From Classical Channels Towards Abstract Models of Computation

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Joint work with Karl Hofmann

ISR
Oxford 2011

Work partially supported by ONR

Overview

- Early work: Woodcock & Roscoe (94?), Lowe (95), Ryan & Schneider…
- Quantitative Information Flow
  - Looping constructs (Malacaria)
  - Min-entropy, Min-capacity (Smith)
- *Goal:* Understand capacity from domain-theoretic, abstract perspective
- Peter Shor: *"Capacity is the most important statistic about a channel"*

Plan of the Talk

Classical channels

- Capacity from binary to n-ary channels
- Domain theory and generalization of Jensen's Inequality

- Categorical properties of inputs, outputs and channels
- Lifting results to abstract models of computation

Channels

- Classical Discrete, Memoryless Channel (DMC)
  - Noisy, lossless
    - Each input produces a probability distribution on the outputs.
  - Capacity gives maximum rate of reliable transmission

Channel Capacity

$C : X \longrightarrow \textsc{Prob}(Y)$ noisy, lossless channel

$$(\forall y \in Y) \ \sum_{x \in X} p(y|x) = 1$$

$C = (p(y|x))$ — $|X| \times |Y|$–stochastic matrix

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$\mathsf{ST}(m,n)$ – family of $m \times n$-stochastic matrices

$\mathsf{ST}(n)$ – Monoid of $n \times n$-stochastic matrices

The Binary Case

Martin, Allwein & Moskowitz:

- Binary channels: $M = \begin{bmatrix} a & 1-a \\ b & 1-b \end{bmatrix}$
  - $M \longleftrightarrow (a,b) \in [0,1] \times [0,1]$
  - $\mathsf{ST}(2)$ – 2×2 stochastic matrices
  - $\mathcal{N} = \{(a,b) \mid a \geq b\}$ – lower triangle in square
    – submonoid of $\mathsf{ST}(2)$

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- Capacity of a binary channel

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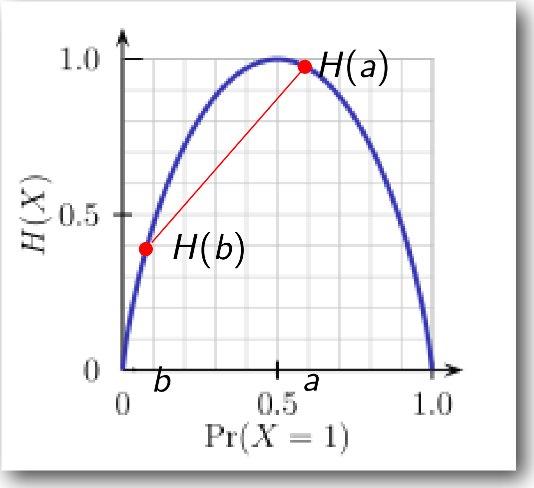
$$Cap(M) = \sup_{x\in[0,1]} H(xa + (1-x)b) - xH(a) - (1-x)H(b)$$



where $H(r) = r\log r - (1-r)\log(1-r)$

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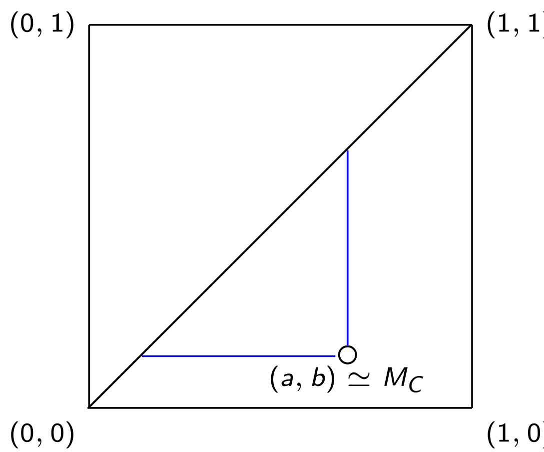

