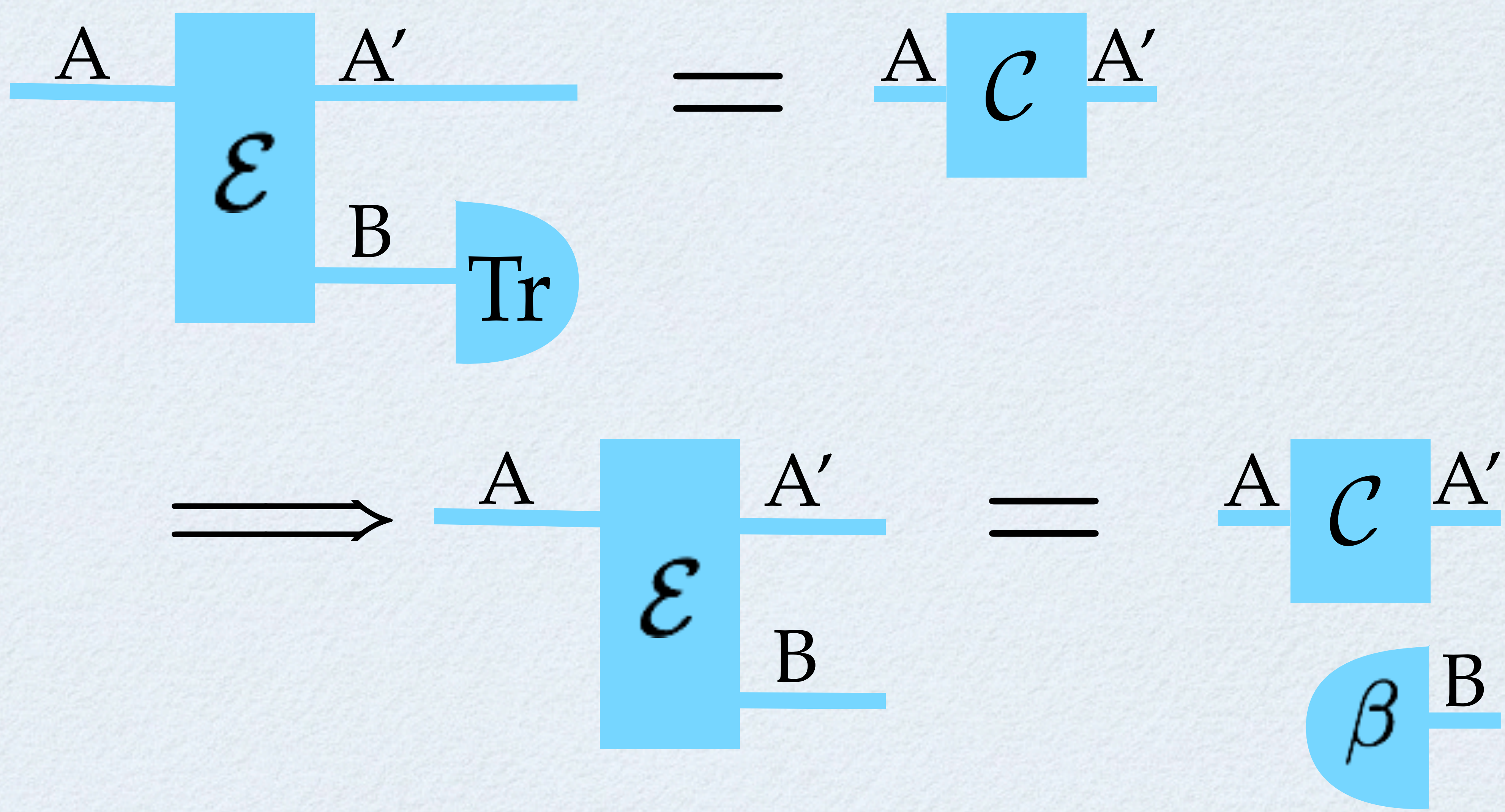
This simulation is unique up to isos on the context.

