(In)Completeness results for the ZX-calculus

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Outline

Introduction

The ZX-calculus is incomplete [Schröder & Zamdzhiev 2014]

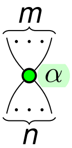
Completeness results
- Stabilizer quantum mechanics
- The single-qubit Clifford+T group

Outlook & Conclusions

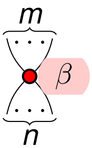
Outline

Elements of ZX-calculus diagrams

- green nodes with *n* inputs and *m* outputs, $\alpha \in (-\pi, \pi]$



- red nodes with *n* inputs and *m* outputs, $\beta \in (-\pi, \pi]$



- Hadamard nodes with one input and one output

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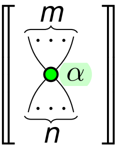

$$:= |0\rangle^{\otimes m} \langle 0|^{\otimes n} + e^{i\alpha} |1\rangle^{\otimes m} \langle 1|^{\otimes n},$$

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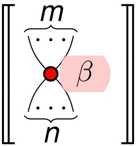

$$:= |+\rangle^{\otimes m} \langle +|^{\otimes n} + e^{i\beta} |-\rangle^{\otimes m} \langle -|^{\otimes n},$$

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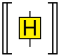
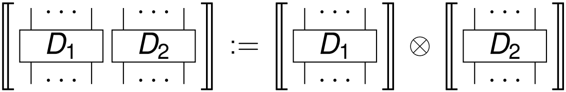


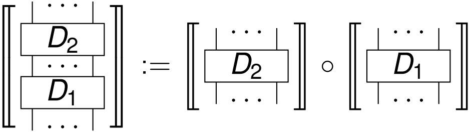
$$:= |+\rangle \langle 0| + |-\rangle \langle 1|$$

Properties of the interpretation functor

- parallel composition:



- serial composition:



- to get the Hermitian adjoint of a diagram, turn it upside-down and flip all the phases: e.g.

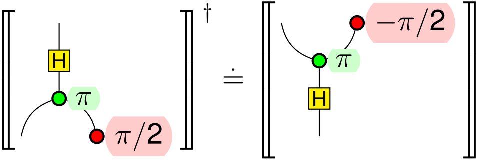
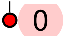


Diagram jargon and conventions

- a phase label of zero is usually left out: e.g.  := 

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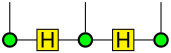
- diagrams with no inputs are *states*: e.g. 

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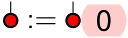
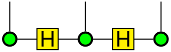
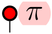
- a phase label of zero is usually left out: e.g. 

- diagrams with no inputs are *states*: e.g. 

- diagrams with no outputs are *effects*: e.g. 

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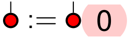
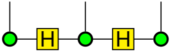
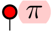
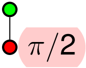
- a phase label of zero is usually left out: e.g. 

- diagrams with no inputs are *states*: e.g. 

- diagrams with no outputs are *effects*: e.g. 

- diagrams with no inputs or outputs are *scalars*: e.g. 

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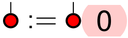
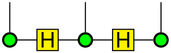
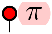
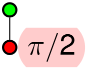
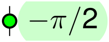
- a phase label of zero is usually left out: e.g. 

- diagrams with no inputs are *states*: e.g. 

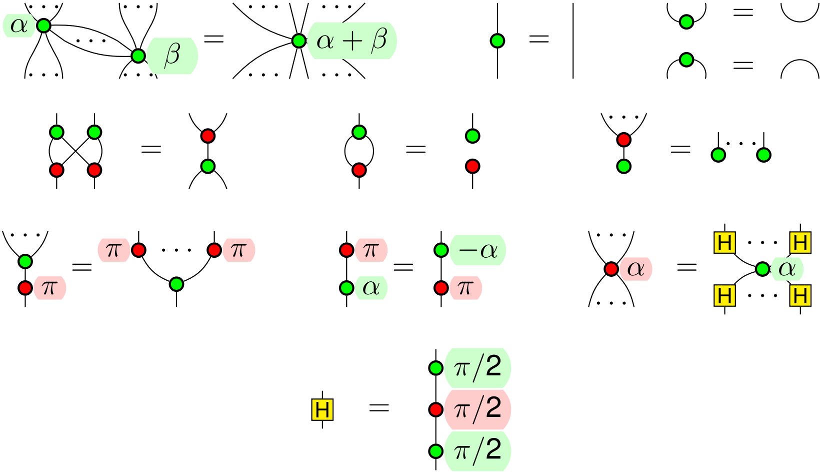
- diagrams with no outputs are *effects*: e.g. 

