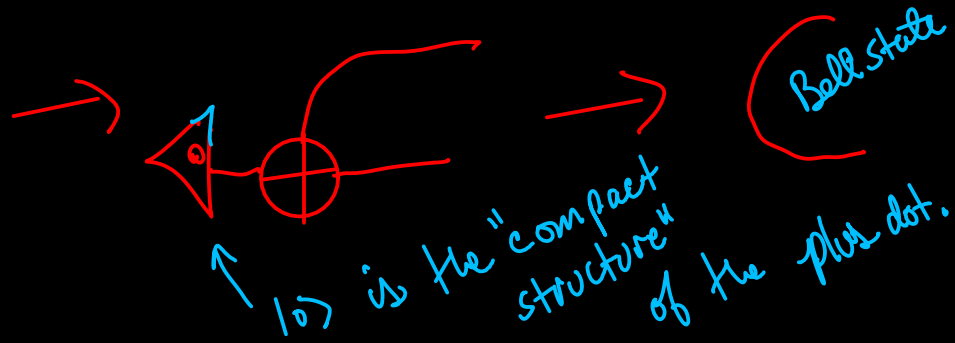
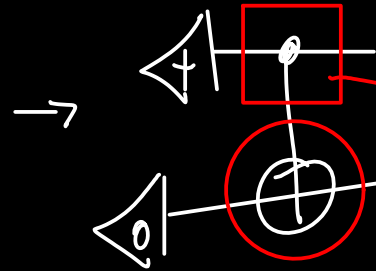
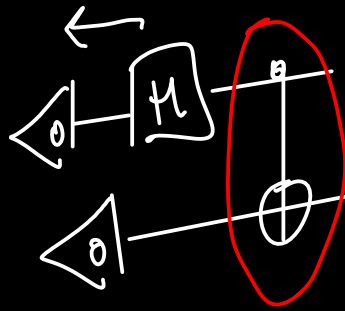


Tuesday 18 May

jacob.biamonte@comlab.ox.ac.uk

~ Last Week
Bell states

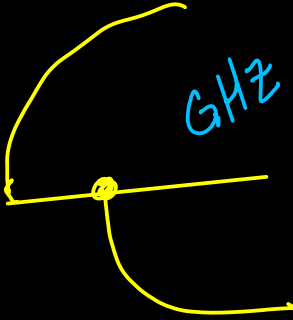
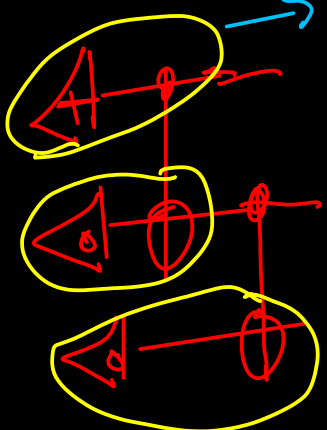


Bell state

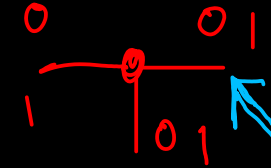
wire

Bell co-state

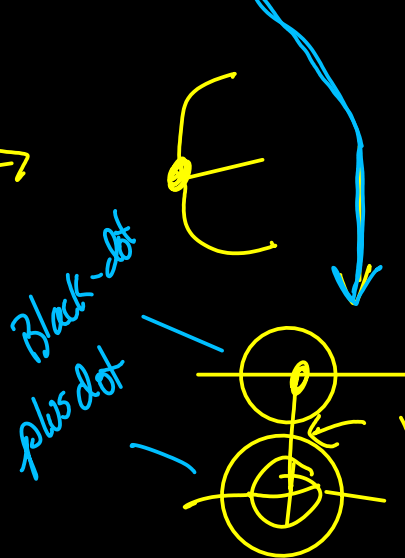
$\Psi_{\text{GHZ}} = |000\rangle + |111\rangle$
 $|1\rangle$ is the compact structure of the black dot.



GHZ state



$0 \rightarrow 00$
 $1 \rightarrow 11$

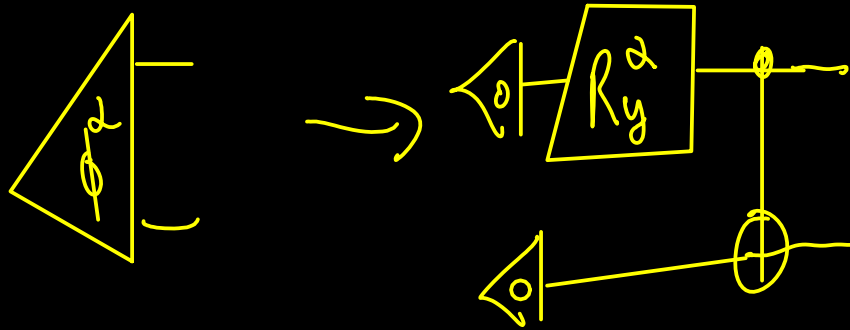


the two-dots (black and plus) are both parts of the CNOT gate.

