2-Categorical Quantum Mechanics

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Categorical Quantum Mechanics 10th Anniversary Workshop
Jericho Tavern, Oxford, UK
18 October 2014

Introduction

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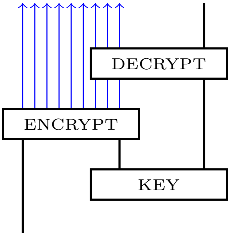

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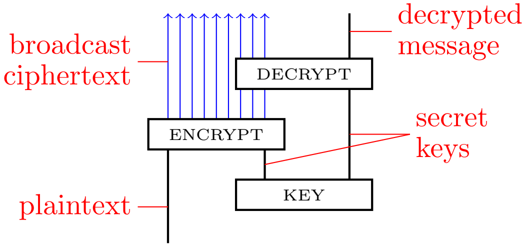

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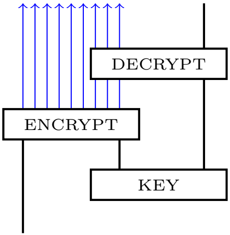

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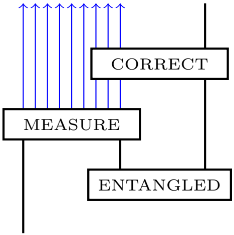

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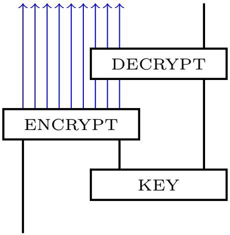

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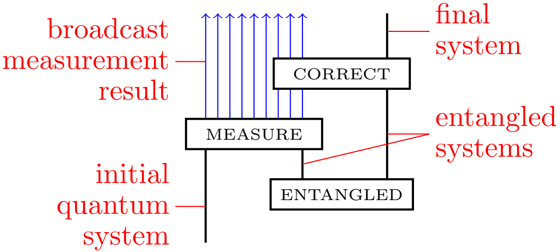

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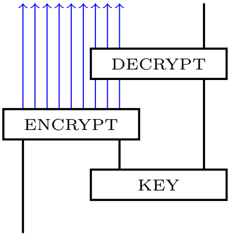

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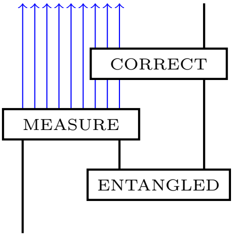

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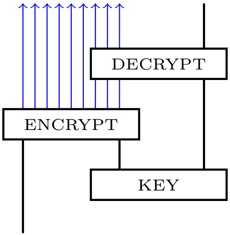

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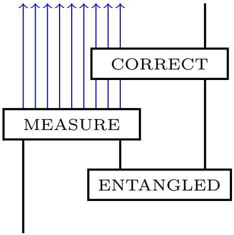
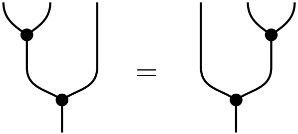


Quantum teleportation

We can make this precise using *2-categorical* quantum mechanics.

Surfaces and logic

We now think about basic properties of copying, comparing and deleting classical information:



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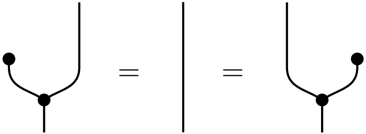

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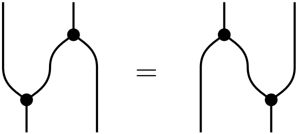
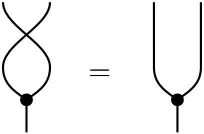


Frobenius law



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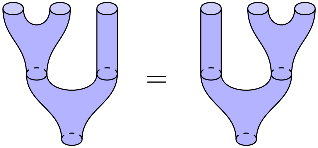

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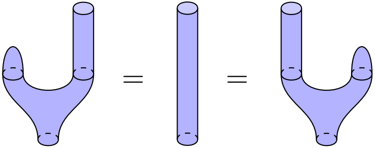

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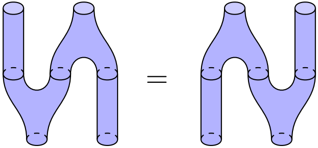

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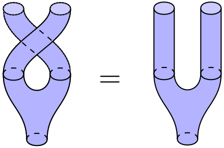

Commutativity

These are the laws obeyed by surfaces up to deformation!
So we change notation and use a **2d topological field theory**.