- diagrams with no inputs or outputs are *scalars*: e.g. 

- a red or green node with one input and one output is also called *phase shift*: e.g. 

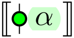
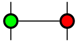
Rules of the ZX-calculus

- ignore non-zero scalar factors ($\llbracket \pi \rrbracket \doteq 0 \doteq \llbracket \pi \rrbracket$)
- only the topology matters



Universality, soundness, and completeness

- *universality*: any pure state, post-selected pure projective measurement, and unitary operation can be expressed: to see this note that ⟦⟧ $\doteq |0\rangle$, ⟦⟧ $\doteq \langle 0|$,

⟦⟧ $\doteq \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$, ⟦⟧ $\doteq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

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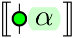
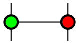
$$[\![$$  $$]\!] \doteq \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix},$$

$$[\![$$  $$]\!] \doteq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- *soundness*: any equality that can be derived graphically can also be derived using matrices; to see this note that

$$[\![\mathrm{LHS}]\!] \doteq [\![\mathrm{RHS}]\!] \quad \text{for all rewrite rules}$$

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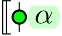
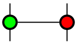
$[\![$$]\!] \doteq \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$, $[\![$$]\!] \doteq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

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- *completeness*: can any equality that can be derived using matrices also be derived graphically, i.e.

does $[\![D_1]\!] \doteq [\![D_2]\!]$ imply $D_1 = D_2$?

Outline

Euler decomposition of arbitrary unitaries

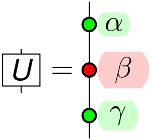
- Any 2x2 unitary U can be written as

$$U \doteq \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \begin{pmatrix} \cos\beta & -i\sin\beta \\ -i\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\gamma} \end{pmatrix}$$

- $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \doteq$ 

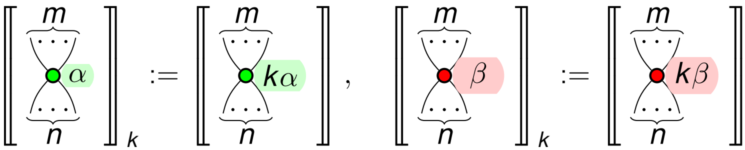
and $\begin{pmatrix} \cos\beta & -i\sin\beta \\ -i\sin\beta & \cos\beta \end{pmatrix} \doteq$ 

- So if the ZX-calculus is complete, then for any U we have to be able to find α, β, γ such that



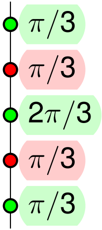
Alternative interpretation functors

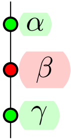
- define $[\![-]\!]_k$ by $[\![\boxed{H}]\!]_k := [\![\boxed{H}]\!]$ and



- if k is an odd integer, the interpretation $[\![-]\!]_k$ is sound

A counterexample for completeness

Consider diagrams D_1 := 

and D_2 := 

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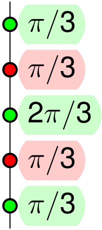
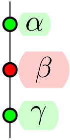
Consider diagrams D_1 := 

and D_2 := 

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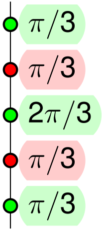
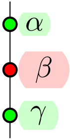
Consider diagrams $\boxed{D_1}$ $:=$ 

and $\boxed{D_2}$ $:=$ 

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- Thus if ZX-calculus is complete, $\boxed{D_1} = \boxed{D_2}$
- Then must also have $\left[\!\left[\boxed{D_1} \right]\!\right]_{-3} \doteq \left[\!\left[\boxed{D_2} \right]\!\right]_{-3}$

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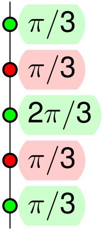
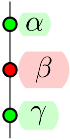
Consider diagrams $\boxed{D_1}$:= 

and $\boxed{D_2}$:= 

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- Then must also have $[\![\boxed{D_1}]\!]_{-3} \doteq [\![\boxed{D_2}]\!]_{-3}$
- But $[\![\boxed{D_1}]\!]_{-3} \doteq I$ and $[\![\boxed{D_2}]\!]_{-3}$ is non-trivial $\Longrightarrow$ contradiction!

