Winning Strategies in Concurrent Games

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A principled way to develop *nondeterministic concurrent strategies* in games within a general model for concurrency. Following Joyal and Conway, a strategy from a game G to a game H will be a strategy in $G^{\perp} \| H$. Strategies will be those nondeterministic plays of a game which compose well with copy-cat strategies, within the model of event structures. Consequences, connections and extensions to winning strategies.

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Event structures

An **event structure** comprises $(E, \leq, \mathrm{Con})$, consisting of a set of *events* E

- partially ordered by $\leq$, the **causal dependency relation**, and

- a nonempty family Con of finite subsets of E, the **consistency relation**,

which satisfy

$$\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,$$

$$\{e\} \in \mathrm{Con} \text{ for all } e \in E,$$

$$Y \subseteq X \in \mathrm{Con} \Rightarrow Y \in \mathrm{Con}, \quad \text{and}$$

$$X \in \mathrm{Con} \;\&\; e \leq e' \in X \Rightarrow X \cup \{e\} \in \mathrm{Con}.$$

Say e, e' are **concurrent** if $\{e, e'\} \in \mathrm{Con} \;\&\; e \not\leq e' \;\&\; e' \not\leq e$.
In games the relation of **immediate dependency** $e \rightarrowtriangle e'$, meaning e and e' are distinct with $e \leq e'$ and no event in between, will play an important role.

Configurations of an event structure

The **configurations**, $\mathcal{C}^\infty(E)$, of an event structure E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall X \subseteq_{\text{fin}} x.\ X \in \text{Con}$ and

Down-closed: $\forall e, e'.\ e' \leq e \in x \Rightarrow e' \in x$.

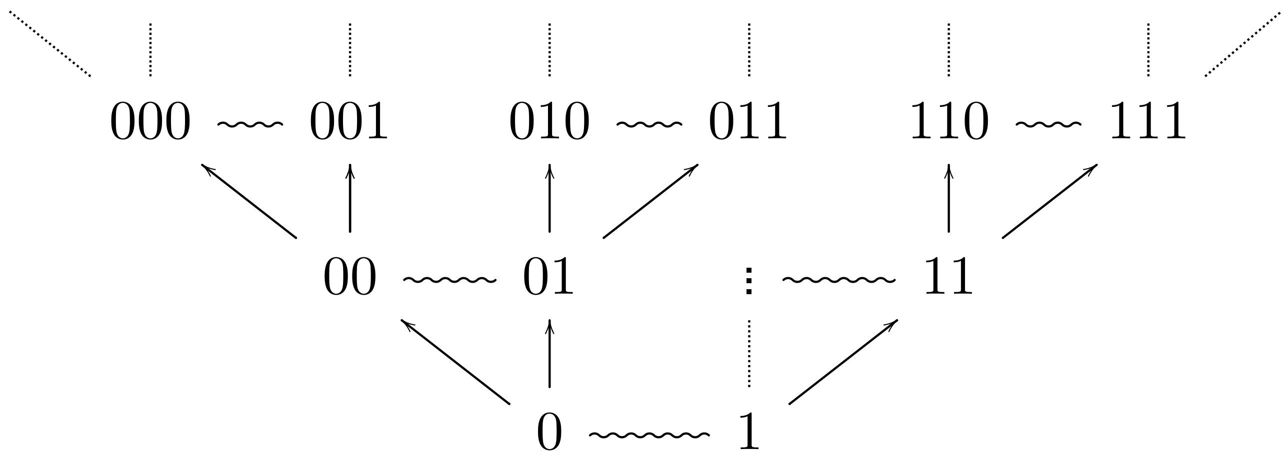
For an event e the set $[e] =_{\text{def}} \{e' \in E \mid e' \leq e\}$ is a configuration describing the whole causal history of the event e.

$x \subseteq x'$, *i.e.* x is a sub-configuration of x', means that x is a sub-history of x'.

If E is countable, $(\mathcal{C}^\infty(E), \subseteq)$ is a dI-domain (and all such are so obtained).