$$\eta = \sum_i |i\rangle\langle i|$$

$$\epsilon = \sum_i \langle i | \langle i |$$

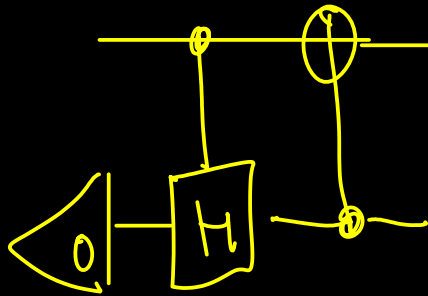
$$\cos\alpha |00\rangle + \sin\alpha |11\rangle$$



"generalised CNOT"

$$X_1 \rightarrow \begin{array}{l} |100\rangle + |010\rangle + |001\rangle \\ |000\rangle + |101\rangle + |110\rangle \end{array}$$

This map is almost a W-map
 $|0\rangle \rightarrow |00\rangle$
 $|1\rangle \rightarrow |01\rangle + |10\rangle$



$$|0\rangle \rightarrow |00\rangle$$

$$|1\rangle \rightarrow |01\rangle + |10\rangle$$

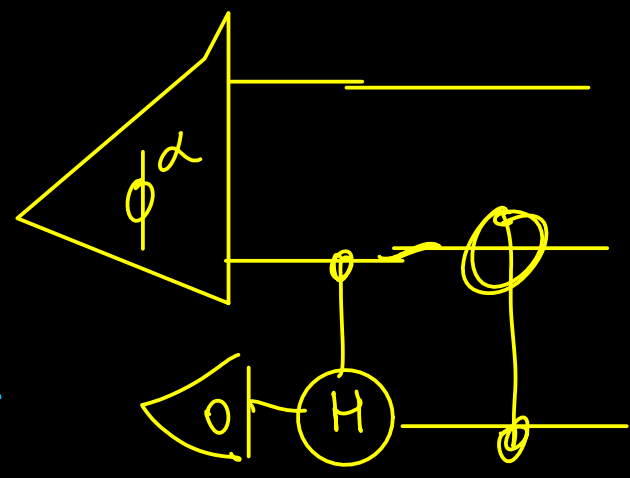
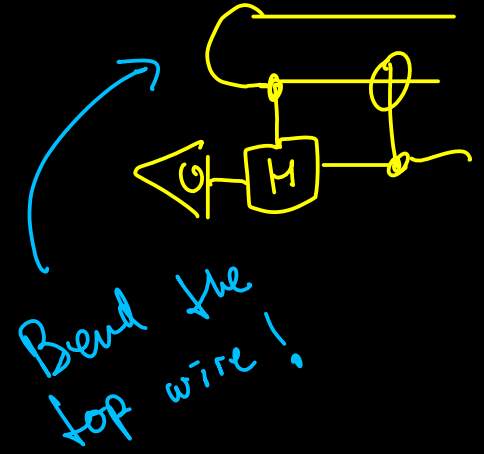
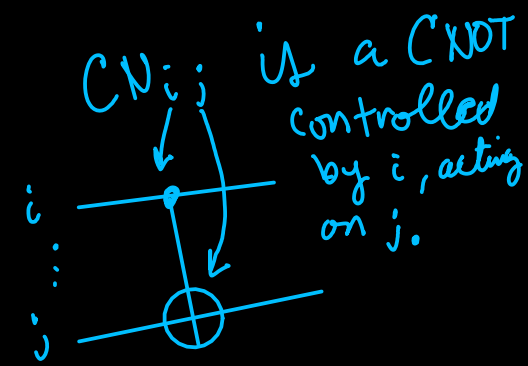
$\sqrt{2}$ by a scale factor!

$$(|00\rangle + |11\rangle)|0\rangle = |000\rangle + |110\rangle$$

circuit input

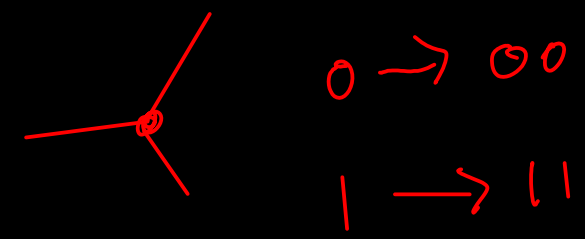
$$|110\rangle \xrightarrow{CH_{23}} \frac{|110\rangle + |111\rangle}{\sqrt{2}} \xrightarrow{CN_{32}}$$

$$\Rightarrow \frac{|110\rangle + |101\rangle + |000\rangle}{\sqrt{2}}$$



$$\rightarrow \frac{|000\rangle + |101\rangle + |010\rangle}{\sqrt{2}}$$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



"GHZ-map"

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha|00\rangle + \beta|11\rangle$$

$$(\alpha, \beta) = (1, 1) |1+\rangle$$

$$\langle + | \otimes | + \rangle = \text{Bell-state from compact structure.}$$

$$|0\rangle \rightarrow |00\rangle$$

$$|1\rangle \rightarrow |01\rangle + |10\rangle$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

"W-map"

$$\Delta_{m \times n} = U_{m \times m} \sum_{m \times n} V_{n \times n}^+$$

unitary

$$= \sum_i |u_i\rangle \theta_i \langle v_i|$$

basis co-basis

— Today

Stabilisers

QM

- Pauli Group
- Clifford Group

- H.P.

