What is a Resource?

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Complexity Resources in Physical Computation
25.viii.2009

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So, why not try other models of computation, such as *analogue computers*?

Geometric formulation.

Descartes tells us that *numerical* problems can often be recast *geometrically*.

So, don't think about finding **numbers** x and y such that $xy = n$ (i.e. $y = n/x$);
think about the **graph** $y = n/x$.

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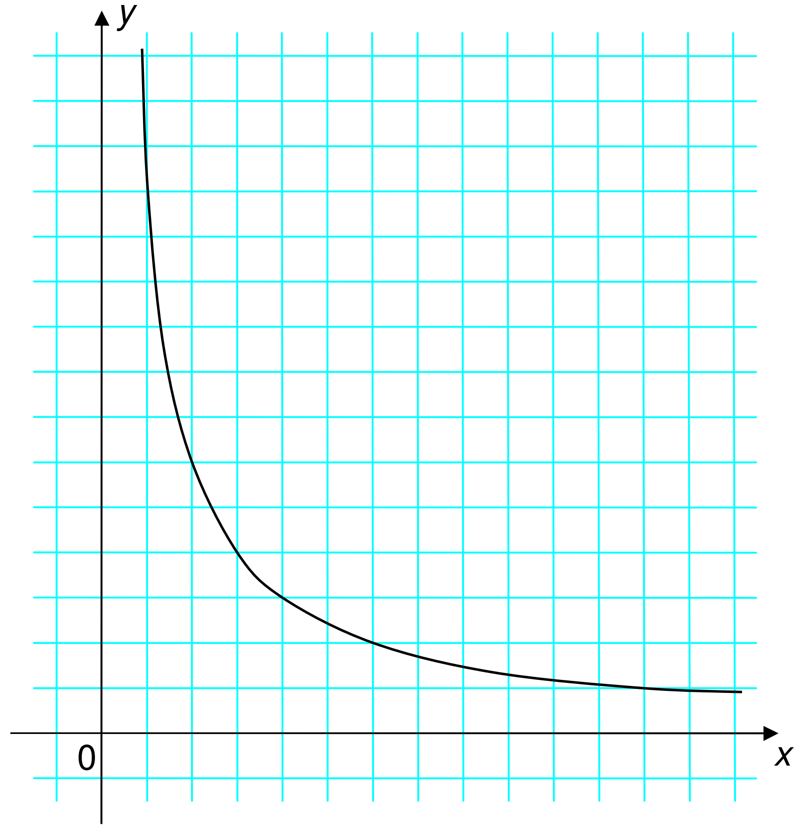
We want to factorize n. So, we want to find *integer* points (x, y) on the curve $y = n/x$. Such x and y are *factors of n*.



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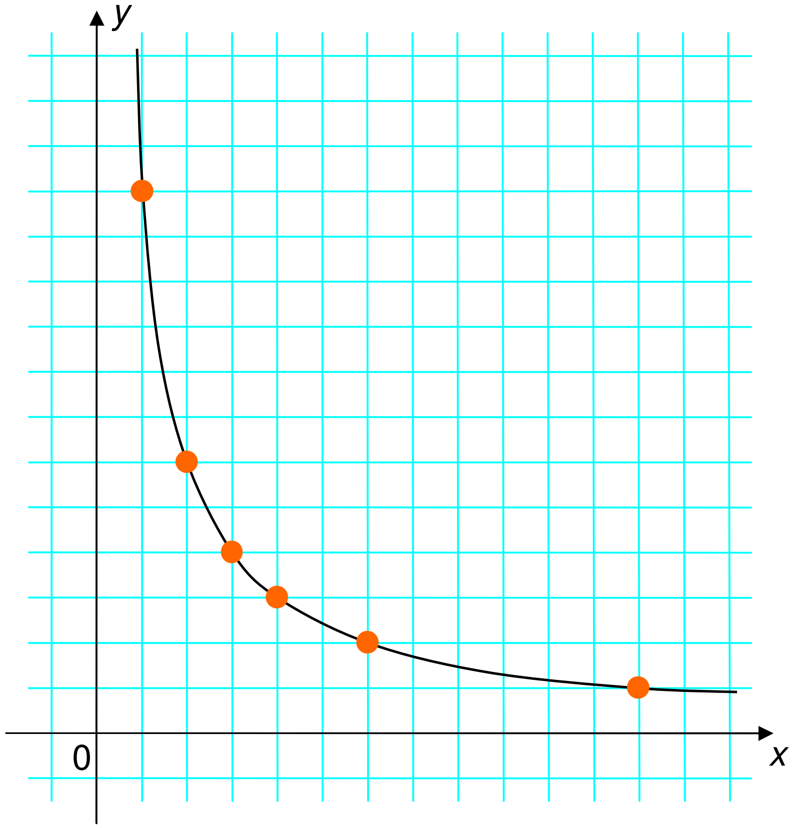