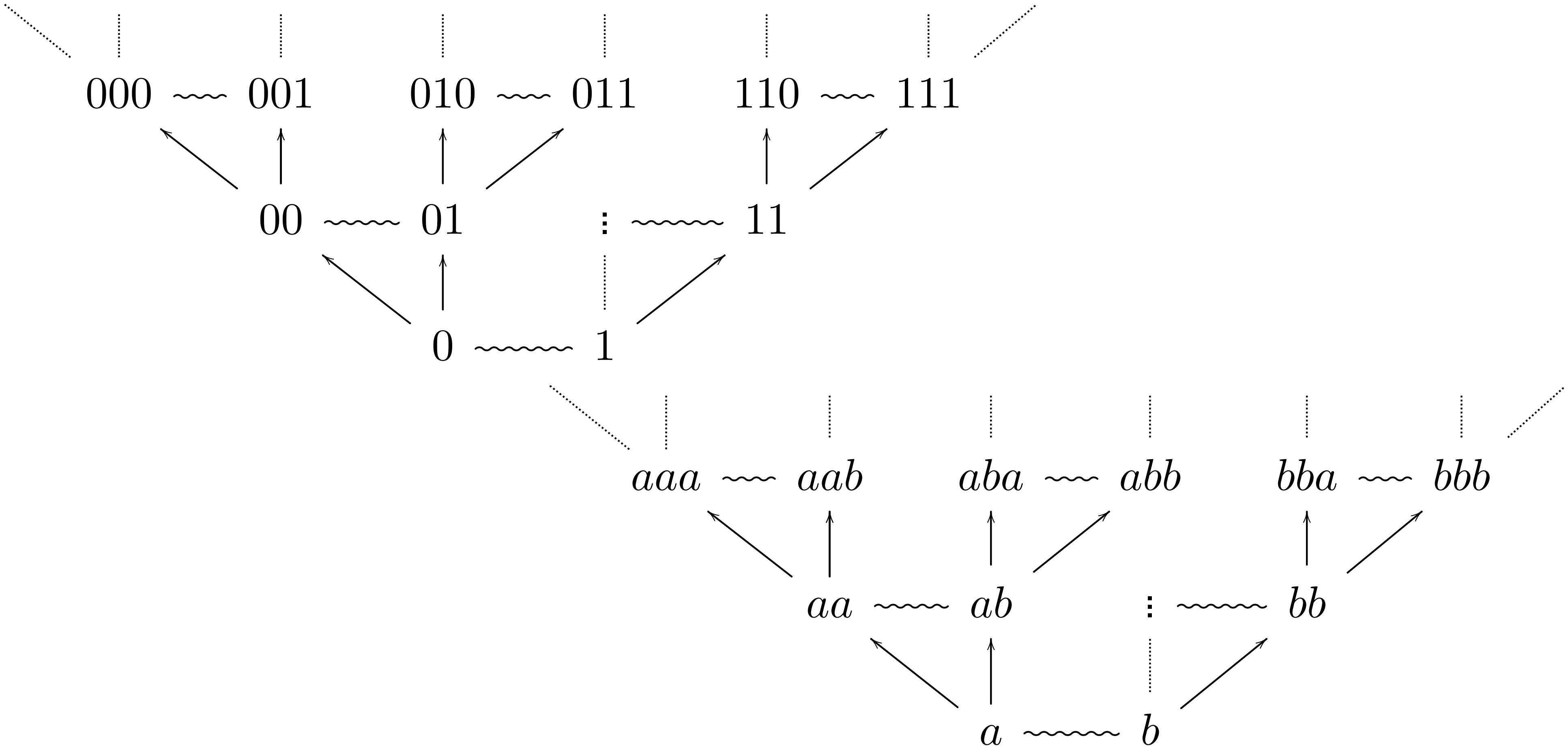
Often concentrate on the **finite configurations** $\mathcal{C}(E)$.

Example: Streams as event structures

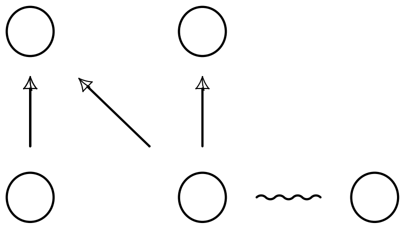


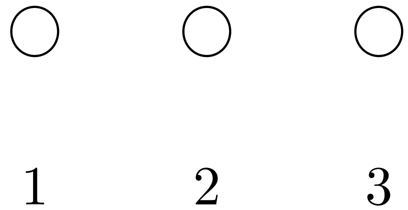
~~~ conflict (inconsistency) ⟶ immediate causal dependency
~~~

Simple parallel composition



Other examples





$$\text{Con} = \{\ \emptyset,\ \{1\}, \{2\}, \{3\},\ \{1,2\}, \{1,3\}, \{2,3\}\ \}$$

Maps of event structures

- Semantics of synchronising processes [Hoare, Milner] can be expressed in terms of universal constructions on event structures, and other models.
- Relations between models via adjunctions.

In this context, a **simulation map** of event structures $f : E \to E'$ is a partial function on events $f : E \rightharpoonup E'$ such that for all $x \in \mathcal{C}(E)$

$fx \in \mathcal{C}(E')$ and

if $e_1, e_2 \in x$ and $f(e_1) = f(e_2)$, then $e_1 = e_2$. *('event linearity')*

Idea: *the occurrence of an event e in E induces the coincident occurrence of the event $f(e)$ in E' whenever it is defined.*

Process constructions on event structures

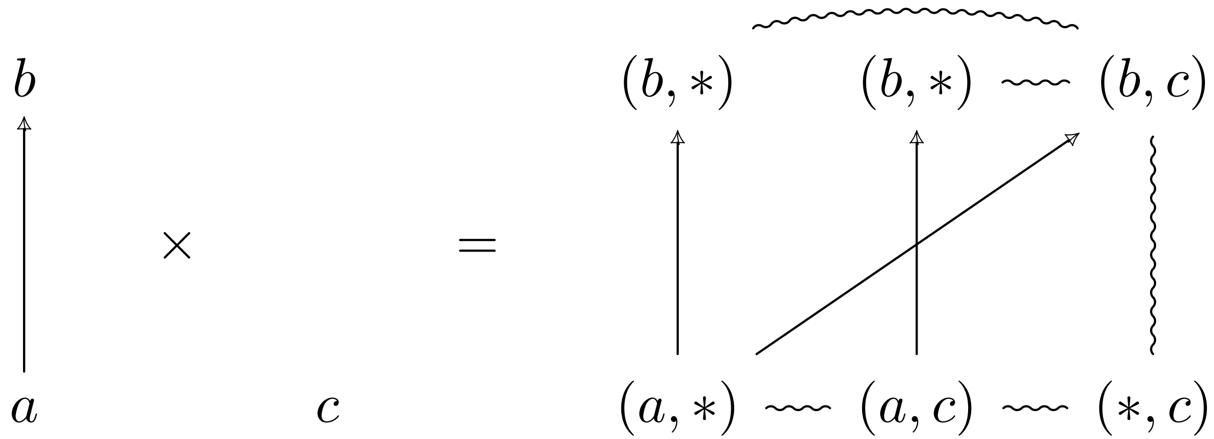
"Partial synchronous" product: $A \times B$ with projections Π_1 and Π_2, *cf.* CCS synchronized composition where all events of A *can* synchronize with all events of B. ($Hard\ to\ construct\ directly\ so\ use\ e.g.\ stable\ families.$)

