Exercises for Chapter 3 of
An Introduction to Description Logic

June 4, 2017

Exercise 1 Prove the following claim:

If the $\mathcal{ALC}$ concept C is satisfiable w.r.t. the $\mathcal{ALC}$ TBox $\mathcal{T}$, then for all $n \geq 1$ there is a finite model $\mathcal{I}_n$ of $\mathcal{T}$ such that $|C^{\mathcal{I}_n}| \geq n$.
Would the claim still be true if $|C^{\mathcal{I}_n}| = n$ were required?

Exercise 2 Show that the extension of $\mathcal{ALC}$ with self restrictions (see Table A.1 on page 229) is more expressive than $\mathcal{ALC}$.

Exercise 3 Define a notion of bisimulation that is appropriate for the DL $\mathcal{ALCN}$ in the sense that elements bisimular w.r.t. this notion behave the same w.r.t. all $\mathcal{ALCN}$ concepts. Use this to show that $\mathcal{ALCQ}$ is more expressive than $\mathcal{ALCN}$.

Exercise 4 Show that $\mathcal{ALCN}$ is closed under disjoint union.

Exercise 5 Show that bisimulations are closed under union and composition, but not under intersection.

Exercise 6 In this exercise we consider bisimulations between an interpretation $\mathcal{I}$ and itself. We call these bisimulations on $\mathcal{I}$. For two elements d_1, d_2 of $\Delta^{\mathcal{I}}$ we write $d_1 \approx_{\mathcal{I}} d_2$ if they are bisimular, i.e., if there is a bisimulation ρ on $\mathcal{I}$ such that $(\mathcal{I}, d_1)\ \rho\ (\mathcal{I}, d_2)$.

1. Show that the relation $\approx_{\mathcal{I}}$ is an equivalence relation on $\Delta^{\mathcal{I}}$.
2. Describe $\approx_{\mathcal{J}}$ for the tree model $\mathcal{J}$ in Figure 3.5 on page 63.
3. Show that there is a bisimulation ρ on $\mathcal{I}$ such that $\rho = \approx_{\mathcal{I}}$.
4. Show that, for finite interpretations $\mathcal{I}$, the relation $\approx_{\mathcal{I}}$ can be computed in time polynomial in the cardinality of $\Delta^{\mathcal{I}}$.

Exercise 7 Consider an interpretation $\mathcal{I}$ and the equivalence relation $\approx_{\mathcal{I}}$ on $\Delta^{\mathcal{I}}$ as introduced in Exercise 6. We denote the $\approx_{\mathcal{I}}$-equivalence class of $d \in \Delta^{\mathcal{I}}$

by $[d]_{\approx_\mathcal{I}}$. Now use $\approx_\mathcal{I}$ in place of $\simeq_S$ in the construction of the filtration $\mathcal{J}$ in Definition 3.14.

Show that the relation

$$\rho = \{(d, [d]_{\approx_\mathcal{I}}) \mid d \in \Delta^\mathcal{I}\}$$

is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$. Show that this implies the following: if $\mathcal{I}$ is a model of the $\mathcal{ALC}$ concept C w.r.t. the $\mathcal{ALC}$ TBox $\mathcal{T}$, then so is $\mathcal{J}$. Why can we not derive the finite model property of $\mathcal{ALC}$ from this fact?

Exercise 8 In this exercise, we consider simulations, which are "one-sided" variants of bisimulations. Given interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$, the relation $\sigma \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is a *simulation* between $\mathcal{I}_1$ and $\mathcal{I}_2$ if

- $d_1 \ \sigma \ d_2$ implies
$$\text{if } d_1 \in A^{\mathcal{I}_1} \text{ then } d_2 \in A^{\mathcal{I}_2}$$
for all $d_1 \in \Delta^{\mathcal{I}_1}$, $d_2 \in \Delta^{\mathcal{I}_2}$, and $A \in \mathbf{C}$;
- $d_1 \ \sigma \ d_2$ and $(d_1, d_1') \in r^{\mathcal{I}_1}$ implies the existence of $d_2' \in \Delta^{\mathcal{I}_2}$ such that
$$d_1' \ \sigma \ d_2' \text{ and } (d_2, d_2') \in r^{\mathcal{I}_2}$$
for all $d_1, d_1' \in \Delta^{\mathcal{I}_1}$, $d_2 \in \Delta^{\mathcal{I}_2}$, and $r \in \mathbf{R}$.