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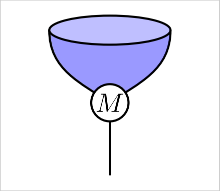

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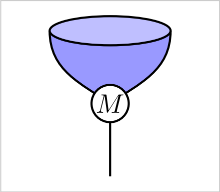

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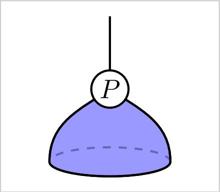

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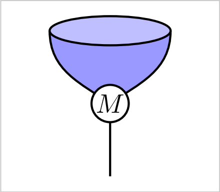

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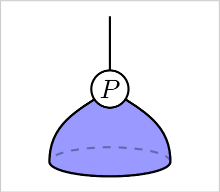

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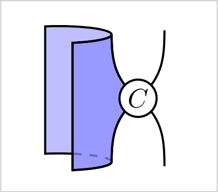

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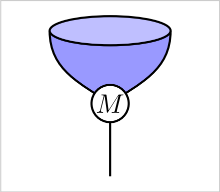

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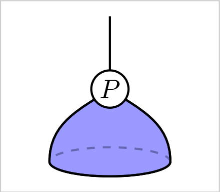

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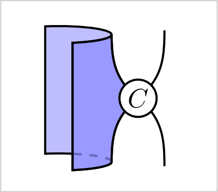

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Also, M and P are inverse.

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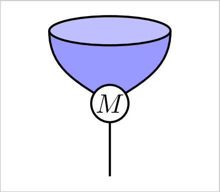

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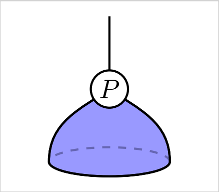

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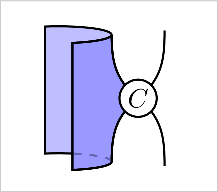

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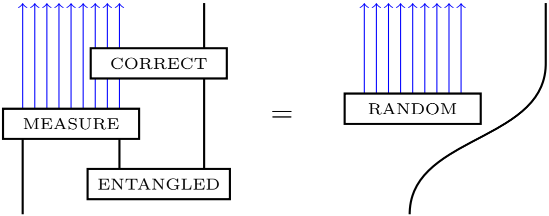
Also, M and P are inverse.

This is a **0-1-2 topological field theory with defects**.

Topological structure

Here is the heuristic quantum teleportation diagram:



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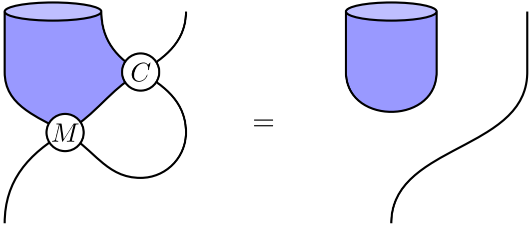

We make it rigorous with this equation between topological defects.

Topological reasoning

We can use the topological formalism to prove interesting things.

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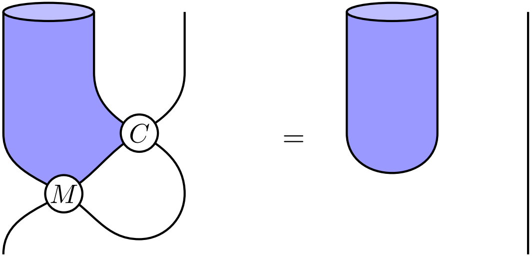

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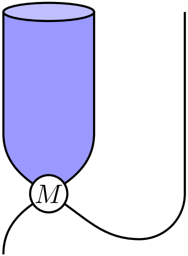
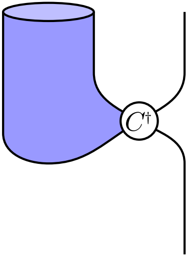
We can use the topological formalism to prove interesting things.