Restriction: $E \restriction R$, the restriction of an event structure E to a subset of events R, has events $E' = \{e \in E \mid [e] \subseteq R\}$ with causal dependency and consistency restricted from E.

Synchronized compositions: restrictions of products $A \times B \restriction R$, where R specifies the allowed synchronized and unsynchronized events.

Projection: Let E be an event structure. Let V be a subset of 'visible' events. The *projection* of E on V, $E{\downarrow}V$, has events V with causal dependency and consistency restricted from E.

Product—an example



Concurrent games

Basics

Games and strategies are represented by **event structures with polarity**, an event structure in which all events carry a polarity $+/-$, respected by maps.

The two polarities $+$ and $-$ express the dichotomy:
player/opponent; process/environment; ally/enemy.

Dual, $E^\perp$, of an event structure with polarity E is a copy of the event structure E with a reversal of polarities; $\overline{e} \in E^\perp$ is complement of $e \in E$, and *vice versa*.

A (nondeterministic) concurrent **pre-strategy** in game A is a total map

$$\sigma : S \to A$$

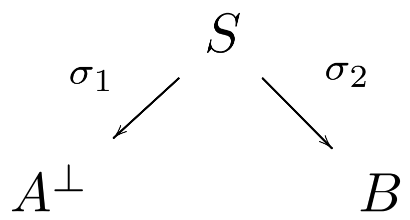
of event structures with polarity (*a nondeterministic play in game* A).

Pre-strategies as arrows

A pre-strategy $\sigma : A \nrightarrow B$ is a total map of event structures with polarity

$$\sigma : S \to A^{\perp} \parallel B \,.$$

It corresponds to a *span* of event structures with polarity



where σ_1, σ_2 are *partial* maps of event structures with polarity; one and only one of σ_1, σ_2 is defined on each event of S.

Pre-strategies are isomorphic if they are isomorphic as spans.

Concurrent copy-cat

Identities on games A are given by copy-cat strategies $\gamma_A : \mathbb{C}\mathbb{C}_A \to A^\perp \parallel A$ —strategies for player based on copying the latest moves made by opponent.

$\mathbb{C}\mathbb{C}_A$ has the same events, consistency and polarity as $A^\perp \parallel A$ but with causal dependency $\leq_{\mathbb{C}\mathbb{C}_A}$ given as the transitive closure of the relation

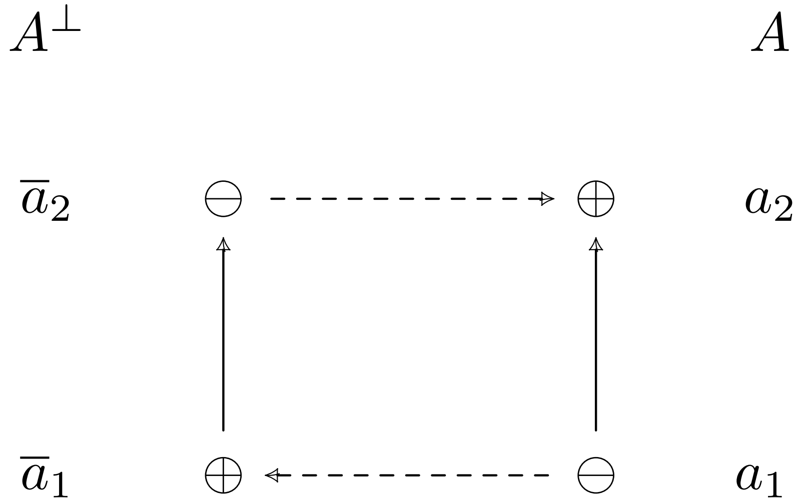
$$\leq_{A^\perp \parallel A} \cup \{(\overline{c}, c) \mid c \in A^\perp \parallel A \;\&\; pol_{A^\perp \parallel A}(c) = +\}$$

where $\overline{c} \leftrightarrow c$ is the natural correspondence between $A^\perp$ and A. The map γ_A is the identity on the common underlying set of events. Then,

$$x \in \mathcal{C}(\mathbb{C}\mathbb{C}_A) \text{ iff } x \in \mathcal{C}(A^\perp \parallel A) \;\&\; \forall c \in x.\; pol_{A^\perp \parallel A}(c) = + \;\Rightarrow\; \overline{c} \in x\,.$$