Further notes on incompleteness

- counterexample involves only line graphs, i.e. very simple structure
- can find similar counterexamples for other types of graphs
- might have to add infinitely many new rewrite rules to complete the general ZX-calculus

But:

- Euler decomposition of general unitaries relies on arbitrary phases being allowed
- can get completeness results for current ruleset by restricting phases

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Stabilizer quantum mechanics

Stabilizer operations:

- preparation of qubits in state $|0\rangle$
- Clifford unitaries, generated by

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \; H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \; \Lambda X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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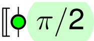
ZX-calculus: $S \doteq [\![$ 

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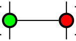
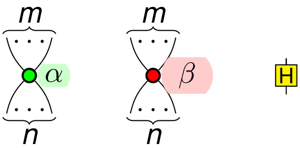
$]\!]$, $\Lambda X \doteq [\![$  $]\!]$
i.e. need to restrict phases to integer multiples of $\pi/2$



where $\alpha, \beta \in \{-\pi/2, 0, \pi/2, \pi\}$

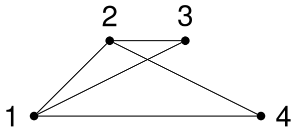
Graph states in the ZX-calculus

Definition

Let G be a finite simple undirected graph. The ZX-calculus diagram for the corresponding graph state consists of:

- for each node in G, a green node with one output, and
- for each edge in G, an edge with a Hadamard node on it.

E.g.



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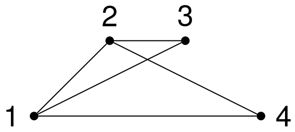

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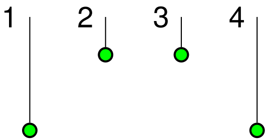

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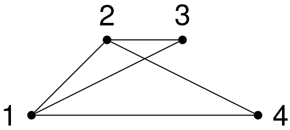

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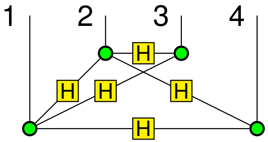

Results about graph states

The local Clifford group consists of all tensor products of the single-qubit Clifford operators $\langle S, H \rangle$.

Theorem (Van den Nest et. al, 2004)

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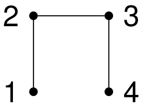
Two graph states are local Clifford-equivalent if and only if they are related by a sequence of local complementations.

A local complementation about a vertex v inverts the subgraph generated by the neighbourhood of v: e.g.



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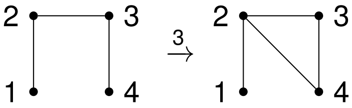
Two graph states are local Clifford-equivalent if and only if they are related by a sequence of local complementations.

A local complementation about a vertex v inverts the subgraph generated by the neighbourhood of v: e.g.



Results about graph states

The local Clifford group consists of all tensor products of the single-qubit Clifford operators $\langle S, H \rangle$.

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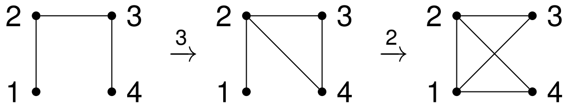
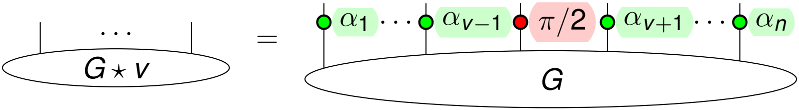


Local complementations in the ZX-calculus

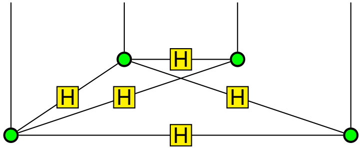
Theorem (Duncan & Perdrix, 2009)

Denote the result of a local complementation about the vertex v in the graph G by $G \star v$. The graph state diagrams for G and $G \star v$ satisfy



where $\alpha_u = -\pi/2$ if $\{u, v\}$ is an edge, $\alpha_u = 0$ otherwise.