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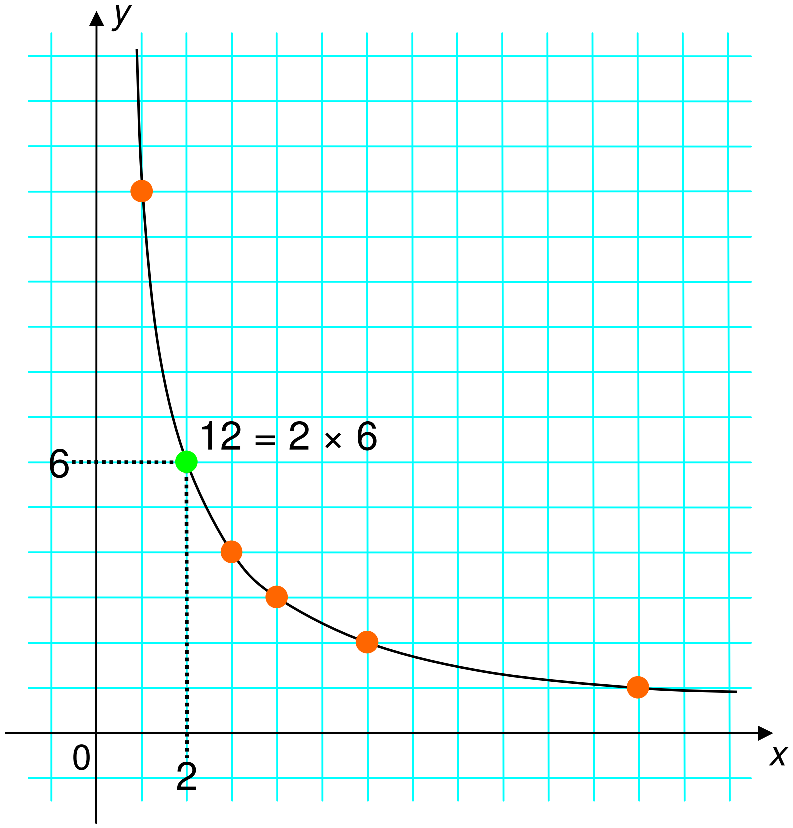

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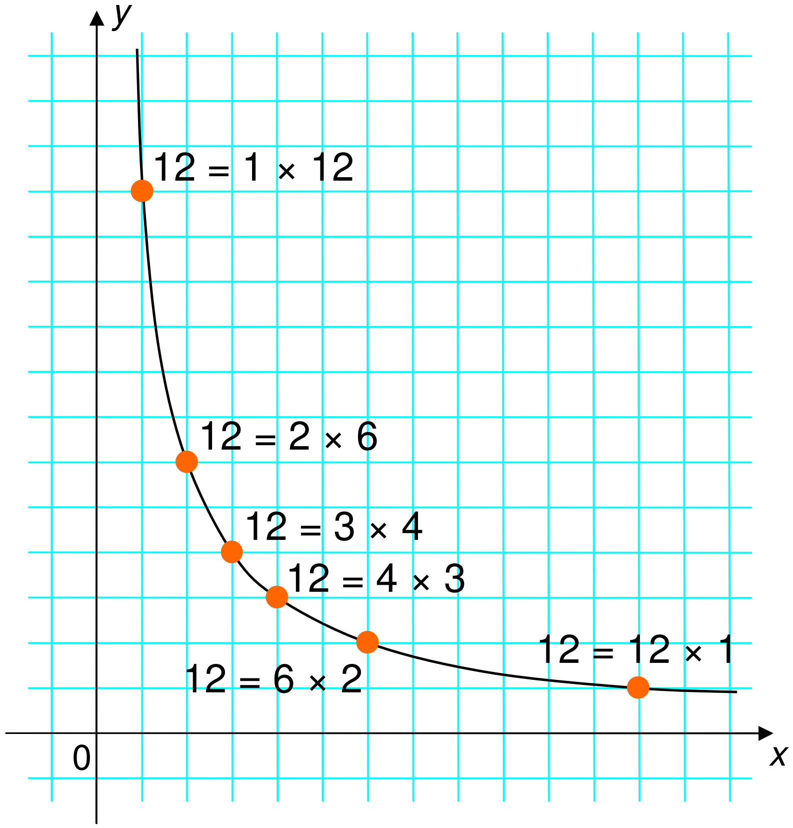

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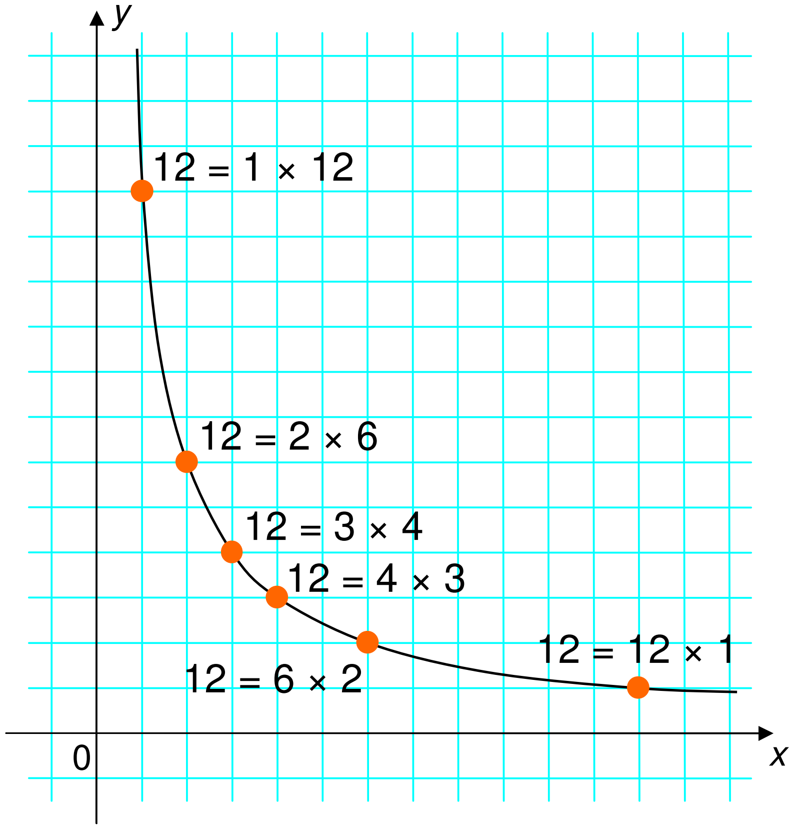
The curve is a hyperbola, and, hence, a *conic section*.