- Frobenius Algebras
- bi-algebras

- Stab for our dots

- Simulation
- cluster model

- Pauli group P

$$P = \{1, X, Y, Z\}$$

n -wise tensor powers of Paul matrices denoted by P^n

then $P^n := \{1, X, Y, Z\}^n$ ←

$$P^n \wedge \begin{matrix} \{ \pm i, \pm 1 \} \\ 1 \quad 1 \end{matrix} \quad (\text{Pauli group product}) \quad \text{closed by the} \quad g_i g_j = \delta_{ij} + i \epsilon_{ijk} \sigma^k$$

- P^n

$$\rho = \frac{1}{2} (1 + \underline{P} \cdot \underline{\sigma})$$

example of operator expansion in the Pauli-basis

$$= \frac{(\text{Tr} \rho) + (\text{Tr} X \rho) X + (\text{Tr} Y \rho) Y + (\text{Tr} Z \rho) Z}{2}$$

~ look online for notes on process tomography and distance measures

<http://www.comlab.ox.ac.uk/activities/quantum/course/notes.html>

$$- \quad X^2 = Y^2 = Z^2 = -iXYZ$$

- Pictures of Q.M.

SP

$$\Psi_t = U_t \Psi_0$$

$$A_t = A_0$$

H.P.

$$\Psi_t = \Psi_0$$

$$A_t = U A_0 U^\dagger$$

$$\langle \Psi_0, A \Psi_t \rangle = \langle U \Psi_0, A U \Psi_0 \rangle = \langle \Psi_0, U^\dagger A U \Psi_0 \rangle$$

real valued $\Rightarrow$

$\langle A^\dagger \rangle = \langle A \rangle$ as $A = A^\dagger$

$$N \psi_0 = \psi_0$$

N is a stabiliser for ψ_0

evolve ψ_0 under U

$$U \psi_0 = U N \psi_0 = U N \mathbb{1} \psi_0 = \underbrace{U N U^\dagger} \underbrace{U \psi_0}$$

ψ_0 evolves as if in the SP.

$$\begin{aligned} N &\xrightarrow{U} U N U^\dagger \\ \psi &\xrightarrow{U} U \psi \end{aligned}$$

$$A_t = U A_0 U^\dagger$$

p^n

N evolves as if in the HP

M stabiliser ψ

$$M \psi = \psi, \quad N \psi = \psi \Rightarrow M N \psi = \psi$$

$$U \psi \Rightarrow M N \xrightarrow{U} U M N U^\dagger = U M U^\dagger U N U^\dagger$$

$$\{x, y, z\}$$

$$xy = iz$$

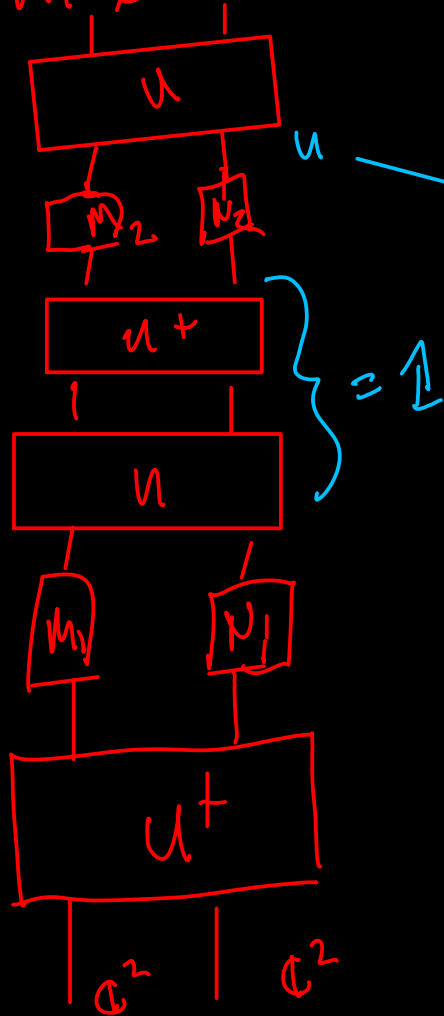
$$\begin{aligned} y &\xrightarrow{U} U y U^\dagger = i U x z U^\dagger \\ &= \underline{i U x U^\dagger} \underline{U z U^\dagger} \end{aligned}$$

— What about P^n ?

