

Thu 13 May

Jacob.Biamonte@comlab.ox.ac.uk

<http://www.comlab.ox.ac.uk/activities/quantum/course/>

- Review
- bra ket duality
- Matrix Rep of States
 - circuit theory
 - ex GHZ circuit
- SVD
 - ex purification
 - state prep
- Bra ket duality w-class circuit

~ Review:

$$H = X_1 X_2$$

(example from last time)

$$U(t) = e^{-it X_1 X_2}$$

$$(X_1 X_2)^2 = 1$$

$$= 1 \cos t - i X_1 X_2 \sin t \quad t = \pi/4$$

$$U_{\pi/4} = (1 - i X_1 X_2) \frac{1}{\sqrt{2}}$$

$$\phi^- = |01\rangle - |10\rangle$$

consider $Z'_1 = e^{-it |1X|} = e^{-it/2 (1 - Z_1)}$

$U_{\pi/4} |01\rangle \rightarrow |01\rangle - i |10\rangle$
↑
remove

$$\phi^- = \underbrace{e^{-i\pi/2 (1 - Z_1)}}_{u_1} \circ \underbrace{e^{-i\pi/4 X_1 X_2}}_{u_2} |01\rangle$$

$$= |01\rangle - |10\rangle$$

$$u_1 u_2 |01\rangle = \phi^-$$

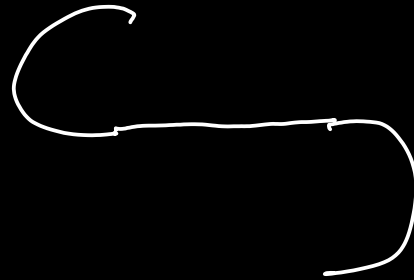
$$|01\rangle - |10\rangle = Z_1 X_1 (|00\rangle + |11\rangle)$$

$$Z_1 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad X_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$M = \sum_i |i\rangle\langle i|$$

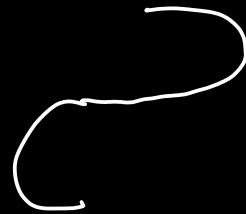
$$E = \sum_i \langle i| \langle i|$$

$$E_{13} M_{12}$$



$$= \text{wire}$$

$$\sum_i S_{ii} =$$

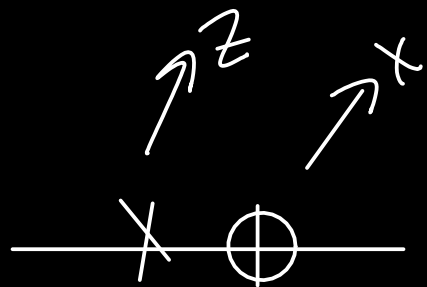


$$= \text{wire}$$

Exercise: compose M, E to get

$\sum_i S_{ii}$, and the identity matrix (both ways)

$$\text{cup} \approx \text{cap} \approx \text{wire}$$



$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \xrightarrow{\text{diagonalise}} \phi^- \quad \left(\begin{array}{l} \text{polarisation} \\ \text{vectors} \end{array} \right)$$

$$\underline{\sigma} = (x, y, z)$$

$$\underline{P} = (P_1, P_2, P_3)$$

$$\underline{\sigma} \cdot \underline{P}$$

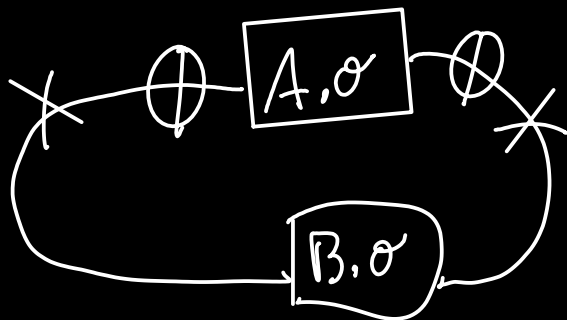
$$|\underline{P}| = 1$$

projection
of spin onto $\underline{P}$

$$\rho = \frac{1}{2} (1 + \underset{\uparrow}{\underline{P}} \cdot \underline{\sigma})$$

$$\langle \underline{A} \cdot \underline{\sigma} \otimes \underline{B} \cdot \underline{\sigma} \rangle_{\phi^-}$$