Apply $C^\dagger$:



$=$



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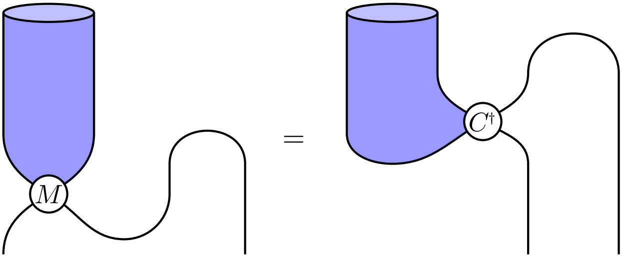

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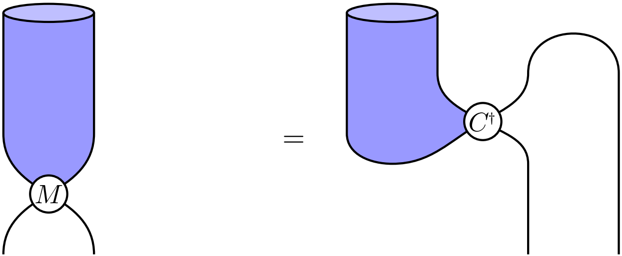

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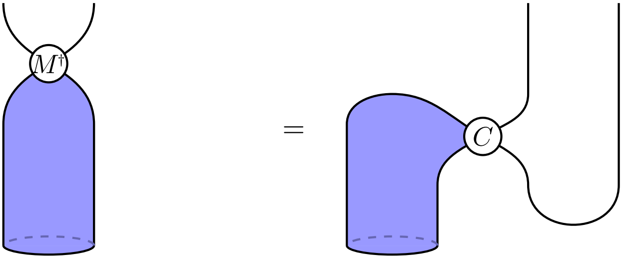
Take adjoints:



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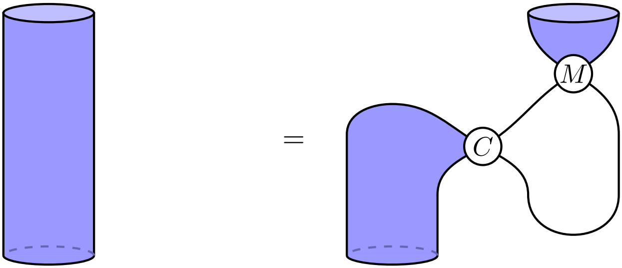
Apply M:



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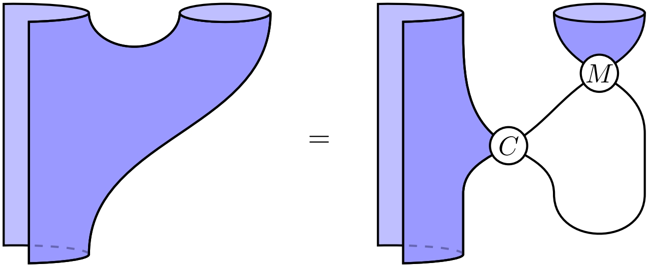
Bend up the surface:



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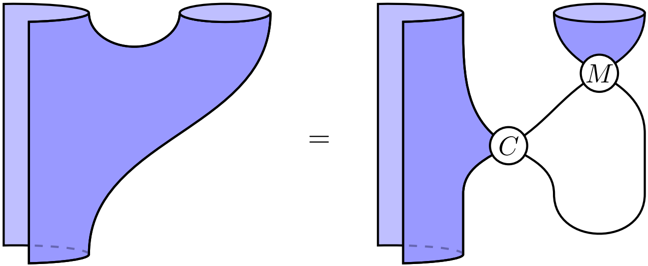

This is dense coding!

So we have a *topological* proof of equivalence with teleportation, independent of the Hilbert space formalism.