So, ***we seek points that are both***

- ***on a* cone *and***
- ***on the* integer grid**.

We implement this cone and grid.



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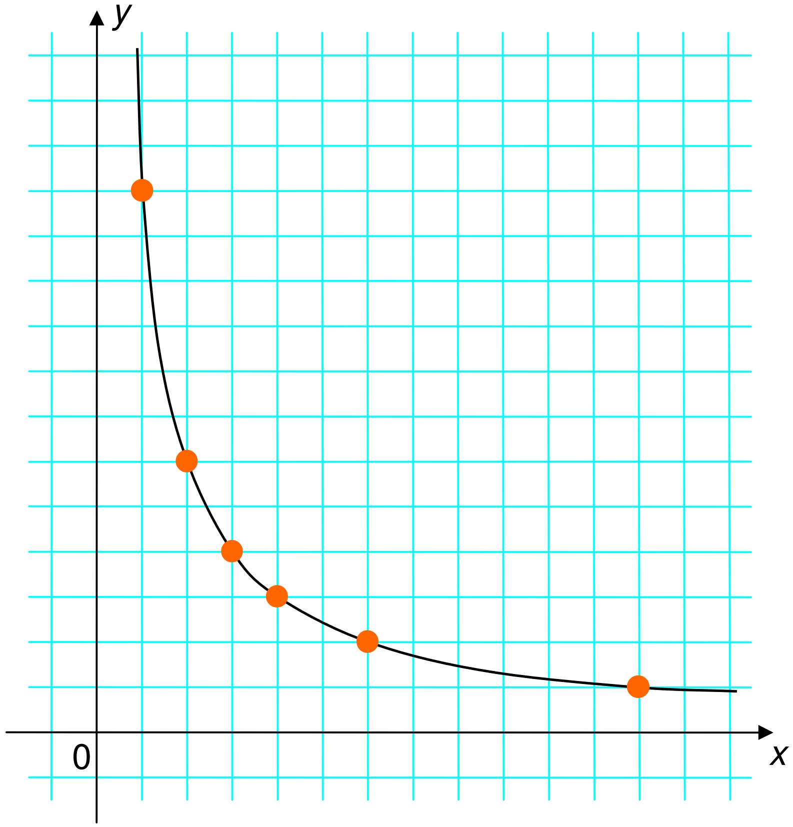

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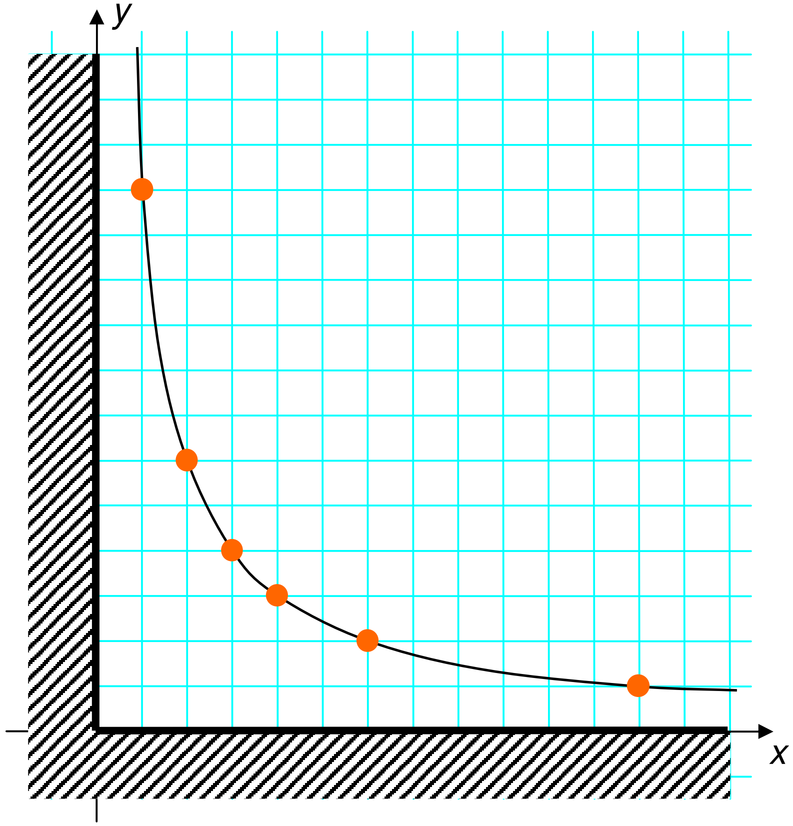

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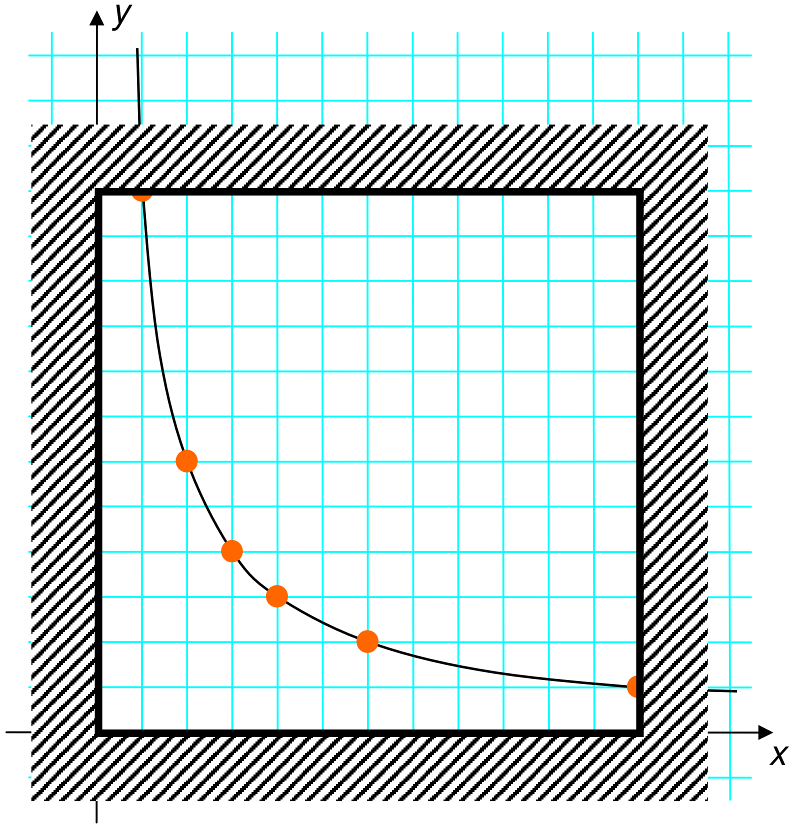