Copy-cat—an example

$\mathbb{C}\mathbb{C}_A$



Composing pre-strategies

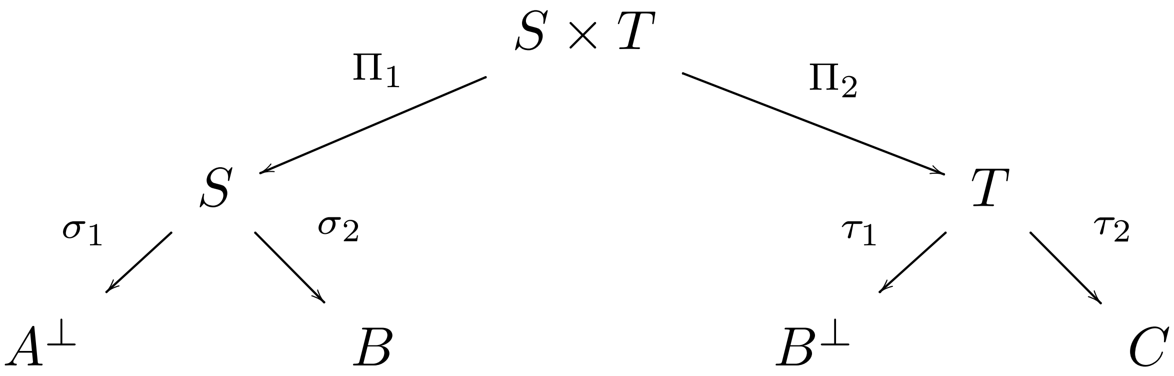
Two pre-strategies $\sigma : A \mathrel{+\!\!\!\!\!\rightarrow} B$ and $\tau : B \mathrel{+\!\!\!\!\!\rightarrow} C$ as spans:

$$
\begin{array}{ccccccc}
 & & S & & & & & T & & \\
 & \sigma_1 \swarrow & & \searrow \sigma_2 & & & \tau_1 \swarrow & & \searrow \tau_2 & \\
A^{\perp} & & & & B & B^{\perp} & & & & C\,.
\end{array}
$$

Their composition

$$
\begin{array}{ccccc}
 & & T \odot S & & \\
 & (\tau\odot\sigma)_1 \swarrow & & \searrow (\tau\odot\sigma)_2 & \\
A^{\perp} & & & & C
\end{array}
$$

where $T \odot S =_{\mathrm{def}} (S \times T \upharpoonright \textbf{Syn}) \downarrow \textbf{Vis}$ where ...

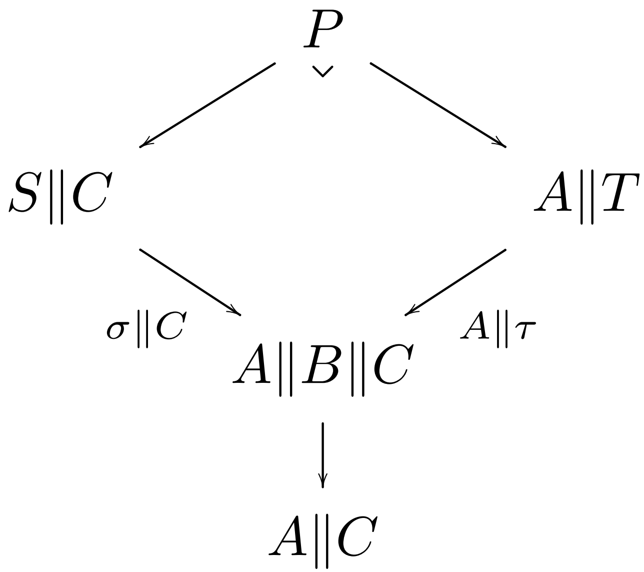


Their composition: $T \odot S =_{\mathrm{def}} (S \times T \upharpoonright \textbf{Syn}) \downarrow \textbf{Vis}$ where

$$
\begin{aligned}
\textbf{Syn} \;=\; & \{p \in S \times T \;\mid\; \sigma_1\Pi_1(p) \text{ is defined } \& \; \Pi_2(p) \text{ is undefined}\} \cup \\
& \{p \in S \times T \;\mid\; \sigma_2\Pi_1(p) = \overline{\tau_1\Pi_2(p)} \text{ with both defined}\} \cup \\
& \{p \in S \times T \;\mid\; \tau_2\Pi_2(p) \text{ is defined } \& \; \Pi_1(p) \text{ is undefined}\}\,, \\
\textbf{Vis} \;=\; & \{p \in S \times T \upharpoonright \textbf{Syn} \;\mid\; \sigma_1\Pi_1(p) \text{ is defined}\} \cup \\
& \{p \in S \times T \upharpoonright \textbf{Syn} \;\mid\; \tau_2\Pi_2(p) \text{ is defined}\}\,.
\end{aligned}
$$

Composition via pullback:

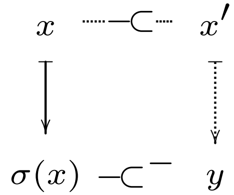
Ignoring polarities, the partial map



has the partial-total map factorization: $P \longrightarrow T \odot S \xrightarrow{\tau \odot \sigma} A \| C \,.$ $\quad$ *[N. Bowler]*

Theorem characterizing concurrent strategies

Receptivity $\sigma : S \to A^{\perp} \parallel B$ is *receptive* when $\sigma(x) \longrightarrow\!\!\!\!\subset^{-} y$ implies there is a *unique* $x' \in \mathcal{C}(S)$ such that $x \longrightarrow\!\!\!\!\subset x'$ & $\sigma(x') = y$.



Innocence $\sigma : S \to A^{\perp} \parallel B$ is *innocent* when it is

$+$-*Innocence:* If $s \rightarrowtriangle s'$ & $pol(s) = +$ then $\sigma(s) \rightarrowtriangle \sigma(s')$ and

$-$-*Innocence:* If $s \rightarrowtriangle s'$ & $pol(s') = -$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.