2–Hilbert spaces

The standard semantics are given by 2–Hilbert spaces.

$$\begin{matrix} & 1 & \sqrt{2}-i \\ 0 & & \\ & i & \ddots \end{matrix}$$

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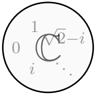

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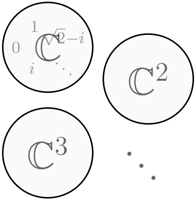

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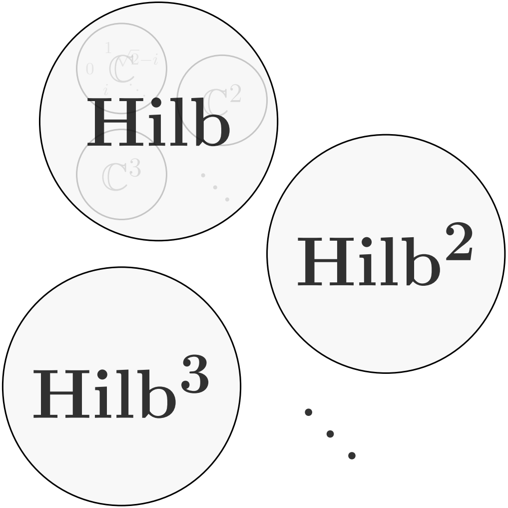

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Let’s think about the relationships between CQM and 2-CQM.

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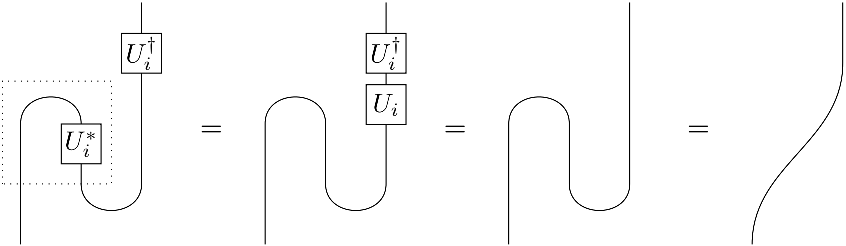
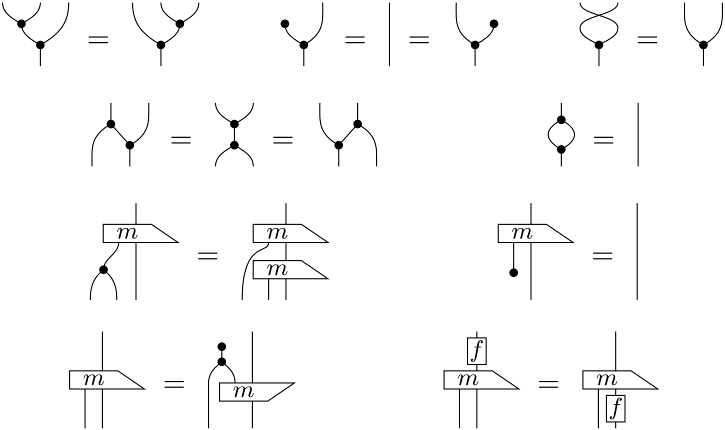
Early work on CQM (SA, BC) handled classical information externally:



Furthermore, extra notation  is required to indicate the measurement basis.

Comparison with 1-CQM

As CQM developed, Frobenius algebras, modules and homomorphisms were introduced to handle classical data and measurement (BC, DP):



Lots of non-geometrical data to check.

Comparison with 1-CQM

There is an immediate connection to 2-CQM.

Definition (Linde Wester)**.** Given a symmetric monoidal dagger-category $\mathbf{C}$, write $\mathbf{2}[\mathbf{C}]$ for the symmetric monoidal bicategory of classical structures, dagger-bimodules and homomorphisms in $\mathbf{C}$.

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So 2-CQM gives a *notation* for ordinary CQM—
just as 1-CQM gives a notation for QM.

Note 2-CQM is strictly more general, since it can be applied in any symmetric monoidal bicategory, not necessarily of the form $\mathbf{2}[\mathbf{C}]$.