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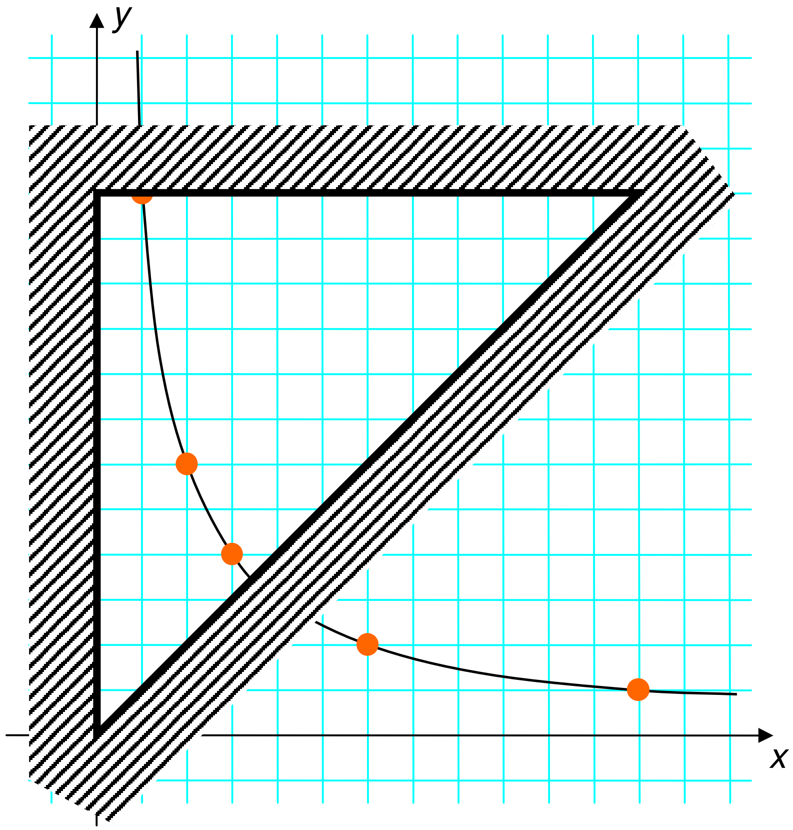

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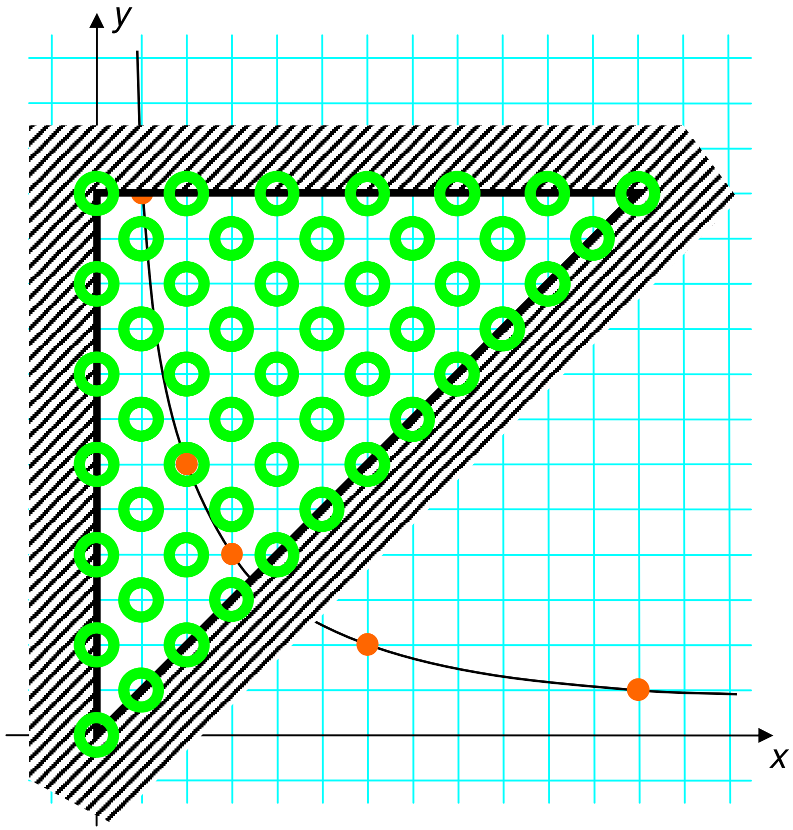