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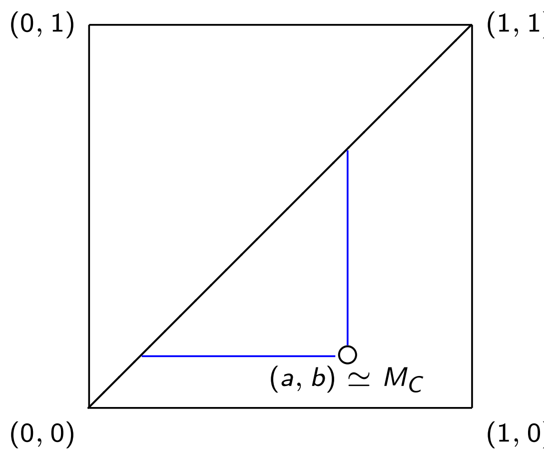

$$\mathcal{N} = \{(a,b) \mid a \geq b\}$$

$(a,b) \mapsto [b,a] \colon \mathcal{N} \to (\mathbf{I}([0,1]), \supseteq)$ order isomorphism

$(a,b) \leq (a',b') \iff (a,b) \supseteq (a',b')$

– defines domain structure on $\mathcal{N}$

$\mathsf{Cap} \colon \mathcal{N} \to [0,1]^{op}$ measures $\mathcal{N}$

Note: $\mathsf{ST}(2)/H(1) \simeq \mathcal{N}$

Topology

$f : \mathbb{R} \to \mathbb{R}$ *convex* if $f(r \cdot x + (1-r)y) \;\leq\; r \cdot f(x) + (1-r) \cdot f(y)$.

Jensen's Inequality: If X is a random variable and f is convex, then $E(f(X)) \geq f(E(X))$, & $E(f(X)) = f(E(X)) \;\Rightarrow\; f$ constant.

Topology

K – compact, convex subset of a locally convex TVS

$f\colon K \to \mathbb{R}$ *strictly concave* if

$$f(r \cdot x + (1-r)y) \;>\; r \cdot f(x) + (1-r) \cdot f(y).$$

Proposition: For $f\colon K \to \mathbb{R}$ TAE:

- f is strictly concave.
- $\forall (r_1, \ldots, r_n) \in [0,1]^n, \forall (k_1, \ldots, k_n) \in K^n$
  $\sum_i r_i = 1 \quad \Rightarrow \quad f(\sum_i r_i k_i) > \sum_i r_i f(k_i).$

Corollary: $M \in \mathsf{ST}(n)$, $(r_,, \ldots, r_n) \in (0,1)^n, \sum_i r_i = 1$ implies $I(pM;p) = H(pM) - \sum_{i \le n} p(i) H(M(i)) > 0.$

Domain Theory

P — partially ordered set

$D \subseteq P$ *directed* if $\forall F \subseteq D$ finite, $\exists d \in D \;.\; F \subseteq {\downarrow} d$

P *directed complete* if every directed subset has a supremum *in* P

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$x, y \in P$, x *approximates* y ($x \ll y$) iff $\forall D \subseteq P$ directed,

$y \leq \sqcup D \;\Rightarrow\; (\exists d \in D)\; x \leq d.$ $\quad {\Downarrow} y = \{x \mid x \ll y\}$

P *continuous* if $(\forall y \in P)\; {\Downarrow} y$ is directed and $y = \sqcup {\Downarrow} y$

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Examples: 1. $(\mathcal{P}(X), \subseteq)$; $A \ll B$ iff A is finite

2. X compact, T_2. $(\Gamma(X), \supseteq)$; $C \ll C'$ iff $C' \subseteq C^\circ$.

3. $K \subseteq \mathbb{R}^n$ compact, convex. $(Con(K), \supseteq)$; $C \ll C'$ iff $C' \subseteq C^\circ$.

Domains and Topology

P a dcpo. $U \subseteq P$ *Scott open* if

1. $U = \uparrow U$, and
2. $(\forall D \subseteq P$ directed$)\ \sqcup D \in U \;\Rightarrow\; D \cap U \neq \emptyset$.

Basic Results: 1. $\Uparrow x = \{y \mid x \ll y\}$ is basis for Scott topology on a domain.

2. P a domain implies P is sober in the Scott topology.

3. $f \colon P \to Q$ is Scott continuous iff f is monotone and preserves sups of directed sets.

