All-versus-Nothing Arguments, Stabiliser Groups, and the "AvN Triple" Conjecture

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Motivation:

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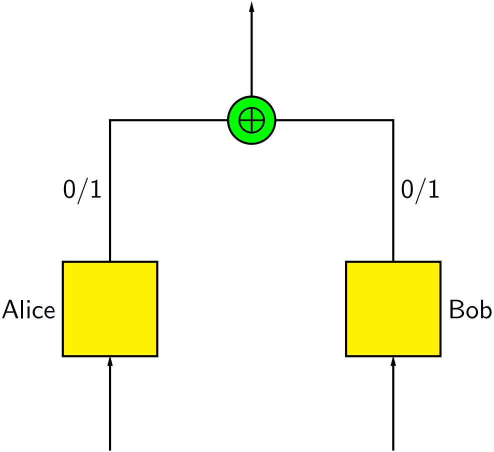
Motivation:

- Understand where AvN sits in the hierarchy of contextuality properties
- Characterise the quantum states which give rise to maximal degrees of non-locality/contextuality.

The XOR Game



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Note that the eigenvalues of the operators XXX etc. are $+1$ and -1.

The **expected values** of these measurements give information about the **parity** of the support.

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However, this can never be the case for any assignment

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Clearly, these are **inconsistent**.

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Proposition

If an empirical model e is AvN, *then it is strongly contextual.*

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A Galois correspondence between Pauli operators and states/vectors in the Hilbert space $\mathbb{C}^n$:

$$gRv \iff gv = v.$$

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The subgroups of $\mathcal{P}_n$ which stabilise non-trivial subspaces must be commutative, and only contain elements with global phases ± 1.

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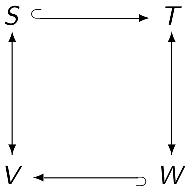

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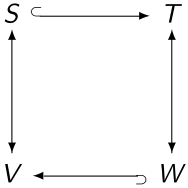

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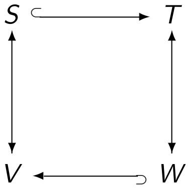

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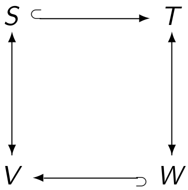

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Note that the correspondence is tight: a rank k subgroup determines a dimension 2^{n-k} subspace.

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Question:

How can we characterise when this happens?

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Define an AvN **triple** in $\mathcal{P}_n$ to be (e, f, g) (order is important) with global phases $+1$, which pairwise commute, and additionally satisfy the following conditions:

(A1) For all $i = 1, \ldots, n$ at least two of e_i, f_i, g_i are the same.

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This means that any state in V_S, where S is the subgroup generated by $\{e, f, g\}$, admits an AvN argument. Note that this is a 2^{n-3}-dimensional space, assuming e, f, g are independent.

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Example from Mermin, yielding a GHZ argument:

$$\begin{array}{ccc} X & Y & Y \\ Y & X & Y \\ Y & Y & X \end{array}$$

Example of 1-dimensional cluster state, $n = 4$:

$$\begin{array}{cccc} X & I & X & Z \\ Z & Y & Y & Z \\ X & I & Y & Y \end{array}$$