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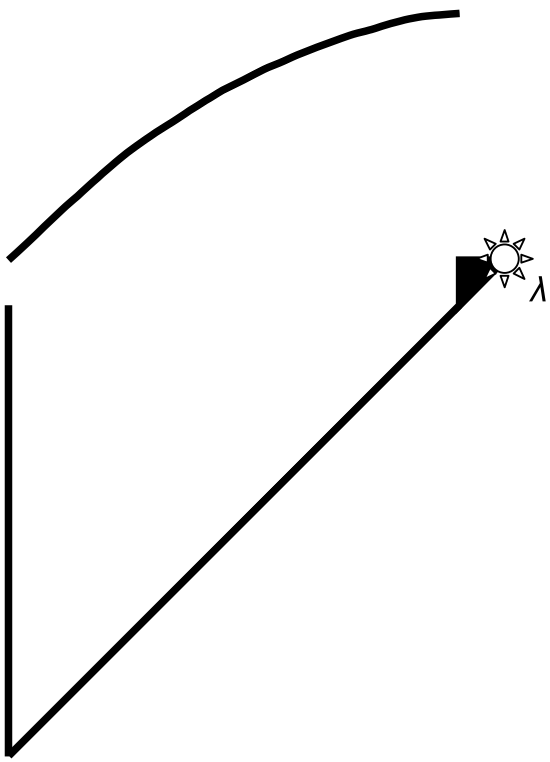
- (*x*, *y*) gives the same factors as (*y*, *x*), so we suppose that ***x* ≤ *y***.
- We assume that *n* is odd, and so need only consider points where *x* and *y* are odd; we implement points where ***x* and *y* have the same parity**.

This is the part of the integer grid that we implement.

Implementing the grid.



We use:

- a *source of waves* of wavelength $\lambda = 2/n$, and
- three *mirrors* (one parabolic, two plane).

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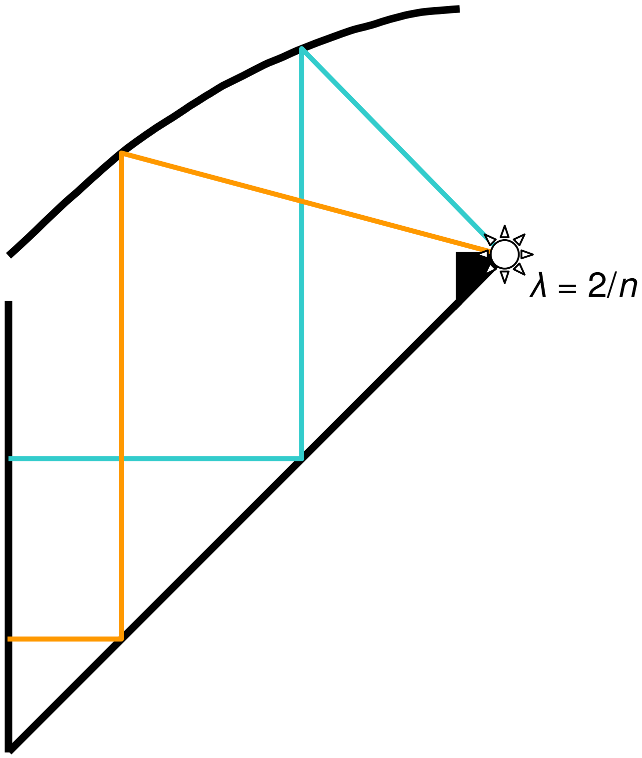

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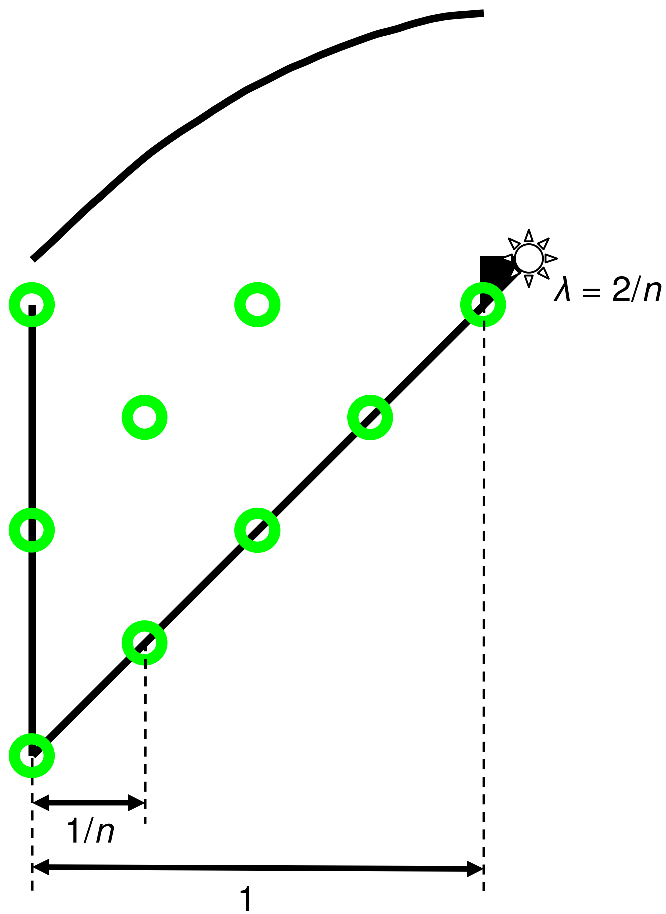

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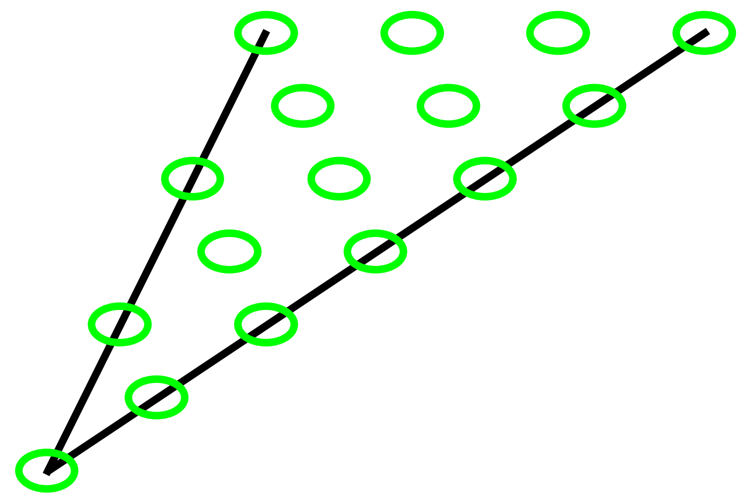
The points of maximal wave activity in the resultant interference pattern model the grid points.