Evaluate this expectation
value (assisted w/ diagrams)



$$Z X (X, Y, Z) X Z \xrightarrow{\sigma}$$

$$\sigma_i \bar{\sigma}_i; \sigma_i = -\sigma_i$$

$$i \neq j$$

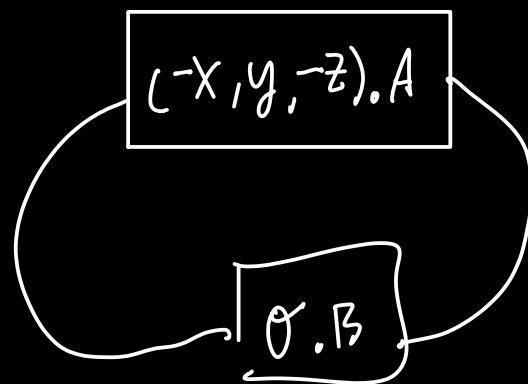
$$i = j \rightarrow \bar{\sigma}_i$$

$\downarrow \$X$

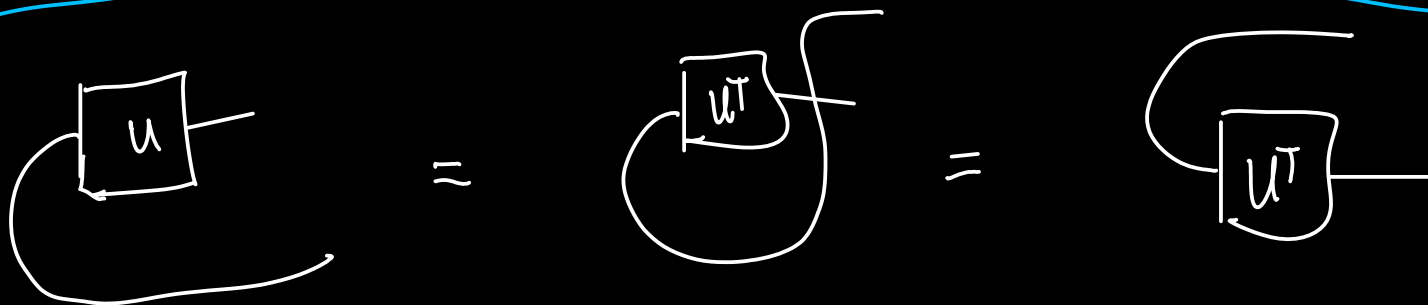
$$Z (X, -Y, -Z) Z$$

$\downarrow \$Z$

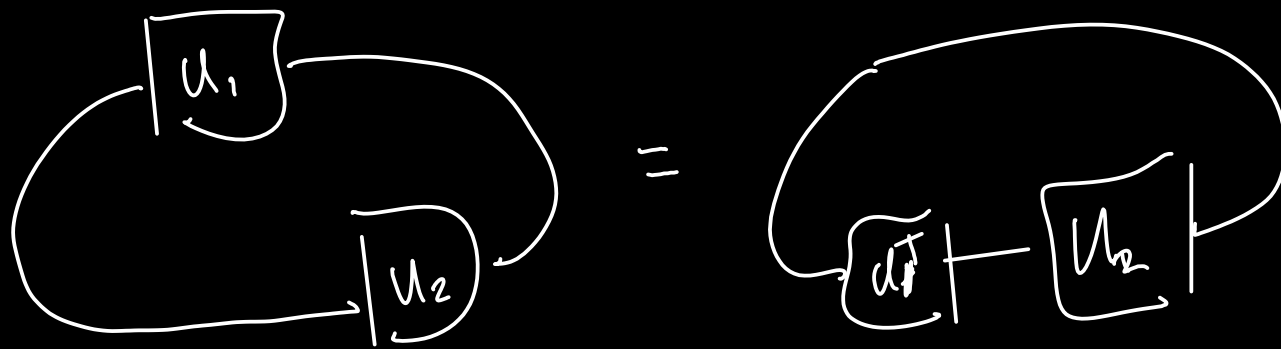
$$(-X, Y, -Z)$$



Recall that:



Bending wires to take transpose!



$$(-x, y, -z) \cdot A$$

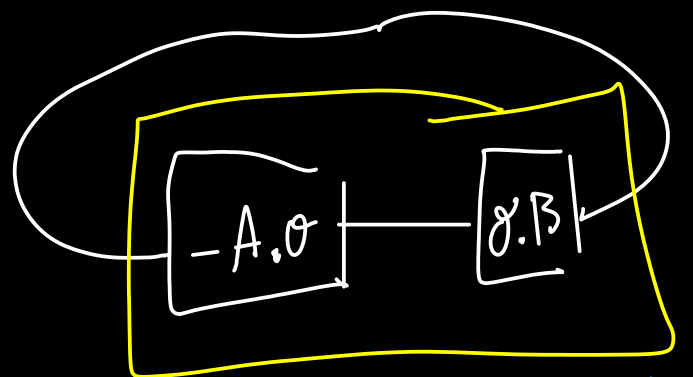
$\downarrow^T$ *transpose*

$$(-x, -y, -z) \cdot A$$

$$y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$\downarrow$

$$-y$$



evaluate this product.