[$\rightarrowtriangle$ *stands for immediate causal dependency*]

Theorem Receptivity and innocence are necessary and sufficient for copy-cat to act as identity w.r.t. composition: $\sigma \odot \gamma_A \cong \sigma$ and $\gamma_B \odot \sigma \cong \sigma$ for all $\sigma : A \nrightarrow B$. *[Silvain Rideau, GW]*

Definition A *strategy* is a receptive, innocent pre-strategy.

$\rightsquigarrow$ A bicategory, $\mathbf{Games}$, whose

objects are event structures with polarity—the games,

arrows are strategies $\sigma : A \nrightarrow B$

2-cells are maps of spans.

The vertical composition of 2-cells is the usual composition of maps of spans. Horizontal composition is given by the composition of strategies $\odot$ (which extends to a functor on 2-cells via the functoriality of synchronized composition).

Strategies—alternative description 1

A strategy S in a game A comprises a total map of event structures with polarity $\sigma : S \to A$ such that
(i) whenever $\sigma x \subseteq^{-} y$ in $\mathcal{C}(A)$ there is a unique $x' \in \mathcal{C}(S)$ so that

$x \subseteq x'$ & $\sigma x' = y$, *i.e.*

$$\begin{array}{ccc} x & \subseteq & x' \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \sigma x & \subseteq^{-} & y\,, \end{array}$$

and
(ii) whenever $y \subseteq^{+} \sigma x$ in $\mathcal{C}(A)$ there is a (necessarily unique) $x' \in \mathcal{C}(S)$ so that

$x' \subseteq x$ & $\sigma x' = y$, *i.e.*

$$\begin{array}{ccc} x' & \subseteq & x \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ y & \subseteq^{+} & \sigma x\,. \end{array}$$

[$\rightsquigarrow$ *strategies as presheaves over "Scott order"* $\sqsubseteq =_{\mathrm{def}} \subseteq^{+} \circ \supseteq^{-}$.]

Strategies—alternative description 2

A strategy S in a game A comprises a total map of event structures with polarity $\sigma : S \to A$ such that

(i) $\sigma x \stackrel{a}{\longrightarrow\!\subset} \ \& \ pol_A(a) = - \Rightarrow \exists ! s \in S.\ x \stackrel{s}{\longrightarrow\!\subset} \ \& \ \sigma(s) = a$, for all $x \in \mathcal{C}(S)$, $a \in A$.

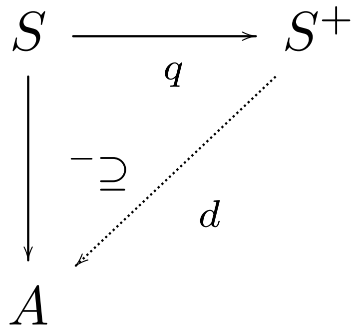
(ii)(+) If $x \stackrel{e}{\longrightarrow\!\subset} x_1 \stackrel{e'}{\longrightarrow\!\subset} \ \& \ pol_S(e) = +$ in $\mathcal{C}(S)$ and $\sigma x \stackrel{\sigma(e')}{\longrightarrow\!\subset}$ in $\mathcal{C}(A)$, then $x \stackrel{e'}{\longrightarrow\!\subset}$ in $\mathcal{C}(S)$.

(ii)(−) If $x \stackrel{e}{\longrightarrow\!\subset} x_1 \stackrel{e'}{\longrightarrow\!\subset} \ \& \ pol_S(e') = -$ in $\mathcal{C}(S)$ and $\sigma x \stackrel{\sigma(e')}{\longrightarrow\!\subset}$ in $\mathcal{C}(A)$, then $x \stackrel{e'}{\longrightarrow\!\subset}$ in $\mathcal{C}(S)$.

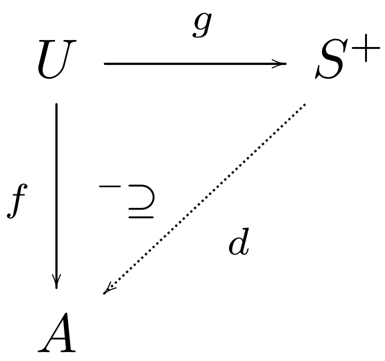
Notation $x \stackrel{e}{\longrightarrow\!\subset} y$ iff $x \cup \{e\} = y \ \& \ e \notin x$, for configurations x, y, event e.
$x \stackrel{e}{\longrightarrow\!\subset}$ iff $\exists y.\ x \stackrel{e}{\longrightarrow\!\subset} y$.

Strategies—alternative description 3, via just $+$-moves

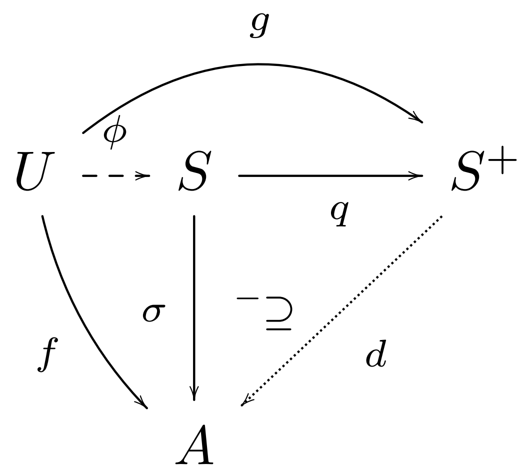
A strategy $\sigma : S \to A$ determines



where q is projection and $d : \mathcal{C}(S) \to \mathcal{C}(A)$ s.t. $d(x) = \sigma[x]$. Universal property showing d determines σ:



$\Rightarrow \exists! \phi$ s.t.