Since λ depends on the input value n, setting the wavelength forms part of the system's input process.

Implementing the cone.



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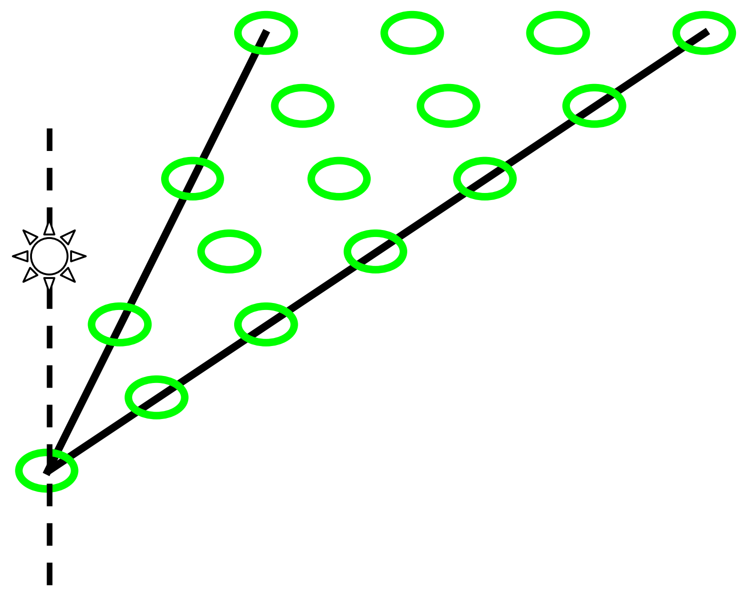

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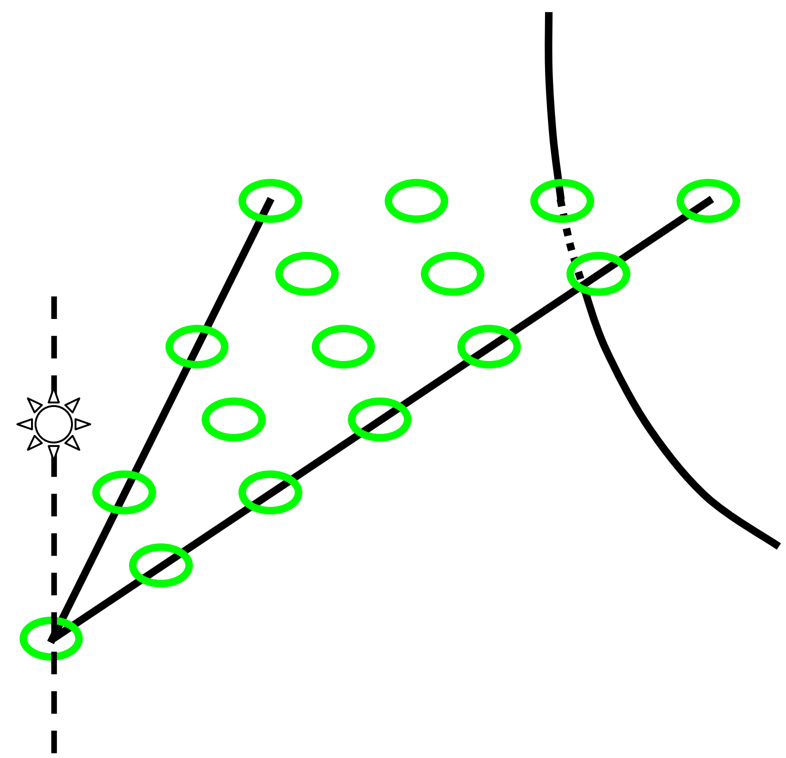

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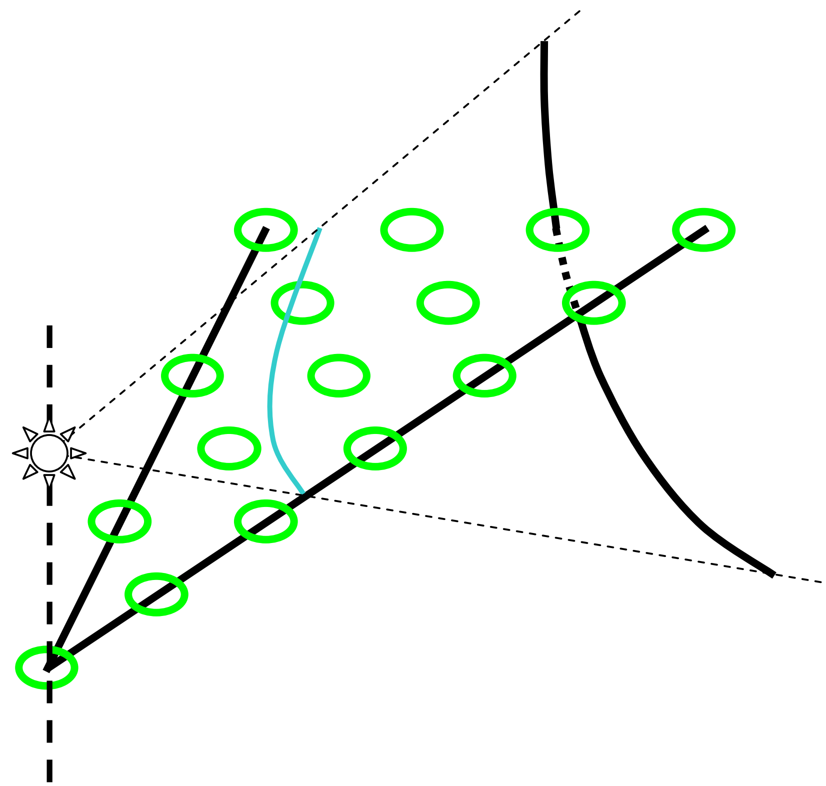