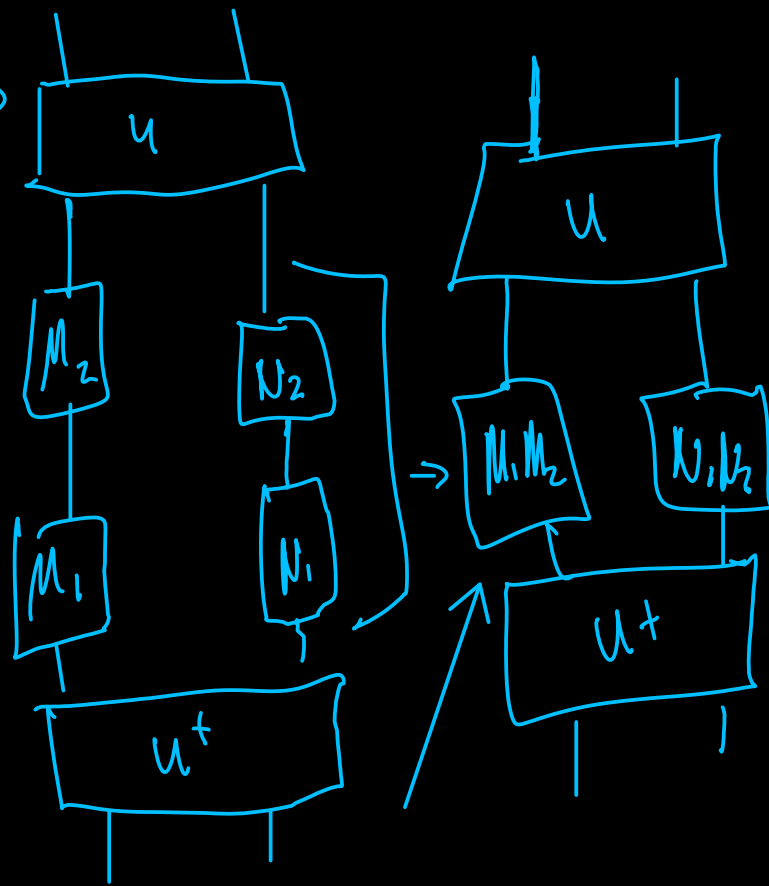
$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$$

$$M_i \otimes N_i \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

$$U M_1 \otimes N_1 U^\dagger$$



$$U M_2 \otimes N_2 U^\dagger$$



(This shows that it is enough to follow Z_1 and X_3 to get the evolution of Z_1, X_3)

$$\{X_1, Z_1, X_2, Z_2\}$$

$$XY = iZ$$

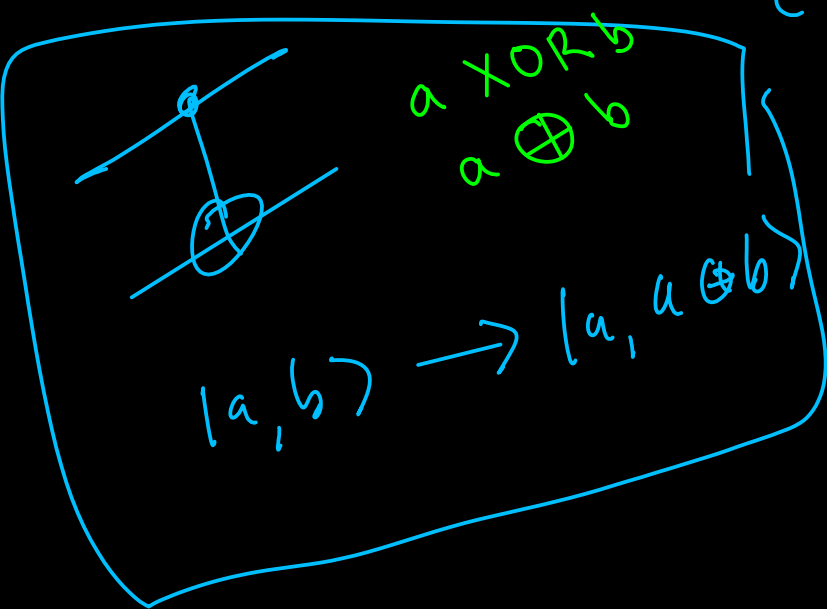
$$\rightarrow \underline{O} = \{X_1, \dots, X_n, Z_1, \dots, Z_n\}$$

$$|\underline{O}| = 2n$$

follow $2n$ operators
to recover full operator
space!

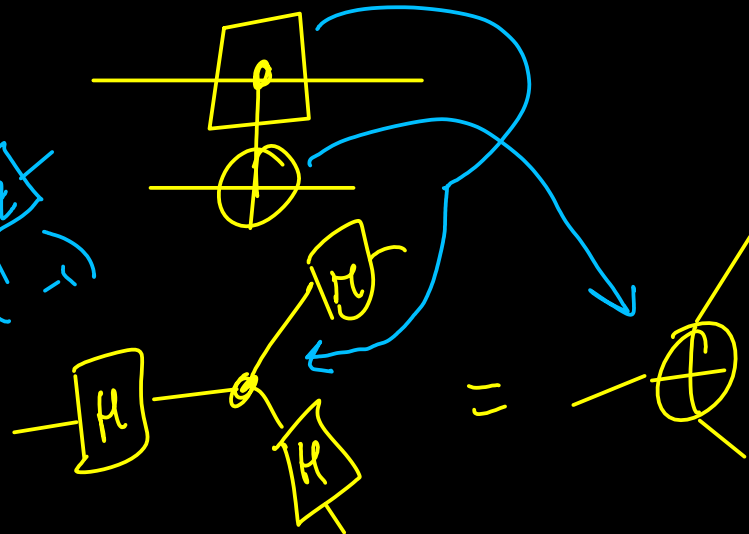
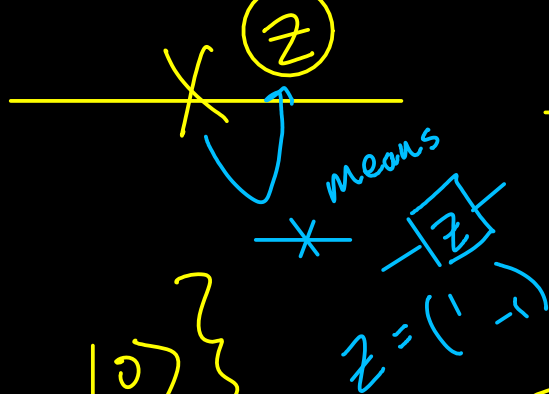
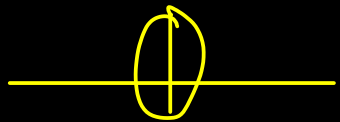
$U \Rightarrow$ restrict to the Clifford group

