Exercises for Chapter 5 of *An Introduction to Description Logic*

June 9, 2017

Exercise 1 Complete the proof of Proposition 5.1 (conversion of acyclic TBoxes into NNF) by establishing correctness of the transformation stated there.

Exercise 2

1. Let $\mathcal{T}$ be an acyclic TBox in NNF, let $\mathcal{T}^{\sqsubseteq}$ be obtained from $\mathcal{T}$ by replacing each definition $A \equiv C \in \mathcal{T}$ with $A \sqsubseteq C$, and let A_0 be a concept name. Prove that A_0 is satisfiable w.r.t. $\mathcal{T}$ iff A_0 is satisfiable w.r.t. $\mathcal{T}^{\sqsubseteq}$.

   Hint: Show how to extend a model of $\mathcal{T}^{\sqsubseteq}$ to a model of $\mathcal{T}$ by making the extension of defined concept names larger, proceeding along the "$\prec$" order defined in the proof of Lemma 5.4.

2. Find an example which shows that the above need not be true when $\mathcal{T}$ is not in NNF.

Exercise 3 Transform the following acyclic TBox into simple form, following exactly the steps used in Section 5.1.1:

$$
\begin{aligned}
\mathsf{Student} &\sqsubseteq \mathsf{Person} \sqcap \exists \mathit{attends}.\mathsf{Course} \\
\mathsf{Person} &\sqsubseteq \mathsf{Male} \sqcup \mathsf{Female} \\
\mathsf{Male} &\sqsubseteq \neg \mathsf{Female}
\end{aligned}
$$

Exercise 4 Use the $\mathcal{ALC}$-worlds algorithm to decide satisfiability of the concept name A_0 w.r.t. the following acyclic TBox:

$$
\begin{array}{lllllllll}
A_0 & \equiv & A_1 \sqcap A_2 & & & & & & \\
A_1 & \equiv & \exists r.A_3 & A_3 & \equiv & P & & & \\
A_2 & \equiv & A_4 \sqcap A_5 & & & & & & \\
A_4 & \equiv & \exists r.A_6 & A_6 & \equiv & Q & & & \\
A_5 & \equiv & A_6 \sqcap A_7 & & & & & & \\
A_6 & \equiv & \forall r.A_1 & & & & & & \\
A_7 & \equiv & \forall r.A_8 & A_8 & \equiv & \forall r.A_9 & A_9 & \equiv & \neg P
\end{array}
$$

Draw the recursion tree of a successful run and of an unsuccessful run. Does the algorithm return a positive or a negative result on this input?

Exercise 5 Determine whether or not Player 1 has a winning strategy in the following finite Boolean games, where in both cases $\Gamma_1 = \{p_1, p_3\}$ and $\Gamma_2 = \{p_2, p_4\}$:

1. $((p_1 \wedge p_3) \rightarrow \neg p_2) \wedge (\neg p_1 \rightarrow p_1) \wedge (\neg p_2 \rightarrow (p_3 \vee p_4))$
2. $(p_1 \vee \neg p_2) \wedge (p_2 \vee p_3) \wedge (\neg p_3 \vee \neg p_4) \wedge (\neg p_1 \vee \neg p_2 \vee p_3 \vee p_4)$

Exercise 6 Use the $\mathcal{ALC}$-elim algorithm to decide satisfiability of

1. the concept name A w.r.t. $\mathcal{T} = \{A \sqsubseteq \exists r.A, \top \sqsubseteq A, \forall r.A \sqsubseteq \exists r.A\}$
2. the concept $\forall r.\forall r.\neg B$ w.r.t. $\mathcal{T} = \{\neg A \sqsubseteq B, A \sqsubseteq \neg B, \top \sqsubseteq \neg\forall r.A\}$

Give the constructed type sequence $\Gamma_0, \Gamma_1, \ldots$. In case of satisfiability, also give the satisfying model constructed in the proof of Lemma 5.10.

Exercise 7 Extend the $\mathcal{ALC}$-elim algorithm to the description logics $\mathcal{ALCH}$ and $\mathcal{ALCI}$. Prove the correctness of the extended algorithms.

Exercise 8 Prove that, in the extension $\mathcal{ALC}_{R^+}$ of $\mathcal{ALC}$ with transitive roles, concept satisfiability w.r.t. TBoxes is decidable. Use a polynomial time reduction to the same problem in $\mathcal{ALC}$. Proceed as follows:

- given an $\mathcal{ALC}_{R^+}$ TBox $\mathcal{T}$, assume w.l.o.g. that $\mathcal{T}$ contains only a single GCI (as well as role transitivity axioms), and that this GCI is of the form $\top \sqsubseteq C_{\mathcal{T}}$;
- consider the $\mathcal{ALC}$ TBox $\mathcal{T}'$ that contains $\top \sqsubseteq C_{\mathcal{T}}$ and, for each $\mathsf{Trans}(r) \in \mathcal{T}$ and each subconcept of (a concept in) $\mathcal{T}$ of the form $\forall r.C$, the additional GCI $\forall r.C \sqsubseteq \forall r.\forall r.C$;
- prove that for all concept names A, the following is true: A is satisfiable w.r.t. $\mathcal{T}$ iff A is satisfiable w.r.t. $\mathcal{T}'$.