$$(\sigma.A)(\sigma.B) = \underline{A} \cdot \underline{B} +$$

$$i \underline{\sigma} \cdot (\underline{A} \wedge \underline{B})$$

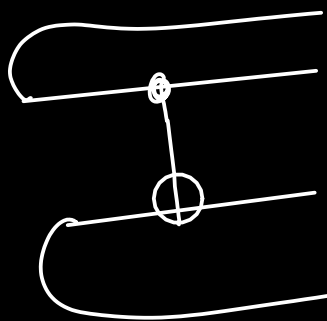
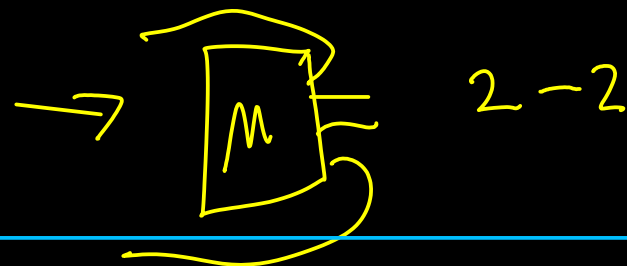
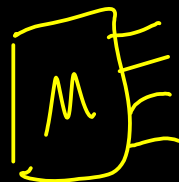
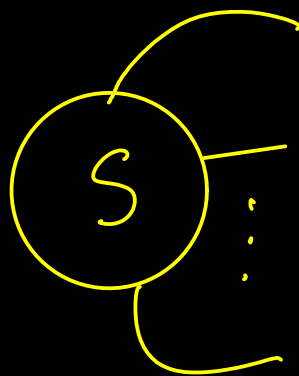
$$-\underline{A} \cdot \underline{B} = -\cos \theta$$

↑
angle between $\underline{A}, \underline{B}$!

$$\sigma_i = 0 \quad \text{show this using } \epsilon, n$$

$|00\rangle + |11\rangle$

— Part I of today



CNOT:

$$|00\rangle \otimes |00\rangle + |01\rangle \otimes |01\rangle + |10\rangle \otimes |11\rangle + |11\rangle \otimes |10\rangle$$

← what state does this diagram represent?