$\{P, H, CNOT\}$ generators of the group.

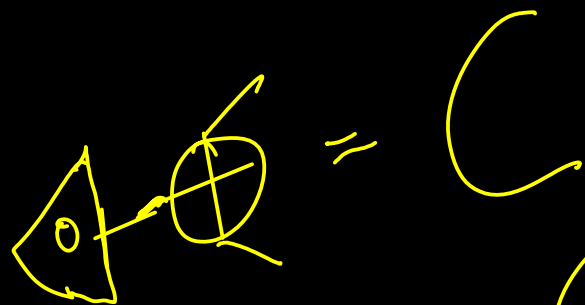
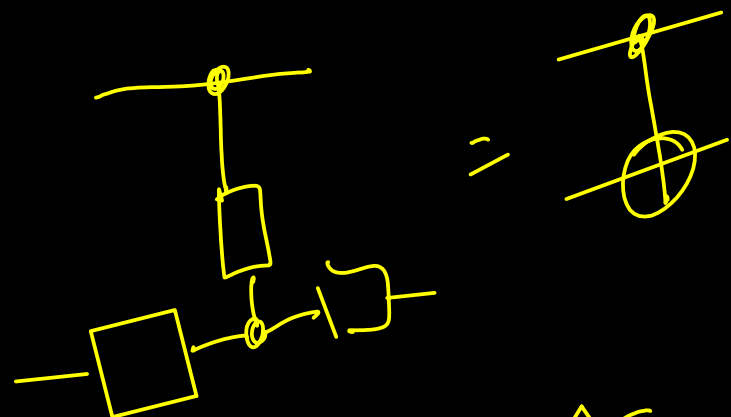


$$P|\psi\rangle = i^\psi |\psi\rangle, \quad \psi \in \{0, 1\}$$

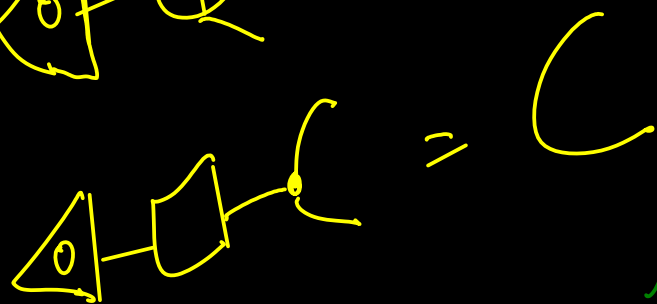
$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



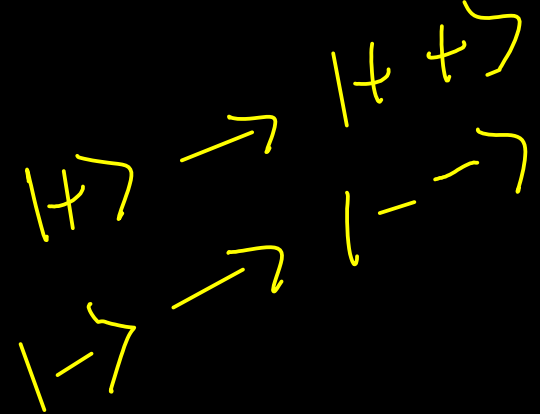
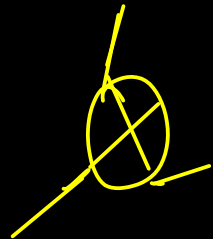
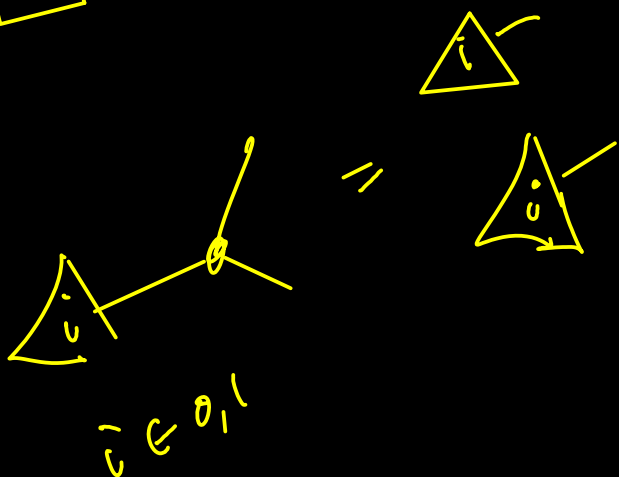
$\{ \text{---} \times \text{---}, H, |0\rangle \}$



compact structures
for plus and
black dots.