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Definition. The symmetric monoidal 2-category **2Hilb** is built from the following structures:

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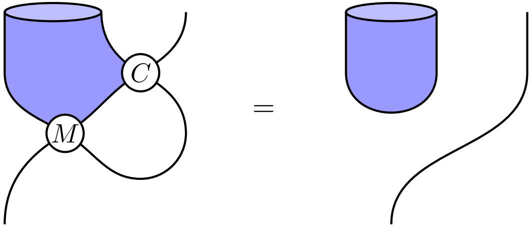
Quantum teleportation

Theorem. Solutions to the teleportation equation in **2Hilb** correspond exactly to quantum teleportation schemes.



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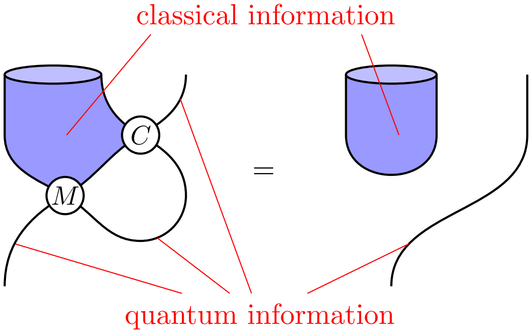

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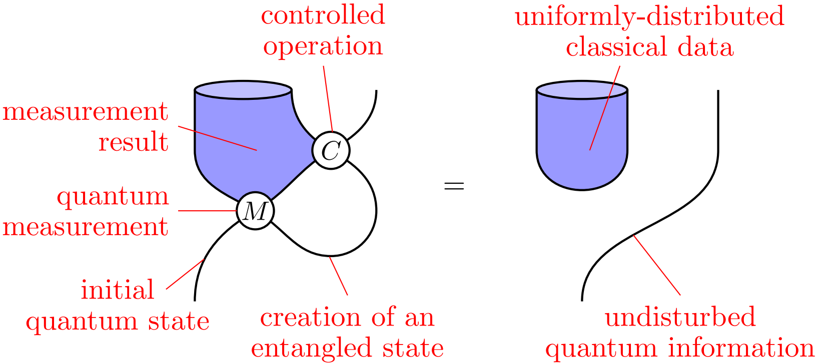

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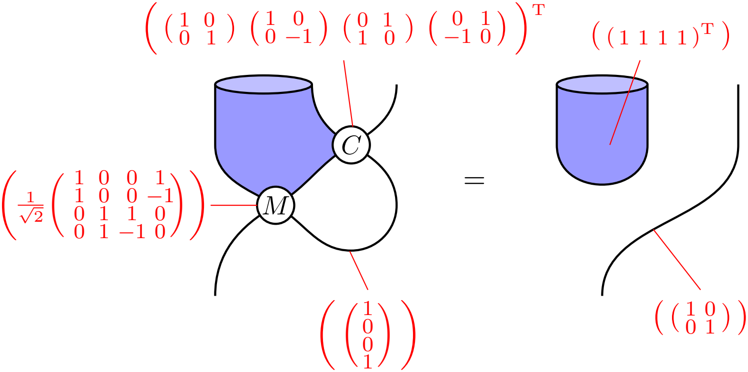


This is exactly the data that would appear in a quantum information textbook.

The Big Picture

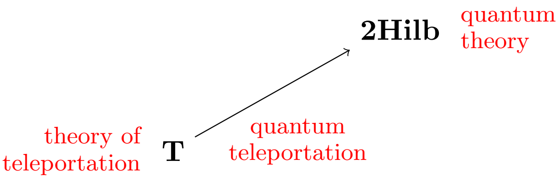
theory of
teleportation **T**

The Big Picture

2Hilb quantum theory

theory of teleportation **T**

The Big Picture



Theorem. Structure-preserving maps $\mathbf{T} \to \mathbf{2Hilb}$ correspond to implementations of quantum teleportation.