- II

$$|GHZ\rangle = |000\rangle + |111\rangle$$

↙ *braket-duality*

$$|0\cancel{0}0\rangle + |1\cancel{1}1\rangle$$

$$|0\cancel{0}0\rangle + |1\cancel{1}1\rangle$$

$$GHZ^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$GHZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$|w\rangle = |0\underline{0}1\rangle + |1\underline{0}0\rangle + |0\underline{1}0\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$

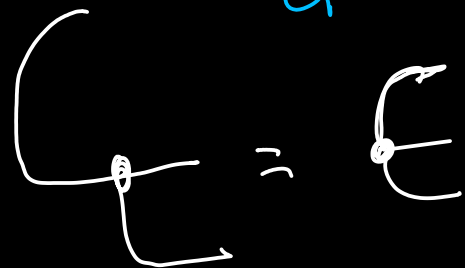
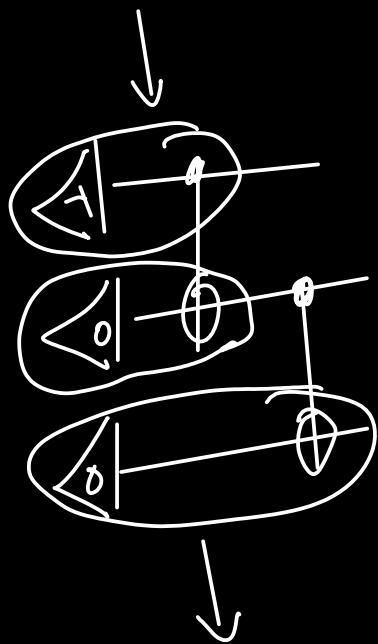
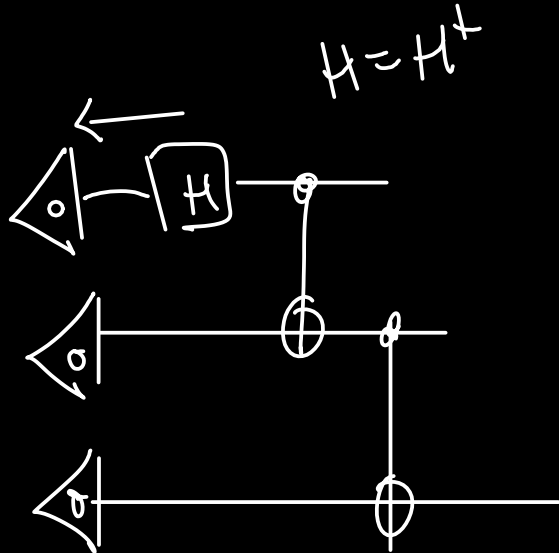
$$\underline{1} \quad |0\rangle \quad |1\rangle \quad \underline{0}$$

$$\begin{pmatrix} \langle \underline{0} | \\ \langle \underline{1} | \\ \langle \underline{2} | \\ \langle \underline{3} | \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|0\rangle | \underline{1} \rangle +$$