We define

$$(\mathcal{I}_1, d_1) \overset{\rightharpoonup}{\sim} (\mathcal{I}_2, d_2) \quad \text{if} \quad \text{there is a simulation } \sigma \text{ between } \mathcal{I}_1 \text{ and } \mathcal{I}_2 \text{ such that } d_1 \ \sigma \ d_2.$$

1. Prove that
   $(\mathcal{I}_1, d_1) \sim (\mathcal{I}_2, d_2)$ implies $(\mathcal{I}_1, d_1) \overset{\rightharpoonup}{\sim} (\mathcal{I}_2, d_2)$ and $(\mathcal{I}_2, d_2) \overset{\rightharpoonup}{\sim} (\mathcal{I}_1, d_1)$.
2. Is the converse of the implication in 1. also true?
3. Consider the DL $\mathcal{EL}$, which has the constructors conjunction, existential restriction, and top (see Chapter 6), and show that for $\mathcal{EL}$ concepts, the following one-sided variant of Theorem 3.2 holds:
   If $(\mathcal{I}_1, d_1) \overset{\rightharpoonup}{\sim} (\mathcal{I}_2, d_2)$, then the following holds for all $\mathcal{EL}$ concepts C:
   $$\text{If } d_1 \in C^{\mathcal{I}_1} \text{ then } d_2 \in C^{\mathcal{I}_2}.$$
4. Which of the constructors disjunction, negation, or value restriction can be added to $\mathcal{EL}$ without losing the property shown in 3.
5. Show that $\mathcal{ALC}$ is more expressive than $\mathcal{EL}$.
6. Show that $\mathcal{ELI}$, the extension of $\mathcal{EL}$ with inverse roles, is more expressive than $\mathcal{EL}$.

Exercise 9 Apply filtration to the tree model $\mathcal{J}$ in Figure 3.5 on page 63.

Exercise 10 Show that the exponential bound given in Theorem 3.16 cannot be improved on. For this purpose, define a sequence $(\mathcal{T}_n, C_n)_{n\geq 1}$ of $\mathcal{ALC}$ TBoxes $\mathcal{T}_n$ and concepts C_n such that

1. the sizes of $\mathcal{T}_n$ and C_n are polynomial in n; and
2. no model of C_n with respect to $\mathcal{T}_n$ can contain less than 2^n elements.

Exercise 11 Show that $\mathcal{ALCN}$ and $\mathcal{ALCI}$ have the bounded model property.

Exercise 12 Show that an $\mathcal{ALC}$ concept C of size n has a tree model (with respect to the empty TBox) of depth at most n, where the depth of a tree model is the length of the longest path from the root to a leaf of the corresponding tree.

Exercise 13 Consider the TBox

$$\mathcal{T} = \{A \sqsubseteq \exists r.A, B \sqsubseteq \exists s.B, \neg A \sqsubseteq \forall s.A, \neg B \sqsubseteq \forall r.B\}$$

and the concept $C = A \sqcap B$.

1. Construct two tree models of C with respect to $\mathcal{T}$ such that the underlying trees are not isomorphic.
2. Apply filtration to these two tree models.
3. What is the minimal cardinality of a model of C with respect to $\mathcal{T}$.

Exercise 14 The tree model property of $\mathcal{ALC}$ also implies the *connected model property*, i.e., if the $\mathcal{ALC}$ concept C is satisfiable with respect to the $\mathcal{ALC}$ TBox $\mathcal{T}$, then its is satisfiable in a connected model, i.e., in a model $\mathcal{I}$ of $\mathcal{T}$ that contains an element d such that $d \in C^{\mathcal{I}}$ and every element of $\mathcal{I}$ is reachable via a role path from d.

Show the connected model property of $\mathcal{ALC}$ without using the tree model property or unravelling.