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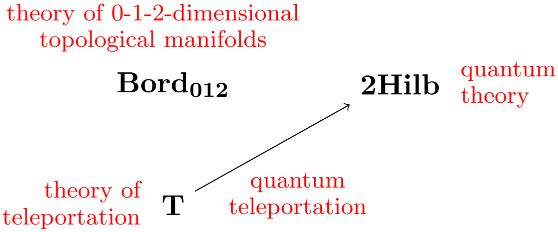
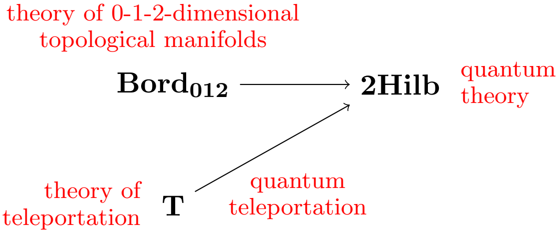


The Big Picture



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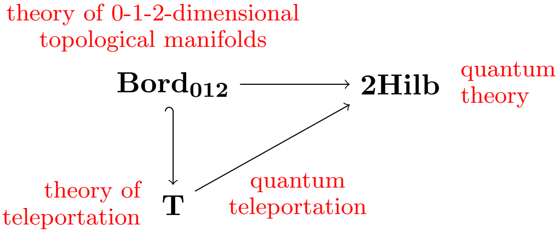

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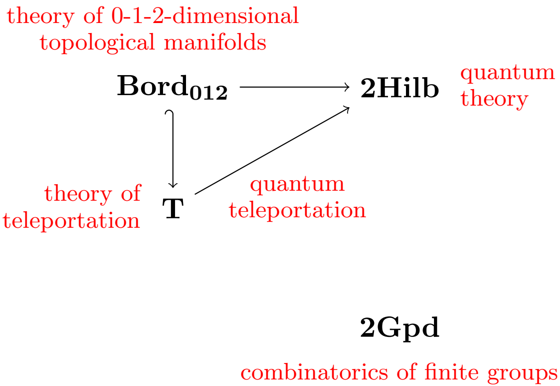

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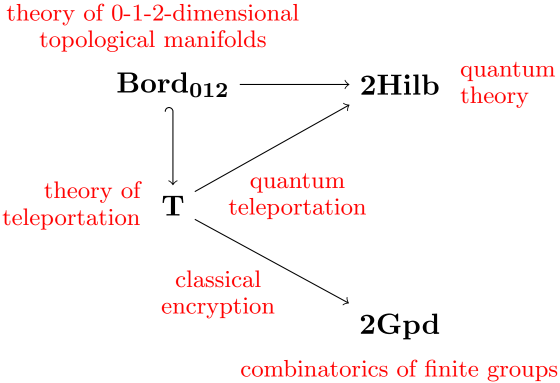
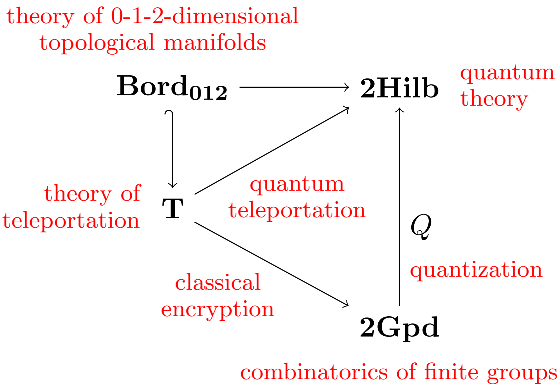


Theorem. Structure-preserving maps $\mathbf{T} \to \mathbf{2Gpd}$ correspond to implementations of encrypted communication via a one-time pad.

The Big Picture



Theorem. The map Q transports encrypted communication into quantum teleportation.

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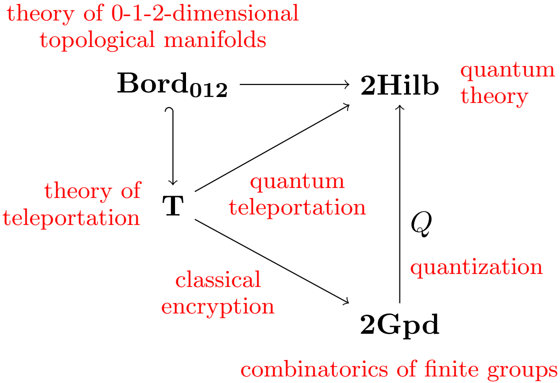
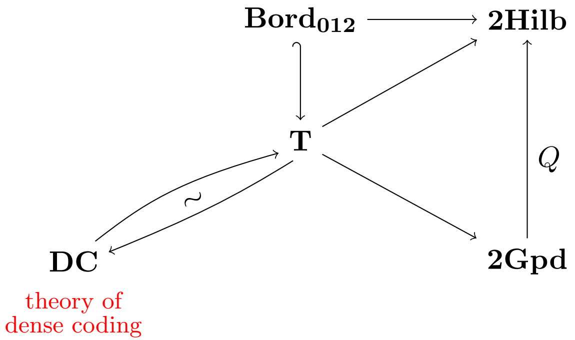