E.g. a local complementation about the 3rd qubit of the previous graph state:



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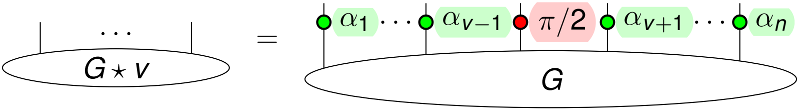

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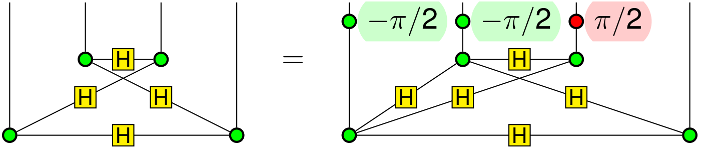
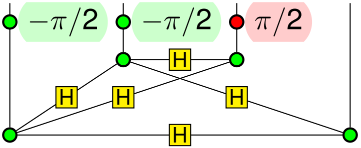


GS-LC diagrams

Definition

A diagram in the stabilizer ZX-calculus is called a *GS-LC diagram* if it consists of a graph state diagram with arbitrary single-qubit Clifford unitaries (i.e. combinations of phase shifts and Hadamards) applied to each output.

E.g.



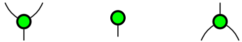
Every state diagram is equal to some GS-LC diagram

Theorem

Any stabilizer ZX*-calculus diagram with no inputs and at least one output can be rewritten to a GS-LC diagram.*

Proof.

- Decompose the diagram into basic spiders



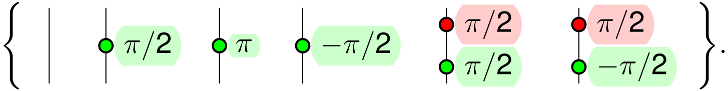
  and single-qubit Clifford unitaries.
- Diagrams with no inputs must contain at least one copy of , this is a GS-LC diagram.
- For each basic element, applying it to a GS-LC diagram yields a diagram that can be rewritten into GS-LC form.
- Thus, by induction, the theorem holds. □

Reduced GS-LC diagrams

Definition

A diagram in the ZX-calculus is called a *reduced GS-LC diagram* if it is a GS-LC diagram and satisfies the following two conditions:

1. All the single-qubit Clifford unitaries belong to the set



2. Two adjacent vertices must not both have single-qubit Clifford unitaries that include red nodes.

Theorem

Any stabilizer ZX-calculus state diagram can be rewritten to a reduced GS-LC diagram.

Comparing reduced GS-LC diagrams

Theorem (inspired by Elliott et al., 2008)

There exists a terminating algorithm that, given a pair of reduced GS-LC diagrams on the same number of qubits, rewrites them to a pair of identical diagrams if and only if the two diagrams represent the same state.

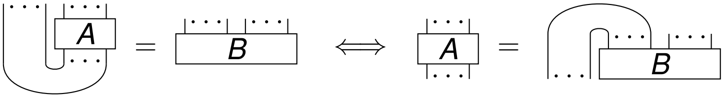
Theorem

The ZX*-calculus is complete for stabilizer state diagrams.*

The Choi-Jamiołkowski isomorphism

Theorem (Choi-Jamiołkowski isomorphism)

For any operator A from n to m qubits and for any $n + m$-qubit state B,



The stabilizer ZX-calculus is complete

Theorem

The ZX-calculus is complete for all stabilizer diagrams.

Proof.

Given two ZX-calculus diagrams with n inputs and m outputs each:

- Apply the Choi-Jamiołkowski isomorphism to get two diagrams with $n + m$ outputs each.
- Bring the diagrams into reduced GS-LC form.
- Apply the comparison algorithm for reduced GS-LC diagrams.
- If this yields a pair of identical diagrams, use the Choi-Jamiołkowski isomorphism to transform the sequence of equal state diagrams back into operators.