plus is a copy
construct on
the +/- basis

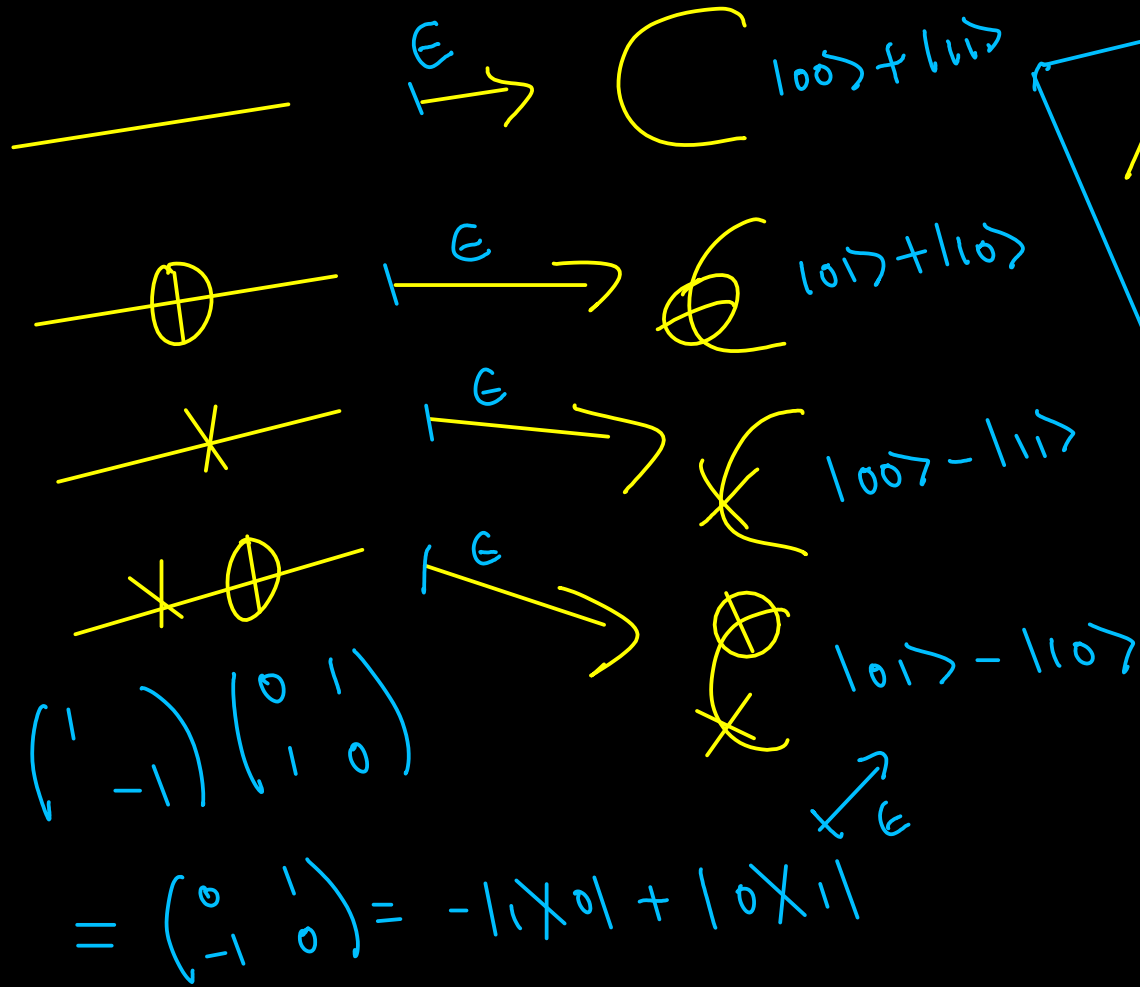




$$\begin{aligned}
 |0\rangle &\rightarrow |11\rangle + |00\rangle \\
 |1\rangle &\rightarrow |01\rangle + |10\rangle
 \end{aligned}$$

XOR

a	b	$a \oplus b$
0	0	0
0	1	1
1	0	1
1	1	0



$$\begin{aligned}
 M &= \sum_{i=0}^1 |i\rangle\langle i| \\
 E &= \sum_{i=0}^1 |i\rangle\langle i|
 \end{aligned}$$

Monoid

$(S, \cdot, e)$
 $\uparrow$ set $\uparrow$ product $\uparrow$ 1

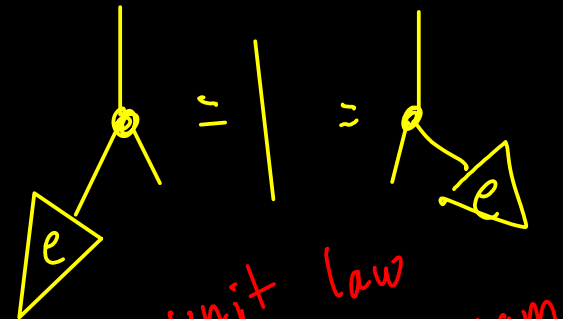
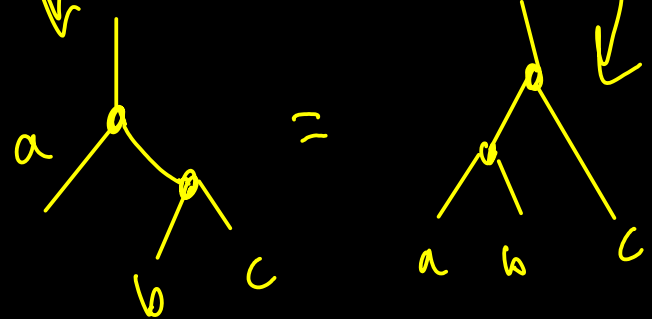
$$a, b \in S \Rightarrow a \cdot b \in S \quad (\text{closed})$$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