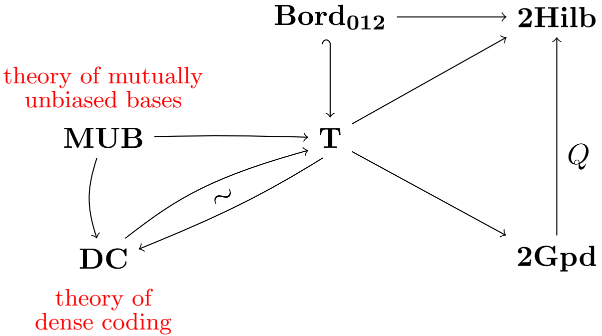
Theorem. The map Q transports encrypted communication into quantum teleportation. Related to Werner's combinatorial construction—and Ben Musto has nice results generalizing this!

The Big Picture



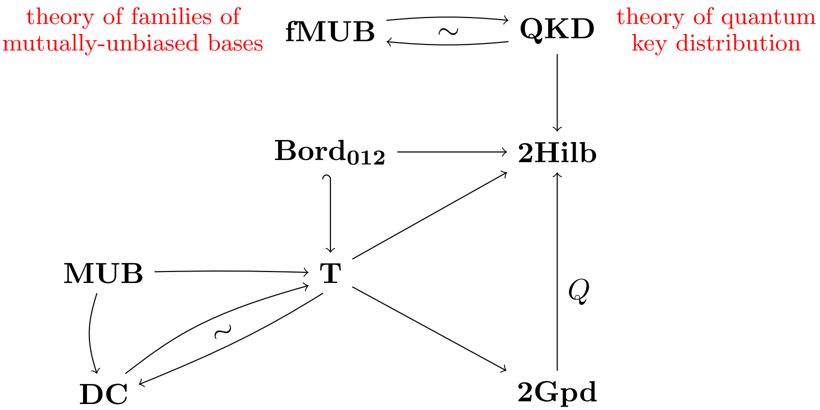
Theorem. Teleportation and dense coding are syntactically equivalent.

The Big Picture



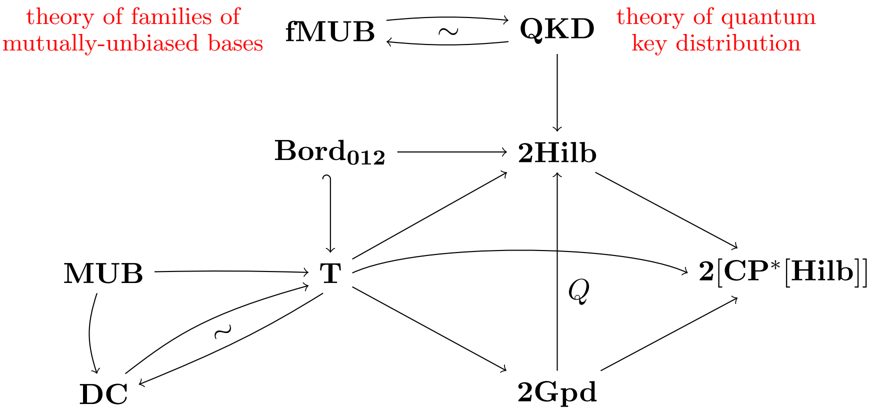
Theorem (Krzysztof Bar, JV)**.** Syntactic construction of teleportation and dense coding from mutually-unbiased bases.

The Big Picture



Theorem (QPL 2014, Krzysztof Bar, JV)**.** Syntactic equivalence between families of MUBs and QKD.