cf. Stinespring-Kraus′ dilation theorem

THE PURITY THEOREM

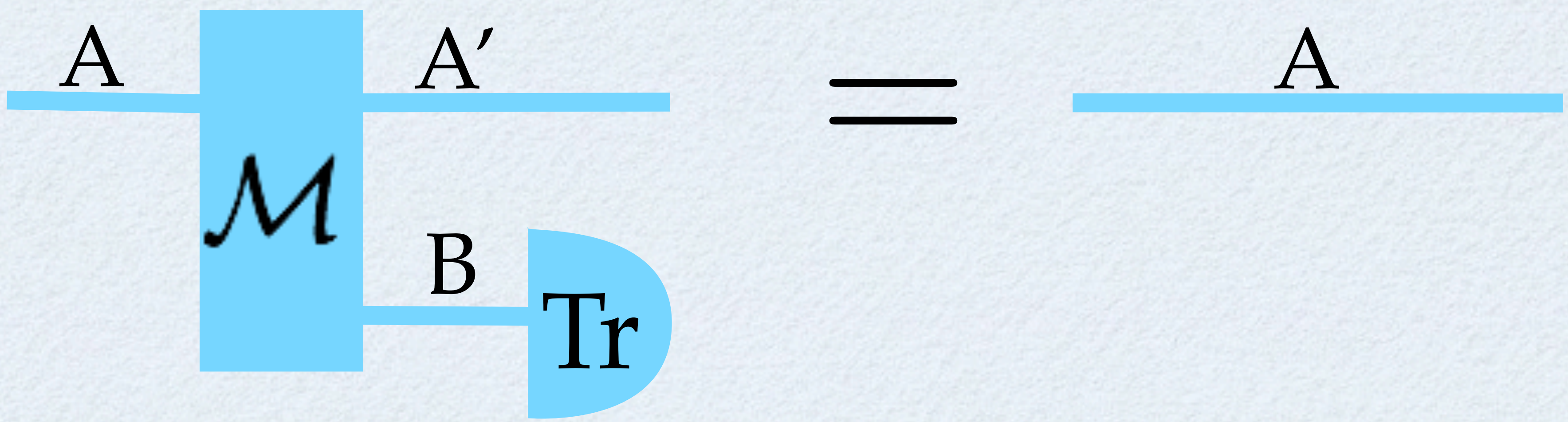
Theorem: under the validity of Purification and Faithfulness
a process is purity-preserving
if and only if it is pure.