& $\sigma\phi = f$ & $q\phi = g$.

Deterministic strategies

Say an event structures with polarity S is *deterministic* iff

$$\forall X \subseteq_{\text{fin}} S.\ Neg[X] \in \text{Con}_S \Rightarrow X \in \text{Con}_S\,,$$

where $Neg[X] =_{\text{def}} \{s' \in S \mid \exists s \in X.\ pol_S(s') = - \ \&\ s' \leq s\}$.
Say a strategy $\sigma : S \to A$ is deterministic if S is deterministic.

Proposition An event structure with polarity S is deterministic iff $x \stackrel{s}{\longrightarrow\!\!\!\!\subset} \ \&\ x \stackrel{s'}{\longrightarrow\!\!\!\!\subset} \ \&\ pol_S(s) = +$ implies $x \cup \{s, s'\} \in \mathcal{C}(S)$, for all $x \in \mathcal{C}(S)$.

Notation $x \stackrel{e}{\longrightarrow\!\!\!\!\subset} y$ iff $x \cup \{e\} = y \ \&\ e \notin x\,,$ for configurations x, y, event e.
$x \stackrel{e}{\longrightarrow\!\!\!\!\subset}$ iff $\exists y.\ x \stackrel{e}{\longrightarrow\!\!\!\!\subset} y$.

Lemma Let A be an event structure with polarity. The copy-cat strategy γ_A is deterministic iff A satisfies

$$\forall x \in \mathcal{C}(A).\ x \stackrel{a}{\longrightarrow\!\!\subset} \ \&\ x \stackrel{a'}{\longrightarrow\!\!\subset} \ \&\ pol_A(a) = + \ \&\ pol_A(a') = - \\ \Rightarrow x \cup \{a, a'\} \in \mathcal{C}(A)\,. \qquad (\ddagger)$$

Lemma The composition $\tau \odot \sigma$ of two deterministic strategies σ and τ is deterministic.

Lemma A deterministic strategy $\sigma : S \to A$ is injective on configurations (equivalently, $\sigma : S \rightarrowtail A$).

$\rightsquigarrow$ sub-bicategory $\mathbf{DetGames}$, equivalent to an order-enriched category.

Related work

Ingenuous strategies Deterministic concurrent strategies coincide with the $receptive$ ingenuous strategies of and Melliès and Mimram.

Closure operators A deterministic strategy $\sigma : S \to A$ determines a closure operator φ on $\mathcal{C}^\infty(S)$: for $x \in \mathcal{C}^\infty(S)$,

$$\varphi(x) = x \cup \{s \in S \mid pol(s) = + \ \& \ Neg[\{s\}] \subseteq x\} .$$

The closure operator φ on $\mathcal{C}^\infty(S)$ induces a *partial* closure operator φ_p on $\mathcal{C}^\infty(A)$ and in turn a closure operator $\varphi_p^\top$ on $\mathcal{C}^\infty(A)^\top$ of Abramsky and Melliès.

Simple games *"Simple games"* of game semantics arise when we restrict $\mathbf{Games}$ to objects and deterministic strategies which are 'tree-like'—alternating polarities, with conflicting branches, beginning with opponent moves.

Stable spans, profunctors and stable functions The sub-bicategory of $\mathbf{Games}$ where the events of games are purely $+$ve is equivalent to the bicategory of stable spans:

$$\begin{array}{ccccc} & & S & & \\ & \sigma_1 \swarrow & & \searrow \sigma_2 & \\ A^{\perp} & & & & B \end{array} \quad \longleftrightarrow \quad \begin{array}{ccccc} & & S^{+} & & \\ & \sigma_1^{-} \swarrow & & \searrow \sigma_2^{+} & \\ A & & & & B \end{array},$$

where S^{+} is the projection of S to its $+$ve events; σ_2^{+} is the restriction of σ_2 to S^{+} is rigid; σ_2^{-} is a *demand map* taking $x \in \mathcal{C}(S^{+})$ to $\sigma_1^{-}(x) = \sigma_1[x]$.
Composition of stable spans coincides with composition of their associated profunctors.

When deterministic (and event structures are countable) we obtain a sub-bicategory equivalent to Berry's **dI-domains and stable functions**.

Winning conditions

A *game with winning conditions* comprises

$$G = (A, W)$$

where A is an event structure with polarity and $W \subseteq \mathcal{C}^\infty(A)$ consists of the *winning configurations* for Player.

Define the *losing conditions* to be $L =_{\mathrm{def}} \mathcal{C}^\infty(A) \setminus W$.
[$\textit{Can generalize to winning, losing and neutral conditions.}$]

Winning strategies

Let $G = (A, W)$ be a game with winning conditions.

A strategy in G is a strategy in A.

A strategy $\sigma : S \to A$ in G is *winning (for Player)* if $\sigma x \in W$, for all $+$-maximal configurations $x \in \mathcal{C}^\infty(S)$.
[A configuration x is $+$-maximal if whenever $x \stackrel{s}{\longrightarrow\!\!\subset}$ then the event s has $-$ve polarity.]

A winning strategy prescribes moves for Player to avoid ending in a losing configuration, no matter what the activity or inactivity of Opponent.