The Big Picture



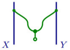
Quantum and classical worlds unified in **2**[**CP***[**Hilb**]]? Partial results in QPL 2014 paper (Chris Heunen, JV and Linde Wester.)

Orbifold completion

Orbifolding is an operation on a quantum field theory that constructs its maximal extension. Recently it has been described in terms of Frobenius algebras in bicategories:

Lemma 2.3. *Suppose that idempotent 2-morphisms split in $\mathcal{B}$ and that A is separable Frobenius. Then $X \otimes_A Y$ exists for all modules X, Y and can be written as the image of the idempotent*

$\pi_A^{X,Y} =$



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Nils Carqueville and Ingo Runkel, "Orbifold completion of defect bicategories", arXiv:1210:6363

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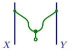

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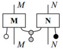
This is formally identical to our $2[-]$ construction:

$(C, \ldots)$, $(D, \ldots)$, $(E, \ldots)$, for all C-D-bimodules **M** and all D-E-bimodules **N**, the following dagger idempotent splits:



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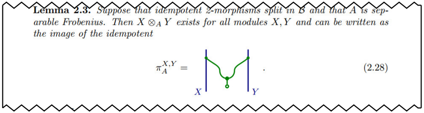
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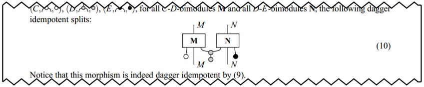
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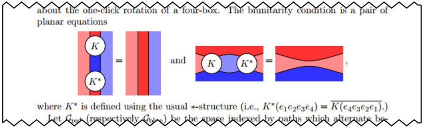
This gives a surprising connection between 2-CQM and quantum field theory.

Connections

In subfactor theory, people are interested in understanding *connections* in planar algebras. These are 2d operators satisfying the following graphical condition:


about the one-click rotation of a four-box. The biunitarity condition is a pair of planar equations

where K^* is defined using the usual $*$-structure (i.e., $K^*(e_1e_2e_3e_4) = \overline{K(e_4e_3e_2e_1)}$.)

Let $\mathcal{G}_{\text{red}}$ (respectively $\mathcal{G}_{\text{blue}}$) be the space indexed by paths which alternate be-

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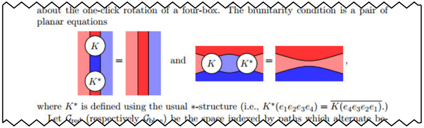

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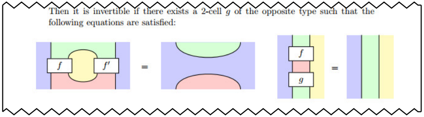
This is identical to our notion of 'completely invertible', which we used to classify teleportation, dense coding and MUBs:



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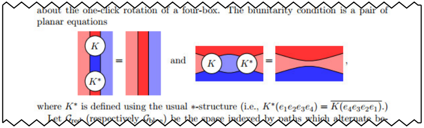
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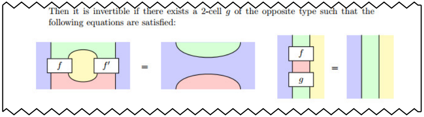
Then it is invertible if there exists a 2-cell g of the opposite type such that the following equations are satisfied:

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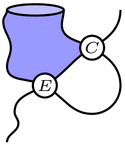
This gives a surprising link between quantum information and subfactor theory, von Neumann algebras, and planar algebra.

Future directions



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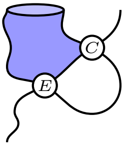

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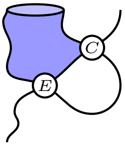

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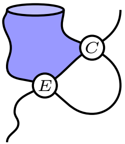

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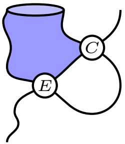

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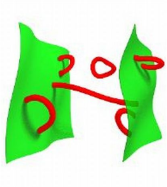

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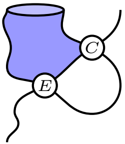

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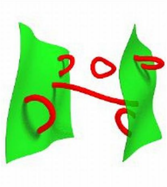

Thank you!