Domains and Topology

Lawson topology: common refinement of Scott topology with weak lower topology – $\{P \setminus \uparrow F \mid F \subseteq P \text{ finite}\}$.

Lawson topology on a domain is T_2; P *coherent* if it is compact.

Examples: 1. $(\mathcal{P}(X), \subseteq)$.

2. $(\Gamma(X), \supseteq)$ — Lawson is Vietoris topology.

3. $K \subseteq \mathbb{R}^n$ compact, convex. $(Con(K), \supseteq)$ coherent.

$$Con_n(K) = \{\overline{\mathrm{conv}}(F) \mid \emptyset \neq F \subseteq K, \ \& \ |F| \leq n\}.$$

Domains and Convexity

$$\Delta^k = \{(r_1, \ldots, r_k) \in [0,1]^k \mid \sum_{i \le k} r_i = 1\} \quad \simeq \ \mathsf{Prob}(\underline{k})$$

$K \subseteq \mathbb{R}^m$ compact convex, $f \colon K \to \mathbb{R}$ concave

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$$(\forall \vec{r} \in \Delta^k, \forall \vec{x} \in K^k)\ \phi_{\vec{r}}(\vec{x}) = \sum_{i \leq n} r_i x_i$$

$$\begin{tikzcd}
K^k \arrow[r, "f^n"] \arrow[d, "\phi_{\vec{r}}"'] \arrow[dr, phantom, "\geq"] & \mathbb{R}^k \arrow[d, "\phi_{\vec{r}}"] \\
K \arrow[r, "f"'] & \mathbb{R}
\end{tikzcd}$$

Domains and Convexity

Proposition: Let $K \subseteq \mathbb{R}^m$ compact, convex, and $f\colon K \to \mathbb{R}_+$ continuous and concave. Then $\widehat{f}\colon (Con_n(K), \supseteq) \to \mathbb{R}_+$ by

$$\widehat{f}(\overline{\mathrm{conv}}(\{x_1, \ldots, x_k\})) = \sup_{(r_1,\ldots,r_k)\in\Delta^k} f(\sum_{i\leq k} r_i x_i) - \sum_{i\leq k} r_i f(x_i)$$

is (Lawson) continuous and monotone.

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is (Lawson) continuous and monotone.

Corollary: For each $n \geq 1$, $\mathsf{cap}\colon (Con_n(\mathsf{Prob}(\underline{n})), \supseteq) \to \mathbb{R}_+^{op}$ by

$$\mathsf{cap}(\overline{\text{conv}}(\{x_1, \ldots, x_k\})) = \sup_{(r_1,\ldots,r_k)\in\Delta^k} H(\sum_{i\leq k} r_i x_i) - \sum_{i\leq k} r_i H(x_i)$$

is (Lawson) continuous and strictly monotone.

On Measurements

$m\colon P \to [0,\infty)^{op}$ *measures the content at* $x \in P$ if $\forall U \subseteq P$ Scott open

$$x \in U \;\Rightarrow\; (\exists \epsilon > 0)\; m_\epsilon(x) \subseteq U$$

where $m_\epsilon(x) = \{y \leq x \mid m(y) < m(x) + \epsilon\}$. m *measures* P if m measures P at each $x \in P$.

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Example: $\mathbb{I}([0,1]) = (\{[a,b] \mid a \leq b \in [0,1]\}, \supseteq)$,

$$m([a,b]) = b - a.$$

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Fact: $f\colon P \to Q$ Scott continuous, P, Q domains $\Rightarrow$

f is proper iff $f^{-1}(\uparrow y) \subseteq P$ is compact $\forall y \in Q$.

Proposition: $m\colon P \to [0,\infty)^{op}$ Scott continuous, P domain, m proper at $x \in P$. TAE:

1. m measures the content at x.
2. m is strictly monotone at x: $y \leq x \;\&\; m(y) = m(x) \;\Rightarrow\; y = x$.

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In particular, if m is Scott continuous and proper, then m measures P iff m is strictly monotone.