This yields a sequence of rewrites which transforms one of the original diagrams into the other. □

Outline

The single-qubit Clifford+T group

An approximately universal group, generated by:

- single-qubit Clifford group $\mathcal{C}_1 = \langle S, H \rangle$, where

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad \text{and} \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- T gate (note $T^2 = S$)

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

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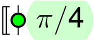
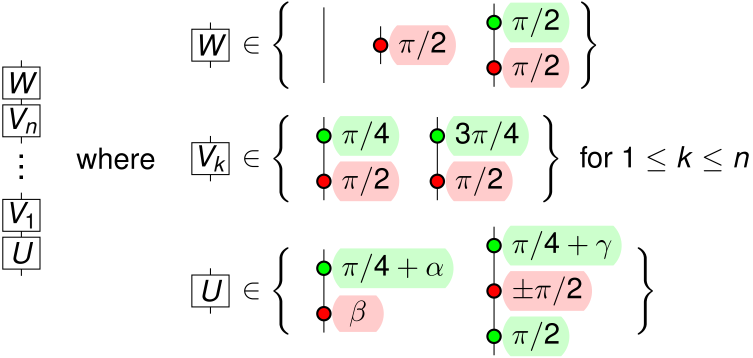
ZX-calculus: $T \doteq [\![$  $]\!]$ and $H \doteq [\![$  $]\!]$

- diagrams are restricted to line graphs (each node has one input and one output)
- phases are restricted to integer multiples of $\pi/4$

The normal form for single-qubit Clifford+T diagrams

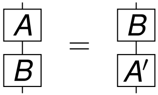
Following [Matsumoto & Amano 2008], any single-qubit Clifford+T diagram is either pure Clifford or it can be written as

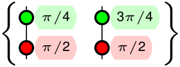


with *n* a non-negative integer and $\alpha, \beta, \gamma \in \{0, \pi/2, \pi, -\pi/2\}$.

Rewriting diagrams into normal form

- write diagram in terms of α where α is a multiple of $\pi/4$, and β with β a multiple of $\pi/2$
- diagrams are rewritten into normal form by pushing phase shift towards the bottom of the diagram
- we say we can "push *A* past *B*" if there is *A*′ such that



  - can push α and π past $\pi/4$
  - can push α and π past $\pi/2$
  - can push π and π past V $\in$ 

Normal form diagrams act non-trivially

Lemma

No normal form diagram represents the identity operator.

- write any single-qubit density operator as $xX + yY + zZ$, where $x, y, z \in \mathbb{R}$ and X, Y, Z are the Pauli matrices
- Clifford unitaries act on the vectors (x, y, z) by permuting the elements and adding minus signs; T sends

$$(x, y, z) \mapsto \frac{1}{\sqrt{2}}\left(x - y, x + y, z\sqrt{2}\right).$$

- if D is a normal form operator, $D\,|0\rangle$ has vector

$$\frac{1}{\sqrt{2^m}}\left(x_1 + x_2\sqrt{2}, y_1 + y_2\sqrt{2}, z_1 + z_2\sqrt{2}\right)$$

  where $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{Z}$
- by parity arguments, can show none of them represent $|0\rangle$

Normal forms are unique

Lemma

The normal form of the inverse of a normal form diagram has the same number of $\pi/4$ *nodes as the original diagram.*

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Theorem

The normal form is unique.

- Suppose D and D' are non-identical normal form diagrams
- Can then show that $D^{-1} \circ D'$ has non-trivial normal-form
- Thus $[\![D]\!] \neq [\![D']\!]$, and by soundness $D \neq D'$

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Corollary

The ZX*-calculus is complete for the single-qubit Clifford+T group.*

Outline

Related work and open questions

- Stabilizer completeness proof carries over to a ZX-like graphical calculus for Spekkens' toy theory (joint work with Ali Nabi Duman)

- can the two completeness results be combined into a completeness proof for the multi-qubit Clifford+T group?
- zero-diagram completeness?
- what happens if we put the scalars back in (possibly still ignoring complex phases)?
- can the completeness results be implemented in `Quantomatic`?

Summary

- ZX-calculus is not complete in general but fragments of it are complete, e.g.
  - line graphs where all phases are multiples of $\pi/4$ (single-qubit Clifford+T group)
  - diagrams where all phases are multiples of $\pi/2$ (stabilizer quantum mechanics)
- ongoing work to extend completeness results

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Thank you!