$$|1\rangle | \underline{0} \rangle + |0\rangle | \underline{2} \rangle$$

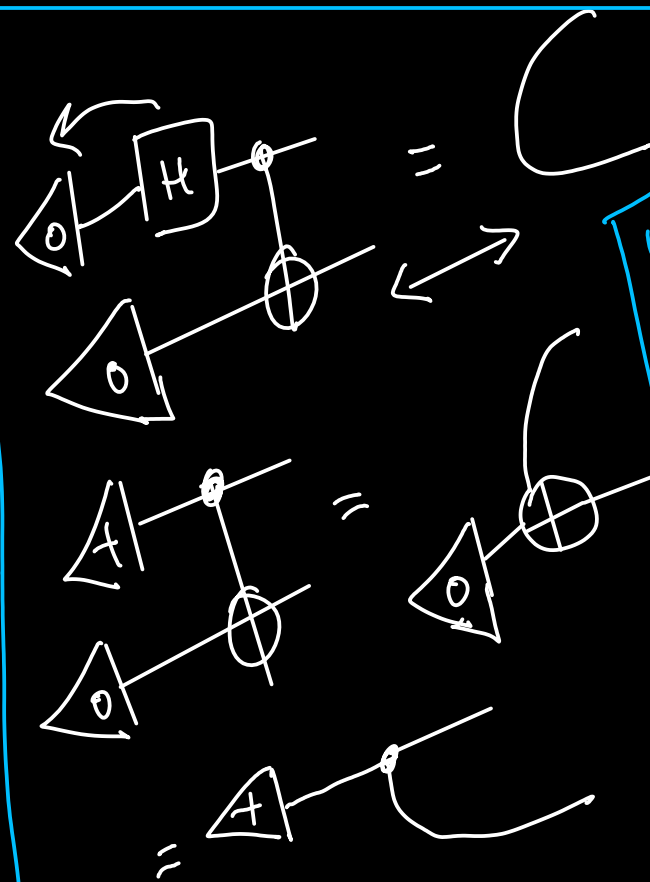
$$\mathbb{C}^2 \otimes \mathbb{C}^4$$



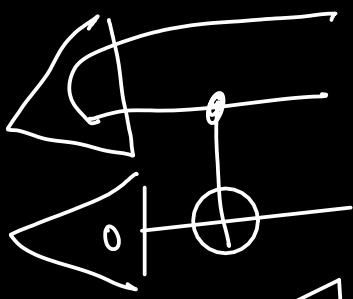
GHZ-state

$$H = \frac{1}{\sqrt{2}} (1, 0, 1) (X, Y, Z)$$

$$= \frac{1}{\sqrt{2}} (X + Z)$$



we talk about compactness next week



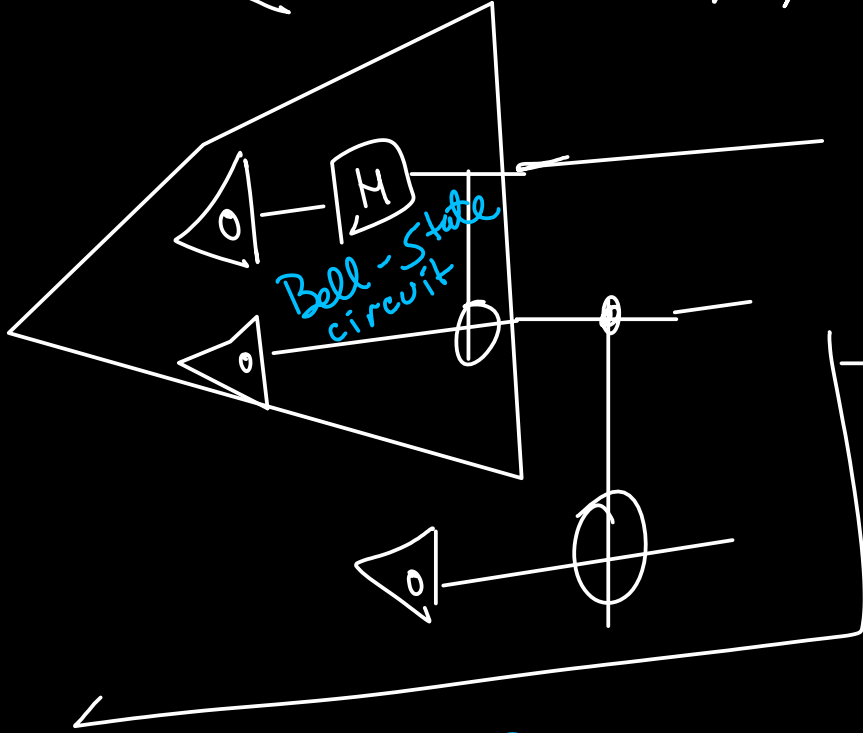
$$|0\rangle \rightarrow |00\rangle$$

$$|1\rangle \rightarrow |11\rangle$$

$$|00\rangle\langle 00| + |11\rangle\langle 11|$$



$$|00\rangle + |11\rangle$$



$$\rho = \rho^\dagger$$

$$\rho: \mathcal{H} \rightarrow \mathcal{H}$$

$$L(\mathcal{H})$$

$$\rho = |\psi\rangle\langle\psi|$$

Aside on SVD

$$\rho = \sum_i p_i |i\rangle\langle i| \quad \sqrt{\rho} = \sum_i \sqrt{p_i} |i\rangle\langle i|$$

$$|\Psi\rangle = \sum_i \sqrt{p_i} |i\rangle_1 |i\rangle_2$$

$$p_i: H_1 \rightarrow H_1$$

$$|\Psi\rangle \in H_1 \otimes H_2$$

$$|\Psi\rangle\langle\Psi| = \sum_{i,j} \sqrt{p_i} \sqrt{p_j} |i\rangle_1 \langle j|_1 \otimes |i\rangle_2 \langle j|_2$$