Characterization via counter-strategies

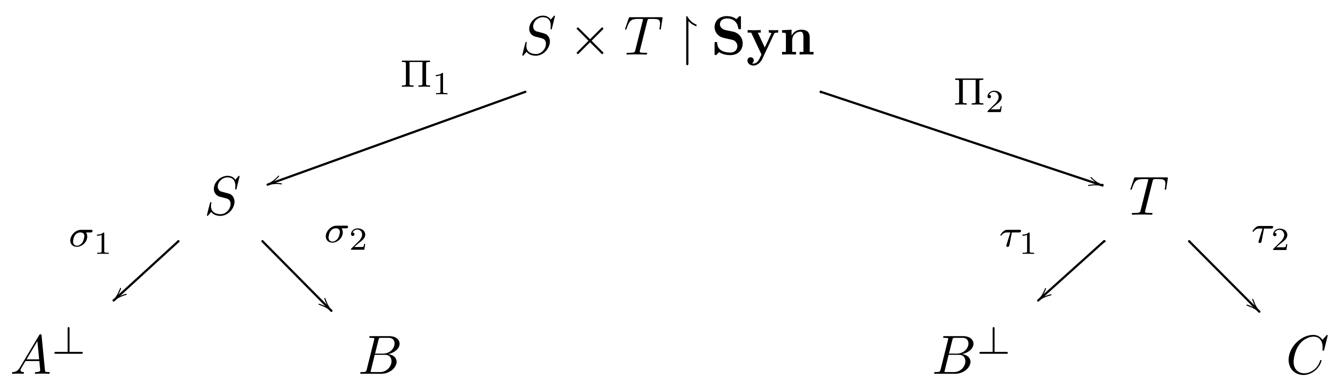
Informally, a strategy is winning for Player if any play against a counter-strategy *of Opponent* results in *a win for Player.*

A *counter-strategy*, *i.e.* a strategy of Opponent, in a game A is a strategy in the dual game, so $\tau : T \rightarrow A^{\perp}$.

What are the *results* $\langle \sigma, \tau \rangle$ of playing strategy σ against counter-strategy τ?

Note $\sigma : \emptyset \nrightarrow A$ and $\tau : A \nrightarrow \emptyset$ ...

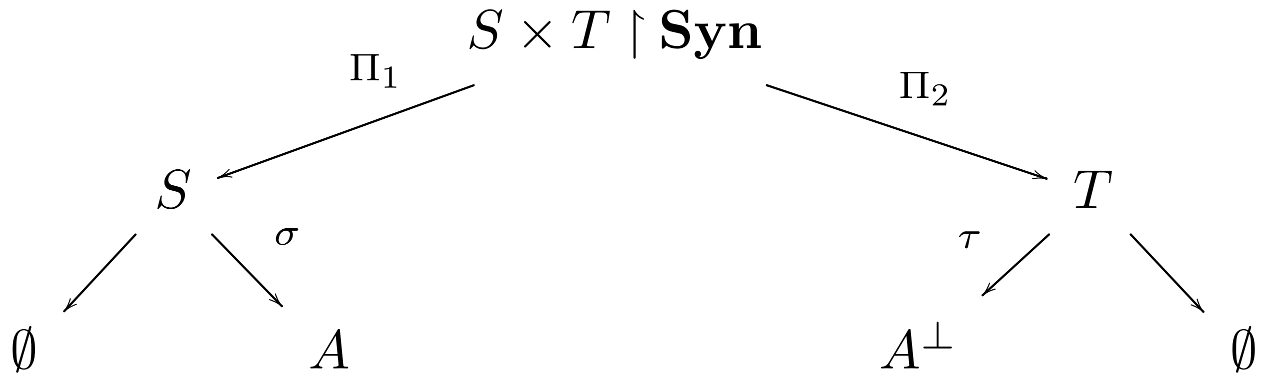
Composition of pre-strategies without hiding



where

$$\begin{aligned}\textbf{Syn} \;=\; &\{p \in S \times T \;\mid\; \sigma_1\Pi_1(p) \text{ is defined } \& \ \Pi_2(p) \text{ is undefined}\} \cup \\ &\{p \in S \times T \;\mid\; \sigma_2\Pi_1(p) = \overline{\tau_1\Pi_2(p)} \text{ with both defined}\} \cup \\ &\{p \in S \times T \;\mid\; \tau_2\Pi_2(p) \text{ is defined } \& \ \Pi_1(p) \text{ is undefined}\}\,.\end{aligned}$$

Special case



where

$$\textbf{Syn} \;=\; \{p \in S \times T \;\mid\; \sigma\Pi_1(p) = \overline{\tau\Pi_2(p)} \text{ with both defined}\}\,.$$

Define **results**, $\langle \sigma, \tau \rangle =_{\mathrm{def}} \{\sigma\Pi_1 z \;\mid\; z \text{ is maximal in } \mathcal{C}^{\infty}(S \times T \restriction \mathbf{Syn})\}\,.$

Characterization of winning strategies

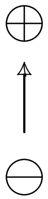
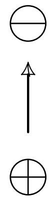
Lemma Let $\sigma : S \to A$ be a strategy in a game (A, W). The strategy σ is a winning for Player iff $\langle \sigma, \tau \rangle \subseteq W$ for all (deterministic) strategies $\tau : T \to A^\perp$.