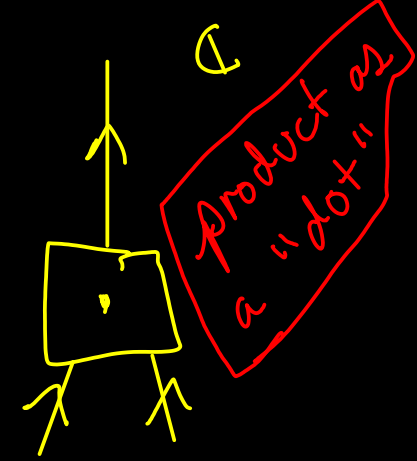
(associative)

$$e \cdot a = a \cdot e = a \in S$$

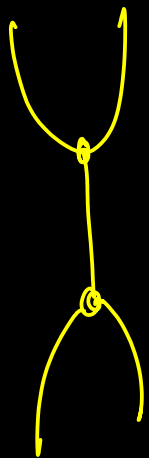
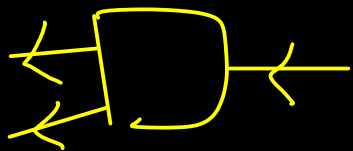
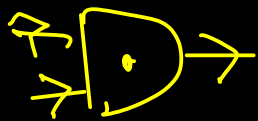
(unit)



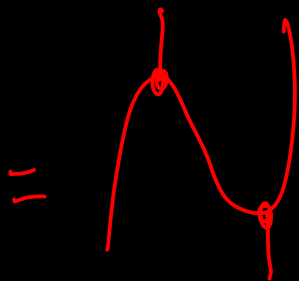
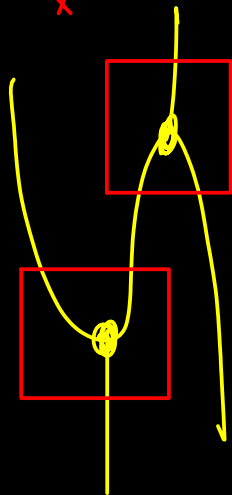
unit law
as a diagram



FA

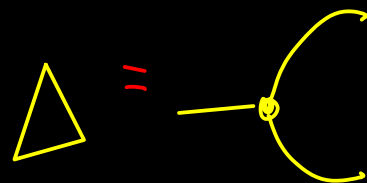
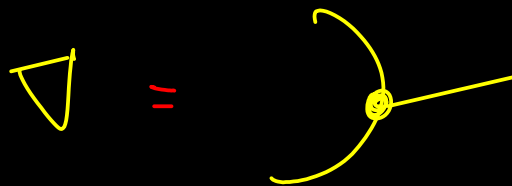


is "dots" satisfy this they are called Frobenius!

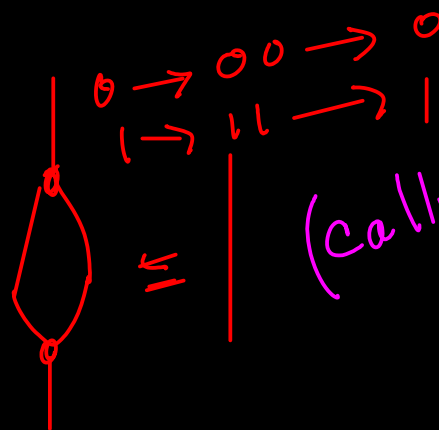
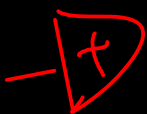


$$X \otimes X \xrightarrow{\Delta} X \xleftarrow{\nabla} I$$

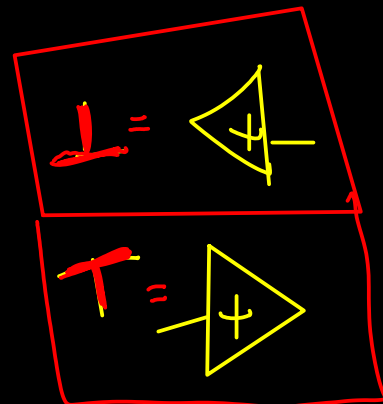
$$X \otimes X \xleftarrow{\Delta} X \xrightarrow{\nabla} I$$



⇒ FA



(called "special")



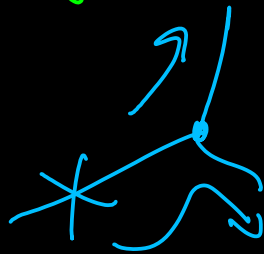
Frobenius algebras

$Z_1 Z_2$
 $Z_2 Z_3$
 $Z_1 \mathbb{I} Z_3$
 $X_1 X_2 X_3$

$- Y Y X$
 $- Y X Y$
 $- X Y Y$

also $\mathbb{I}$

These are the stabilisers of the black-dot on qubits.