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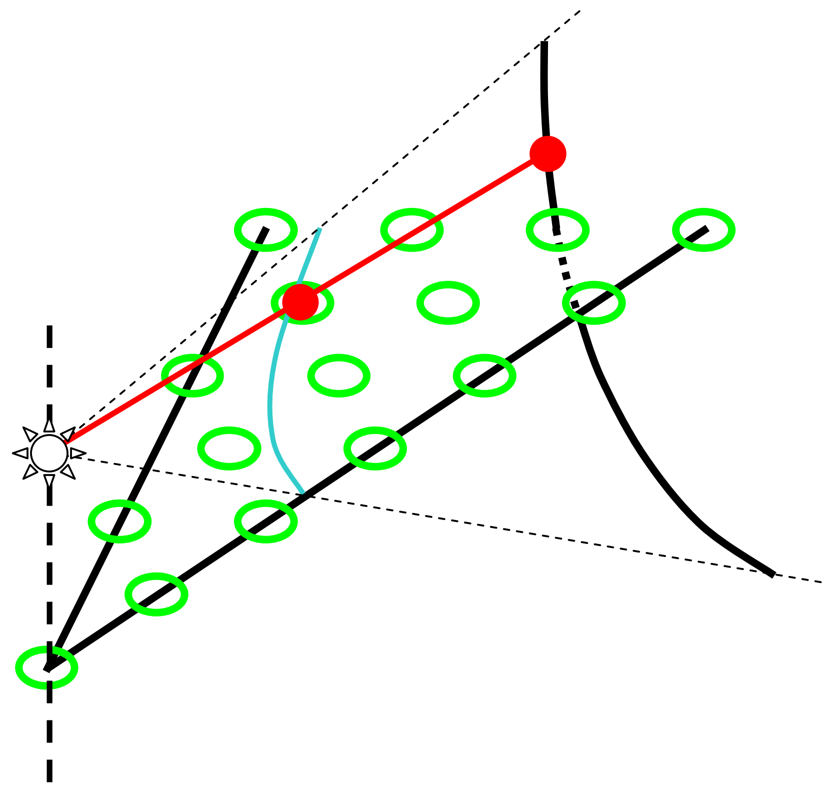
If this point is also an integer point, then the radiation will appear diminished at the sensor.



Finding factors.

- **Input.** Set parameters that depend on n: wavelength of first source, height of second source, height of sensor.
- **Processing.** Waves propagate and produce interference pattern (esp. on sensor).
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Time/space complexity.

- **Input.** Given n, calculate values (e.g. $\lambda = 2/n$) of input parameters.
- **Output.** Convert coordinates of dark spots, which encode factors of n, into factors.

These steps take *polynomial* time and space (via Turing machine).

Everything else is* constant *time/space!

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Precision.

Precision complexity captures robustness against I/O imprecision. Our system's precision complexity is *exponential*.

That is, as *n* increases, the precision with which parameters must be manipulated and measured increases *exponentially*.

How should we measure non-Turing computers' complexity?

Simply by considering all relevant resources (not just the Turing-type ones).

Given a computer, the key question is:

what resources *does the computation consume,*
and in ***what quantities****?*

Obviously still consider *algorithmic* measures (time, space, etc.) but,

especially if these seem too good to be true,

consider whether anything else *non-algorithmic (*precision, energy, etc.) is being consumed.

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- **Precision** (e.g. in analogue, quantum, chemical and optical computers).
- **Thermodynamic cost** (in irreversible computations, where entropy increases).
- **Energy** (e.g. in mechanical/analogue and quantum-adiabatic computers).
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- **Weight** (e.g. in chemical computers).
- **Etc.**

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We can then define the *complexity function* corresponding to a given resource:

$$TC_\Phi(n) := \sup\left\{T_\Phi(x) : |x| = n\right\}.$$

Blum's axioms.

These are conditions that a resource may or may not satisfy.

The axioms ensure:

1. that a resource is **defined** precisely at inputs at which the computation being measured is defined, and
2. that it is a [Turing-]**decidable** problem to determine whether a given value is indeed the measure of resource corresponding to a given input.

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In my work (where the resources are deterministic, even if the computers being measured aren't), the axioms should hold.

But they alone aren't enough…

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We'd like to be able to say that resource X is '***relevant***' for the optical computer, and that Y is for the chemical computer; then we can $\mathcal{O}$-compare X and Y.

Dominance formalizes this idea of 'relevance'.

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Resource A is *dominant* if, for all B,

$$AC_{\Phi} \in \mathscr{O}(BC_{\Phi}) \implies BC_{\Phi} \in \mathscr{O}(AC_{\Phi}) \, .$$

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We define *dominance* relative to a set of resources .

- $\mathcal{R}$ is a set of *resources*.

Resource $A \in \mathcal{R}$ is $\mathcal{R}$-*dominant* if, for all $B \in \mathcal{R}$,

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That is,

$\mathcal{R}$-dominant resources are those that $\mathcal{O}$-exceed all resources with which they are $\mathcal{O}$-comparable.

$\mathcal{R}$-complexity.