Its proof uses a key lemma:

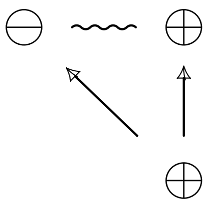
Lemma Let $\sigma : S \to A^\perp \| B$ and $\tau : B^\perp \| C$ be receptive pre-strategies. Then,

$$z \in \mathcal{C}^\infty(S \times T \restriction \mathbf{Syn}) \text{ is } +\text{-maximal iff}$$
$$\Pi_1 z \in \mathcal{C}^\infty(S) \text{ is } +\text{-maximal } \& \ \Pi_2 z \in \mathcal{C}^\infty(T) \text{ is } +\text{-maximal.}$$

Examples

 with $W = \{\emptyset, \{\ominus, \oplus\}\}$ has a winning strategy.  , $W = \{\{\oplus\}\}$ has not.

$\ominus \rightsquigarrow \oplus$ has a winning strategy only if W comprises all configurations.

 the empty strategy is winning if $\emptyset \in W$.

Operations on games with winning conditions

Dual $G^\perp = (A^\perp, W_{G^\perp})$ where, for $x \in \mathcal{C}^\infty(A)$,

$$x \in W_{G^\perp} \text{ iff } \overline{x} \notin W_G \,.$$

Parallel composition For $G = (A, W_G)$, $H = (B, W_H)$,

$$G \| H =_{\text{def}} (A \| B,\ W_G \| \mathcal{C}^\infty(B) \cup \mathcal{C}^\infty(A) \| W_H)$$

where $X \| Y = \{\{1\} \times x \cup \{2\} \times y \mid x \in X \ \& \ y \in Y\}$ when X and Y are subsets of configurations. To win is to win in either game. Unit of $\|$ is $(\emptyset, \emptyset)$.

Derived operations

Tensor Defining $G \otimes H =_{\text{def}} (G^{\perp} \| H^{\perp})^{\perp}$ we obtain a game where to win is to win in both games G and H—so to lose is to lose in either game. More explicitly,

$$(A, W_A) \otimes (B, W_B) =_{\text{def}} (A \| B,\ W_A \| W_B)\,.$$

The unit of $\otimes$ is $(\emptyset, \{\emptyset\})$.

Function space With $G \multimap H =_{\text{def}} G^{\perp} \| H$ a win in $G \multimap H$ is a win in H conditional on a win in G:

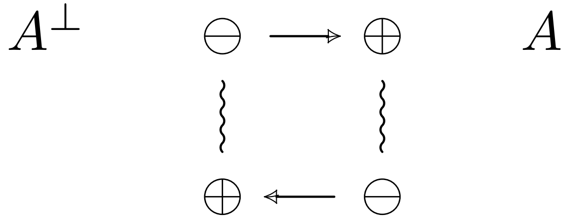
Proposition Let $G = (A, W_G)$ and $H = (B, W_H)$ be games with winning conditions. Write $W_{G \multimap H}$ for the winning conditions of $G \multimap H$. For $x \in \mathcal{C}^{\infty}(A^{\perp} \| B)$,

$$x \in W_{G \multimap H} \quad \text{iff} \quad \overline{x_1} \in W_G \Rightarrow x_2 \in W_H\,.$$

The bicategory of winning strategies

Lemma Let σ be a winning strategy in $G \multimap H$ and τ be a winning strategy in $H \multimap K$. Their composition $\tau \odot \sigma$ is a winning strategy in $G \multimap K$.

But copy-cat need not be winning: Let A consist of $\oplus \rightsquigarrow \ominus$. The event structure $\mathbb{C}_A$:



Taking $x = \{\ominus, \ominus\}$ makes x $+$-maximal, but $\overline{x}_1 \in W$ while $x_2 \notin W$.

A robust sufficient condition for copy-cat to be winning: copy-cat is deterministic.
[The Aarhus lecture notes give a necessary and sufficient condition.]
$\rightsquigarrow$ bicategory of games with winning strategies.

Two applications

Total strategies: To pick out a subcategory of *total* strategies (where Player can always answer Opponent) within simple games.

Determinacy of concurrent games: A necessary condition on a game A for (A, W) to be determined for all winning conditions W: that copy-cat γ_A is deterministic. Not sufficient to ensure determinacy w.r.t. all Borel winning conditions. Think sufficient for determinacy if winning conditions W are $closed$ w.r.t. local Scott topology, and in particular for finite games $[sketchy\ proof]$.

There must be many more!

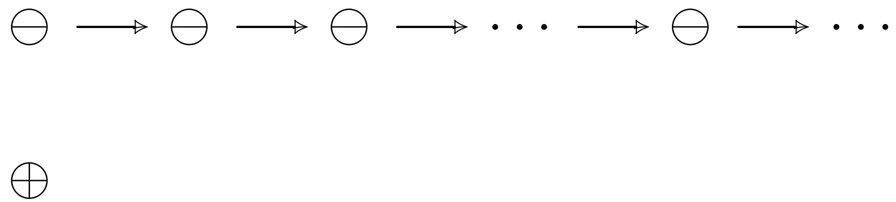
Aarhus Lecture notes: http://daimi.au.dk/∼gwinskel/

A next step: $back\text{-}tracking$ in games via "copying" monads in event structures with symmetry.

Counterexamples to Borel determinacy

(1) $\ominus \rightsquigarrow \oplus$ with $W = \{\{\oplus\}\}$, copy-cat is nondeterministic.

(2)



where Player wins iff

Opponent plays finite no. of $\ominus$ moves and Player does nothing
or
Opponent plays all $\ominus$ moves and Player the single $\oplus$ move.