$$= \sum_{i,j} \sqrt{p_i p_j} |i\rangle\langle j|_1 \otimes |i\rangle\langle j|_2$$

$$\text{tr} |\Psi\rangle\langle\Psi| = \sum_{i,j,k} \sqrt{p_i p_j} \langle k|_2 |i\rangle\langle j|_1 \otimes |i\rangle\langle j|_2 |k\rangle_2$$

$$\begin{aligned}
 & \downarrow \sum_{i,j,k} \sqrt{P_i P_j} |iXj| \otimes \underbrace{\langle k|iXj|k \rangle}_{\in \mathbb{C}} \\
 &= \sum_{i,j,k} \sqrt{P_i P_j} |iXj| \otimes \underbrace{\langle j|j \rangle}_{\delta_{ij}} \\
 &= \sum_i P_i |iX i|
 \end{aligned}$$

$$\begin{aligned}
 \langle \rho \rangle_A &= \text{tr } A \rho = \sum_{i,j} \langle j| A P_i |iX i| j \rangle_{\delta_{ij}} \\
 &= \sum_i P_i \langle i| (A | i \rangle)
 \end{aligned}$$

$$\langle \underline{\Psi} | (A \otimes I) \underline{\Psi} \rangle =$$

$$\sum_{i,j} \sqrt{P_i} \sqrt{P_j} \langle i | \langle i | (A \otimes |j\rangle \langle j|)$$

$$= \sum_{i,j} \sqrt{P_i P_j} \langle i | A | j \rangle \delta_{ij} = \sum_i P_i \langle i | A | i \rangle$$