(As before, $\mathcal{R}$ is a set of *resources* and Φ is a *computer*.)

The $\mathcal{R}$-*complexity* of Φ, denoted $\mathcal{B}_{\mathcal{R},\, \Phi}$, is the complexity function given by:

$$\mathcal{B}_{\mathcal{R},\, \Phi}(n) \ := \ \Sigma_{A \text{ is } \mathcal{R}\text{-dominant}} \, AC_{\Phi}(n) \ .$$

We sum 'relevant' resources (and no others).

This captures 'overall complexity'.

Why Blum's axioms aren't enough.

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- As far as Blum's axioms are concerned, S and S' are **legitimate resources**.
- Each is an **internally consistent** measure of space usage.
- $S(x) \mapsto S'(x)$ is an **isotone** mapping: ordering inputs by their values of S has the same result as ordering by values of S'.

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But then there's dominance…

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We can engineer which resource appears more important. By applying to the more slowly growing, non-dominant resource a sufficiently fast-growing, monotonic function (e.g. $n \mapsto 2^n$), this resource becomes dominant. **We don't want this!**

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Space, depending on how we measure it, can be either *more* or *less* relevant than time!

We can engineer which resource appears more important. By applying to the more slowly growing, non-dominant resource a sufficiently fast-growing, monotonic function (e.g. $n \mapsto 2^n$), this resource becomes dominant. **We don't want this!**

So, let's restrict resources (so that, e.g., S is valid but S' is not) to stop this sort of thing. We do this with ***normalization***.

Normalization—motivation.

Recall resources S (the number of tape-cells) and S' (= 2^S).

What values can these resources take?

- We can write to any number of cells, then halt. So S maps surjectively to $\mathbb{N}$.
- Hence, S' maps surjectively to $\{1, 2, 4, 8, \ldots\}$.

The former property—mapping onto $\mathbb{N}$—seems the more natural (quite literally!), and we use it as our blueprint for normalization.

Normalization—definition.

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- Let A be a resource that can take as its subscript any computing system $\Phi \in \mathcal{C}$ (A_Φ: $X_\Phi \to \mathbb{N}$). Define the $\mathcal{C}$-*normalized form* of A to be the resource $A^\mathcal{C}$ given by $A^\mathcal{C}{}_\Phi$: $X_\Phi \to \mathbb{N}$,

$$A^\mathcal{C}{}_\Phi(x) := |\{\, A_\Psi(y) : \Psi \in \mathcal{C} \text{ and } y \in X_\Psi \text{ and } A_\Psi(y) < A_\Phi(x)\}|\ .$$

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- Resource A is $\mathcal{C}$-*normal* if $A = A^\mathcal{C}$ (i.e. if $A_\Phi(x) = A^\mathcal{C}{}_\Phi(x)$ for all $\Phi \in \mathcal{C}$ and $x \in X_\Phi$).

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In words, $A^\mathcal{C}{}_\Phi(x)$ is the number of distinct values less than $A_\Phi(x)$ taken by A (as it ranges over all computers in $\mathcal{C}$ and all input values).

This is a measure of ‘how much use A makes’ of the natural numbers less than $A_\Phi(x)$.

Normalization—example.

Revisiting our example of S (the number of tape-cells) and S' (= 2^S), we have that

$$S'^{\mathfrak{T}} = S^{\mathfrak{T}} = S$$

(and that $T^{\mathfrak{T}} = T$),

where $\mathfrak{T}$ is the class of Turing machines.

So, if we use only *normal* resources, we may validly compare S and T, but not S' and T.

Normalization—properties.

- Normalization is strictly **isotone**:

$$A_{\Phi}(x) < A_{\Psi}(y) \text{ if and only if } A^{\mathcal{C}}{}_{\Phi}(x) < A^{\mathcal{C}}{}_{\Psi}(y)\ .$$

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- **Characterization** of normal resources:

Resource A is normal if and only if its image set (over all computers and input values) $\{A_{\Psi}(y) : \Psi \in \mathfrak{C} \text{ and } y \in X_{\Psi}\}$ is an 'initial segment' $\{i \in \mathbb{N} : i < n\}$ for some $n \in \mathbb{N} \cup \{\infty\}$.

Normalization—intuition.

Non-normalized: cardinal.
Normalized: ordinal.

If we let resources be *any* functions that satisfy Blum's axioms and have codomain $\mathbb{N}$, then we are effectively dealing with ***cardinals***: we are *counting* time steps, units of energy or similar—we have an intrinsic *unit of measurement*.

This is resource-dependent and not conducive to resource-heterogeneous comparison (e.g., how many time steps should we deem of equivalent cost/value to one tape cell?).

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But if we allow only $\mathcal{C}$-*normal* resources, then we have ***ordinals***: 0 represents the *least* resource consumption, 1 the *second-least*, 2 the *third-least*, and so on; this is independent of resources and units. Comparison then seems fair and equal-footed.

Concluding comments.

- With *Turing machines*, we know what resources to consider.
- With *unconventional computers*, it's not so obvious.
- And even when we've identified our unconventional resources, there are comparison difficulties. Hence ***dominance***.
- But to make dominance work, we need to restrict our notion of resource (more than Blum's axioms do). Hence ***normalization***.
- This restriction stops certain 'deceptive' complexity behaviour, e.g. 'dominance engineering' via application of quickly growing, isotone functions.
- The restriction also renders resources 'ordinal', not 'cardinal', allowing seemingly fairer comparison.

Questions?

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A paper, ***Beyond Blum: What is a Resource?***, related to parts of this talk is to appear in the *International Journal of Unconventional Computation*.

We thank the EPSRC for its generous support. This workshop and the research described in this talk are funded by EPSRC grant

Complexity and Decidability in Unconventional Computational Models.