Exercise 9 Consider the following infinite Boolean games and determine whether Player 2 has a winning strategy. If this is the case, give the strategy. Otherwise describe how Player 1 has to play to win. In both games, the initial configuration t_0 assigns false to all variables.

1. $\varphi = (p_1 \wedge p_2 \wedge \neg q_1) \vee (p_3 \wedge p_4 \wedge \neg q_2) \vee (\neg(p_1 \vee p_4) \wedge q_1 \wedge q_2)$, $\Gamma_1 = \{p_1, \ldots, p_4\}$, $\Gamma_2 = \{q_1, q_2\}$
2. $\varphi = ((p_1 \leftrightarrow \neg q_1) \wedge (p_2 \leftrightarrow \neg q_2) \wedge (p_1 \leftrightarrow p_2)) \vee ((p_1 \leftrightarrow q_1) \wedge (p_2 \leftrightarrow q_2) \wedge (p_1 \leftrightarrow \neg p_2))$, $\Gamma_1 = \{p_1, p_2\}$, $\Gamma_2 = \{q_1, q_2\}$

Exercise 10 Are the following variations of infinite Boolean games also ExpTime-hard?

1. Player 1 wins if the constructed truth assignment falsifies the formula, instead of satisfying it;
2. Player 2 starts instead of Player 1;
3. The variables are not assigned to a specific player; instead, the active player can choose any variable and assign it a truth value; variables can be chosen multiple times.

Exercise 11 Prove Lemma 5.12 (correctness of the reduction of the existence of winning strategies in infinite Boolean games to concept satisfiability in $\mathcal{ALC}$ w.r.t. general TBoxes).

Exercise 12 The *universal role* is a role name u whose interpretation is fixed as $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ in any interpretation $\mathcal{I}$. Let $\mathcal{ALC}^u$ denote the extension of $\mathcal{ALC}$ with the universal role. Show that concept satisfiability in $\mathcal{ALC}^u$ is ExpTime-complete (without TBoxes).

Hint: for the upper bound, find a polynomial-time reduction to concept satisfiability in $\mathcal{ALCOI}$; for the lower bound, reduce concept satisfiability in $\mathcal{ALC}$ w.r.t. general TBoxes.

Exercise 13 Let a *cardinality axiom* be of the form $(C \geq n)$ or $(C \leq n)$ where C is a concept and $n \geq 0$. An interpretation $\mathcal{I}$ satisfies $(C \geq n)$ if the cardinality of $C^{\mathcal{I}}$ is at least n, and likewise for $(C \leq n)$. Determine the complexity of concept satisfiability w.r.t. general TBoxes in $\mathcal{ALC}$ with cardinality axioms.

Exercise 14 For $i \geq 0$, let $\mathcal{ALCOIQ}_i$ denote the fragment of $\mathcal{ALCOIQ}$ in which only i nominals are available. Prove that there is a polynomial time reduction from the satisfiability of $\mathcal{ALCOIQ}$ concepts to the satisfiability of $\mathcal{ALCOIQ}_1$ concepts (both without TBoxes).

Exercise 15 Complete the proof of Proposition 5.22 (undecidability of $\mathcal{ALC}$ with local role value maps) by showing that D is satisfiable if and only if C is satisfiable.

Exercise 16 Construct a two-register machine that, started with register contents n and m, terminates after finitely many steps with the first register containing $n + m$. Is it also easy to multiply two numbers with only two registers?

Exercise 17 Find out for which of the following concrete domains D_i, concept satisfiability w.r.t. general TBoxes is decidable:

1. Δ^{D_1} is the set of all non-negative numbers and Φ^{D_1} contains a unary predicate $=_k$ for each $k \in \Delta^{\mathsf{D}_1}$ with $(=_k)^{\mathsf{D}_1} = \{k\}$.

2. Δ^{D_2} is the set of all finite words over the alphabet $\{a, b\}$ and Φ^{D_2} contains the ternary concatenation predicate $\circ$ and the binary equality predicate $=$, with the obvious extensions. In particular, $\circ^{\mathsf{D}_2} = \{(u, v, w) \mid w = u \circ v\}$.

Exercise 18 If you feel brave, prove undecidability of $\mathcal{ALC}$ with global role value maps by reducing the halting problem for two-register machines instead of the tiling problem.