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For each $n \geq 1$, $\mathsf{cap}\colon (Con_n(\mathsf{Prob}(\underline{n})), \supseteq) \to \mathbb{R}_+^{op}$ measures $(Con_n(\mathsf{Prob}(\underline{n})), \supseteq)$ at $\{\{x\} \mid x \in \mathsf{Prob}(\underline{n})\}$.

Relating *cap* to *Cap*

For each $n \geq 1$, $\mathsf{cap}\colon (Con_n(\mathsf{Prob}(\underline{n})), \supseteq) \to \mathbb{R}_+^{op}$ by

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$$C \in \mathsf{ST}(n) \;\Rightarrow\; Cap(C) = \sup_{p\in\textsc{Prob}(\underline{n})} H(pC) - H(pC \mid p)$$

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$$= \mathsf{cap}(\overline{conv}(\{C(1), \ldots, C(n)\}))$$

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$$C \leq C' \;\Leftrightarrow\; C(\textsc{Prob}(\underline{n})) \supseteq C'(\textsc{Prob}(\underline{n})) \implies \mathsf{Cap}(C) \geq \mathsf{Cap}(C')$$

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$$C \equiv C' \;\Leftrightarrow\; C(\textsc{Prob}(\underline{n})) = C'(\textsc{Prob}(\underline{n}))$$

$$\Leftrightarrow\; S(n) \circ C = S(n) \circ C'$$

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Theorem:

- $Con_n(\mathsf{Prob}(m)), \supseteq)$ is a domain.
- $\mathsf{cap}\colon (Con_n(\mathsf{Prob}(m)), \supseteq) \to \mathbb{R}^{op}$ is a measurement
- $(\mathsf{ST}(m)/\equiv, \leq) \simeq (Con_n(\mathsf{Prob}(m)), \supseteq)$ *qua* domain.
- $C \equiv C' \implies \mathsf{Cap}(C) = \mathsf{Cap}(C')$

Where Channels Come From

$$\text{Monad I}: \ \mathsf{Prob}\colon \mathsf{Comp} \to \mathsf{CompConv}_{LC}$$

$$\mathsf{Comp} \ - \ \text{Compact}, \mathrm{T}_2 \text{ spaces}, \ \text{continuous maps}$$

$$\mathsf{CompConv}_{LC} \ - \ \text{compact}, \ \text{convex} \ \subseteq \ TVS_{LC}, \ \text{affine maps}$$

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$$\mathsf{Prob}(X)_s = \{\sum_{i \leq n} r_i \delta_{x_i} \mid x_i \in X, \sum_i r_i = 1\}$$

$$\mathsf{Prob}(X) = \langle \mathsf{Prob}(X)_s \rangle \text{ where } \textsc{Prob}(X) \subseteq \mathcal{M}(X) = C(X, \mathbb{R})^*$$

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$$\mathsf{ST}(m, n) = \mathsf{Aff}(\textsc{Prob}(\underline{m}), \textsc{Prob}(\underline{n})) \simeq \{\widehat{C} \mid C\colon \underline{m} \to \textsc{Prob}(\underline{n})\}$$

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$$\text{Monad II}: \ \mathsf{Prob}\colon \mathsf{CMon} \to \mathsf{CAM}_{LC}$$

$$\mathsf{CMon} \ - \ \text{Compact}, \mathrm{T}_2 \text{ monoids}, \ \text{continuous monoid maps}$$

$$\mathsf{CAM}_{LC} \ - \ \text{compact}, \ \text{convex}, \ \text{monoids} \ \subseteq \ TVS_{LC},$$
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$$\text{In particular}: \ \ \mathsf{ST}(m) \ = \mathsf{Prob}([\underline{m} \to \underline{m}])$$

$$= \{\sum_{i \leq m} r_i \delta_{f(x_i)} \mid \sum_i r_i = 1 \ \wedge \ f\colon \underline{m} \to \underline{m}\}$$

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$$\text{Adjunction III}: \ \mathsf{Prob}\colon \mathsf{CGrp} \to \mathsf{CAM}_{LC} \colon H$$

$$H(S) = H(1_S) \quad H(\phi) = \phi|_{H(1_S)} \colon S \to T$$

$$\mathsf{CMGrp} \ - \ \text{Compact groups, continuous group homomorphisms}$$

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$$\text{In particular}: \quad \mathsf{DST}(m) = \mathsf{Prob}(S(m))$$