$$= \sum_i P_i A_{ii}$$

$$= \text{tr} AP$$

$$U = P D P^+ \quad \text{spectral decomposition}$$

$$V_{m \times n} = U_{m \times m} \sum_{m \times n} V_{n \times n}^+$$

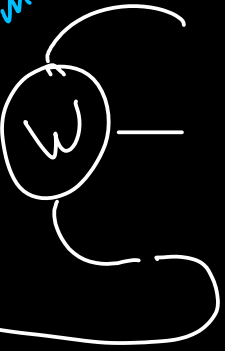
singular value decomposition

$$|w\rangle = |00\rangle + |01\rangle + |10\rangle$$

w-state

$$N_1 |w\rangle = |000\rangle + |110\rangle + |101\rangle$$

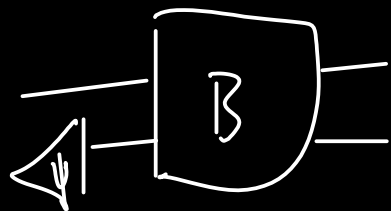
Bend a wire into a turn
w into a map



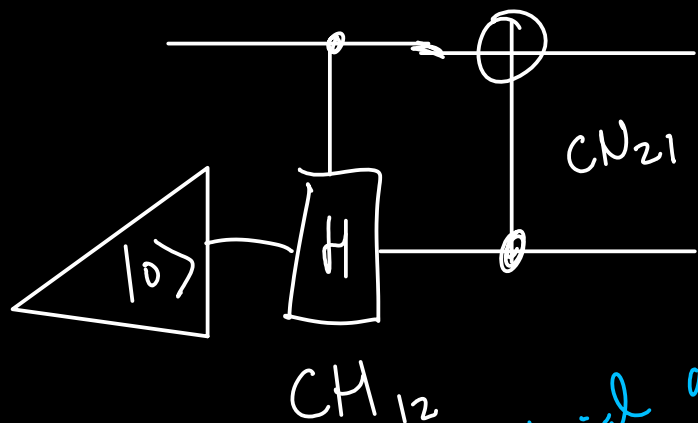
$$|0\rangle \rightarrow |00\rangle$$

$$|1\rangle \rightarrow |01\rangle + |10\rangle$$

as a map
w-state has this
action



we are looking for circuit
components for B



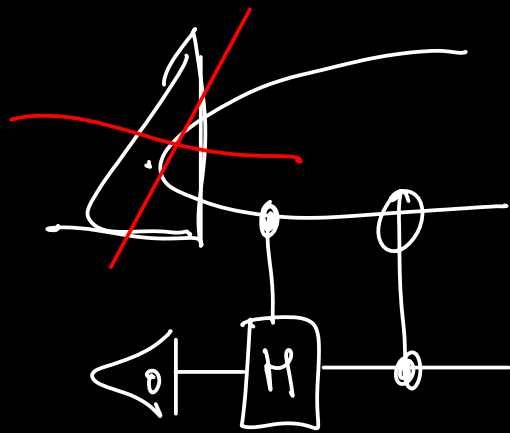
a little

trial and
error gives
this circuit

$$|0\rangle|0\rangle \rightarrow |00\rangle$$

$$|1\rangle \rightarrow |1\rangle|0\rangle \xrightarrow{CH_{12}} |1\rangle(|0\rangle + |1\rangle)$$

$$\xrightarrow{CN_{21}} |10\rangle + |11\rangle$$



this creates the state
 $|000\rangle + \frac{1}{\sqrt{2}}(|110\rangle + |101\rangle)$

$$(|00\rangle + |11\rangle)|0\rangle = |000\rangle + |110\rangle$$

$$\begin{aligned} & \downarrow |000\rangle \\ & |110\rangle \xrightarrow{CH_{23}} |1\rangle|1\rangle(|0\rangle + |1\rangle) \\ & \xrightarrow{CN_{32}} (|110\rangle + |101\rangle)^{1/\sqrt{2}} \end{aligned}$$

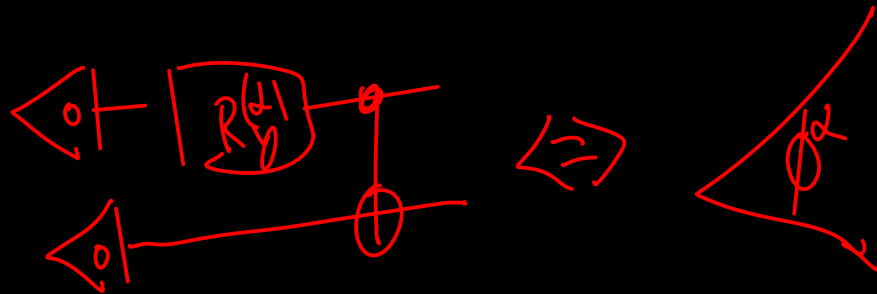
$$\cos \theta |00\rangle + \sin \theta |11\rangle$$