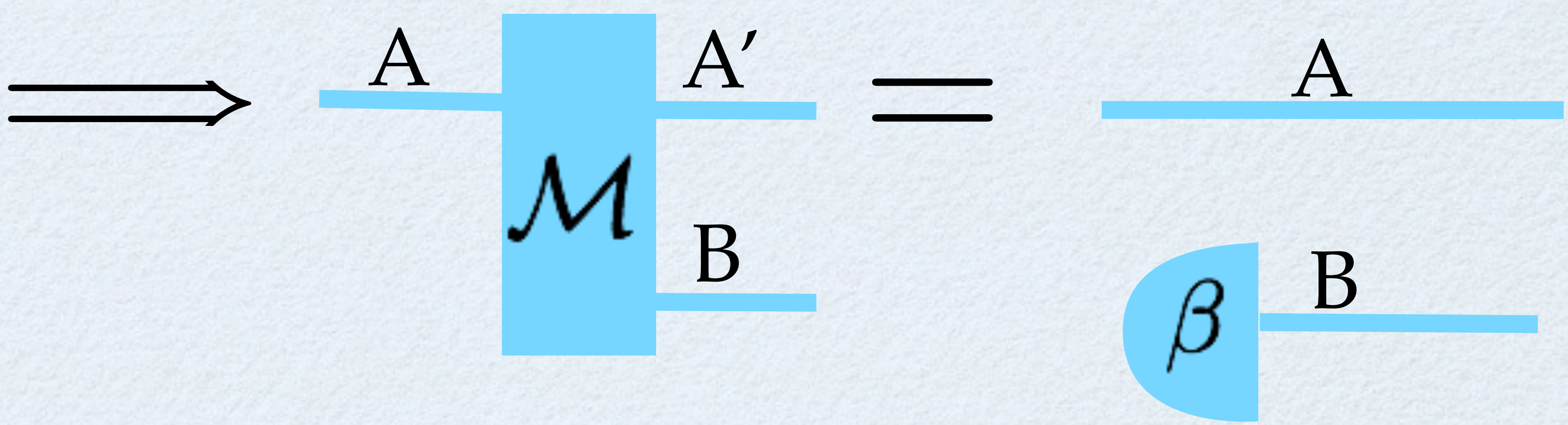
Definition: $\mathcal{E}$ is pure iff



COROLLARIES OF THE PURITY THEOREM

- No Information Without Disturbance



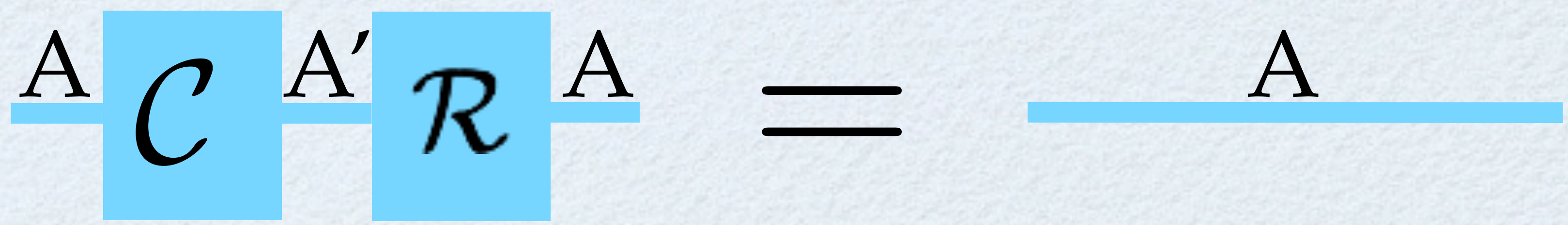


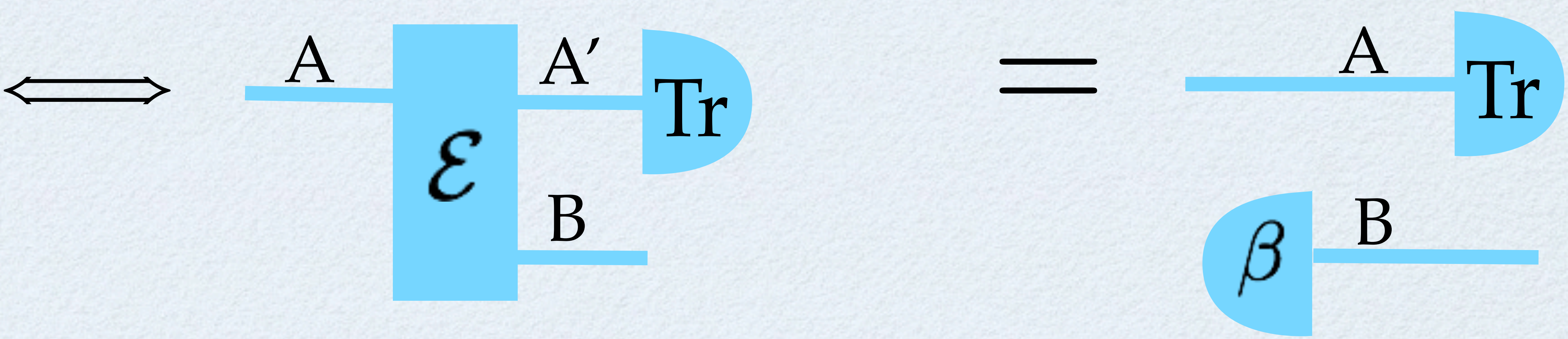
- No Cloning

MORE COROLLARIES OF THE PURITY THEOREM

- Error correction balance

$\exists \mathcal{R} : A' \to A$ s. t.





for every extension $\mathcal{E}$.

CONTRIBUTIONS TO THE CQM PROGRAMME

- Identifies the category of pure processes

- No need of real-valued probabilities

  all results shown here hold also for Schumacher-Westmoreland modal quantum theory on finite fields

- So far, no need of postselected processes

  Everything happens inside the deterministic causal category.

CONTRIBUTIONS TO THE AXIOMATIZATION PROGRAMME

- No restriction to finite dimensions
  e.g. everything holds also for finite infinite dimensional systems

- No no need of "Local Tomography"
  all results hold also for QT on real Hilbert spaces

- No need of "Atomicity of Composition"
  one of the axioms of CDP2010 was that the product of two pure processes yield a pure process
  In the new framework, this is subsumed by the Purity Theorem

WHAT IS HERE AND WHAT IS NOT

• Entanglement	(Y)
• No Cloning	(Y)
• No Information Without Disturbance	(Y)
• Teleportation	(N)
• Steering	(Y/N)
• Existence of perfectly-correlating states	(N)
• Ancilla-assisted process tomography	(Y)
• Reversible simulation of irreversible processes	(Y)
• No Bit Commitment	(Y)
• Principle of Delayed Measurement	(Y/N)
• No Programming Theorem	(Y)
• Error correction balance	(Y)
• Structure of no-signalling channels	(Y)

...

CONCLUSIONS

CONCLUSIONS

- Categorical definition of pure states:
  states/processes that are independent of the context

- Categorical definition of faithful states/processes:
  states/processes that can be in a tomgraphically complete set of contexts

- Purification + Faithfulness imply
  -Pure State-Transformation Isomorphism
  -Reversible process simulation
  -Purity theorem

Thank you
for your attention!