More Generally

$$\text{Monad I}: \ \mathsf{Prob}\colon \mathsf{LocComp} \to \mathsf{CompConv}_{LC}$$

$$X \ \mapsto \ \mathsf{Prob}(X)$$

$$\text{Monad II}: \ \mathsf{Prob}\colon \mathsf{LCMon} \to \mathsf{CAM}_{LC}$$

$$S \mapsto \mathsf{Prob}(S)$$

In particular: This works for discrete spaces

Toward Abstract Models

$$\begin{tikzcd} \mathsf{Comp} \arrow[rr, bend left, "\mathsf{Prob}"] \arrow[rr, phantom, "\perp"] & & \mathsf{CompConv}_{LC} \arrow[ll, bend left] \end{tikzcd}$$

Toward Abstract Models

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathsf{Comp} \arrow[r, bend left, "\mathsf{Prob}"] \arrow[r, phantom, "\perp"] \arrow[d, bend right, "\mathcal{P}_U"'] \arrow[d, phantom, "\dashv"] & \mathsf{CompConv}_{LC} \arrow[l, bend left] \\
\mathsf{BCD}_{\Lambda} \arrow[u, bend right, "\Lambda"']
\end{tikzcd}$$

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\mathsf{Comp} \arrow[r, bend left, "\mathsf{Prob}"] \arrow[r, phantom, "\perp"] \arrow[d, bend right, "\mathcal{P}_U"'] \arrow[d, phantom, "\dashv"] & \mathsf{CompConv}_{LC} \arrow[l, bend left] \arrow[d, bend right, "\mathcal{P}_U"'] \arrow[d, phantom, "\vdash"] \\
\mathsf{BCD}_\Lambda \arrow[u, bend right] \arrow[r, bend left, "\mathsf{Prob}"] \arrow[r, phantom, "\perp"] & \mathsf{Coh}_\Lambda \arrow[l, bend left] \arrow[u, bend right]
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No Distributive Law

Toward Abstract Models

$$
\begin{tikzcd}
\mathsf{Comp} \arrow[r, bend left, "\mathsf{Prob}"] \arrow[d, bend right, "\mathcal{P}_U"'] \arrow[d, phantom, "\dashv"] & \mathsf{CompConv}_{LC} \arrow[l, bend left] \arrow[l, phantom, "\perp"] \arrow[dr, bend left, "Con"] & \\
\mathsf{BCD}_\Lambda \arrow[u, bend right] \arrow[r, bend left, "\mathsf{Prob}"] & \mathsf{Coh}_\Lambda \arrow[l, bend left] \arrow[l, phantom, "\perp"] \arrow[r, bend left, "\mathsf{Pow}_U"] & \mathsf{BCD}_\Lambda \arrow[l, bend left] \arrow[l, phantom, "\perp"] \arrow[ul, bend left]
\end{tikzcd}
$$

Toward Abstract Models

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\begin{tikzcd}[column sep=huge, row sep=huge]
\mathsf{Comp} \arrow[r, bend left, "\mathsf{Prob}"] \arrow[d, bend right, "\mathcal{P}_U"'] \arrow[r, phantom, "\perp"] \arrow[d, phantom, "\dashv"] & \mathsf{CompConv}_{LC} \arrow[l, bend left] \arrow[rd, bend left, "Con"] & \\
\mathsf{BCD}_\Lambda \arrow[u, bend right] \arrow[r, bend left, "\mathsf{Prob}"] \arrow[r, phantom, "\perp"] & \mathsf{Coh}_\Lambda \arrow[l, bend left] \arrow[r, bend left, "\mathsf{Pow}_U"] \arrow[r, phantom, "\perp"] & \mathsf{BCD}_\Lambda \arrow[l, bend left] \arrow[lu, bend left]
\end{tikzcd}
$$

$\mathcal{T}hank\ You!!$

$\mathcal{Q}uestions?$