$$e^{-i2\theta Y}$$

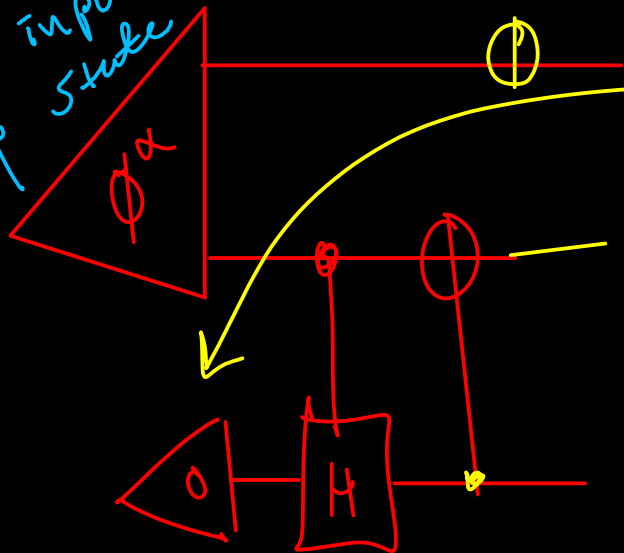
$$R_y(\alpha) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

we need
a new input
state.

$$|0\rangle \rightarrow \cos \theta |0\rangle + \sin \theta |1\rangle$$



new input state



$$(\alpha|00\rangle + \beta|11\rangle)|0\rangle$$

$$\alpha|000\rangle \rightarrow \alpha|000\rangle$$

$$\beta|110\rangle \xrightarrow{CH_{23}} \beta|11\rangle(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}$$

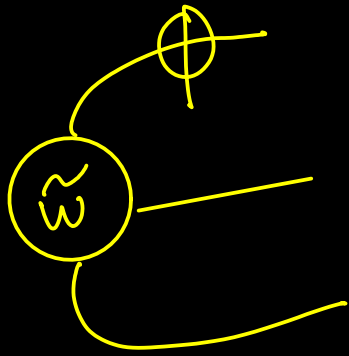
$$\xrightarrow{CH_{32}} \beta(|110\rangle + |101\rangle)\frac{1}{\sqrt{2}}$$

$$\alpha|000\rangle + \beta/\sqrt{2}|101\rangle + \beta/\sqrt{2}|110\rangle = |\bar{w}\rangle$$

$$\frac{1}{\sqrt{3}}(|000\rangle + |101\rangle + |110\rangle)$$

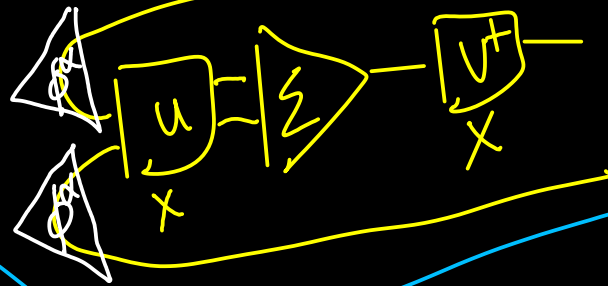
Exercise, (simple) find

α, β and hence θ in $\cos\theta|00\rangle + \sin\theta|11\rangle$ to normalise $|\bar{w}\rangle$ find α, β



$$W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$W = U \Sigma V^\dagger$$



$W = |00\rangle + |01\rangle + |10\rangle$
has only $-ZZZ$
as a stabiliser

We will talk
more about
stabilisers later.

G/HZ has $\rightarrow Z_i Z_i$
as stabilisers $\rightarrow X_i X_i X_k$
generators so

$$\{Z_1, Z_2, Z_3, Z_1 Z_2, Z_2 Z_3, Z_1 Z_3\}$$

$$\{X_1 X_2, X_2 X_3, X_1 X_3\}$$

as stabilisers ($\mathbb{I}$ as well)