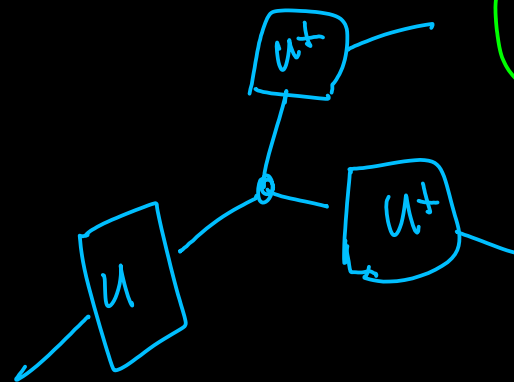
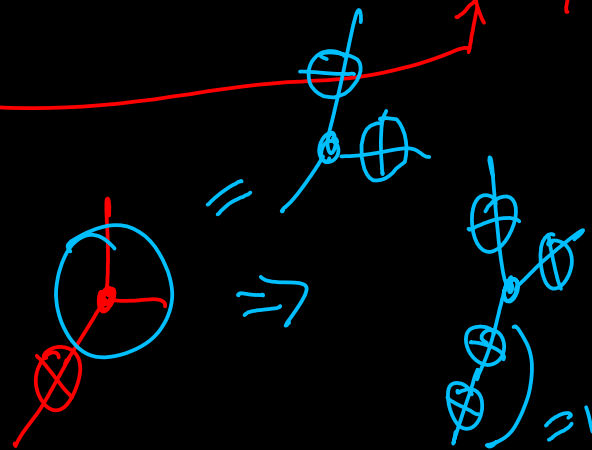


$UUU (XXX) U^\dagger U^\dagger U^\dagger$

Stabilisers transform as dots do.

$|000\rangle + |111\rangle$

$N\psi = \psi$



It can be shown that if a given dot is Frobenius then that dot transformed under U is also.

Back
to stabilisers!

$$\{Z_1, \dots, Z_n\}$$

$$X_1, \dots, X_n\}$$

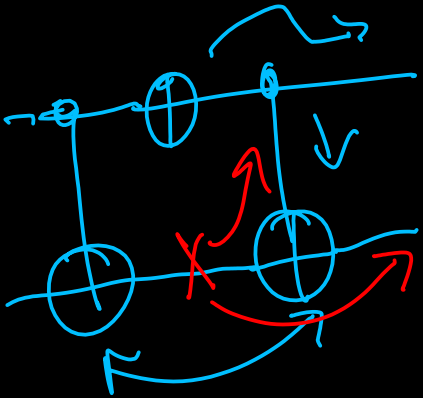
$$PXP^\dagger \rightarrow Y$$

$$PZP^\dagger \rightarrow X$$

$$HXH \rightarrow Z$$

$$HZH \rightarrow X$$

We need to know
how Z and X evolve
in the H.P. under
the Clifford group.

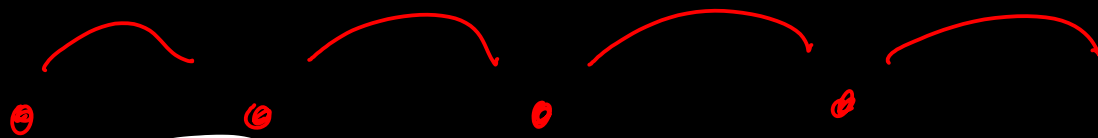


- spins X $+1$ $|+\rangle$

Z $(+1 \uparrow)$ $(-1 \downarrow)$

-1 $|-\rangle$

qubit-spins can point up or down



← 1D spin chain

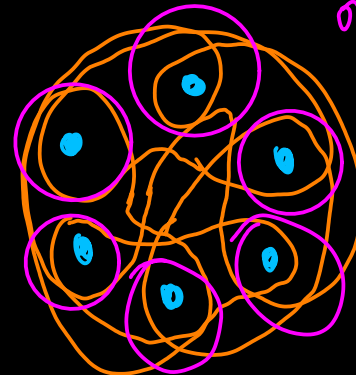
1D Cluster Model

$$\mathbb{I} - |00\rangle\langle 00| + - |11\rangle\langle 11|$$

$$H_c = -J \sum_i Z_i X_{i+1} Z_{i+2} + B \sum_i Z_i$$

acts on triples

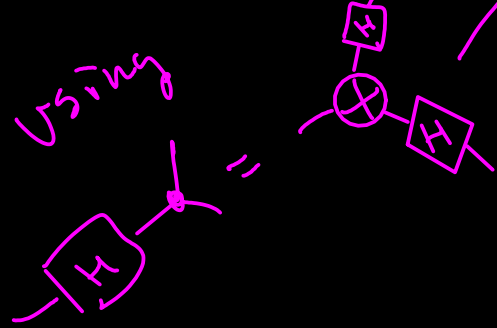
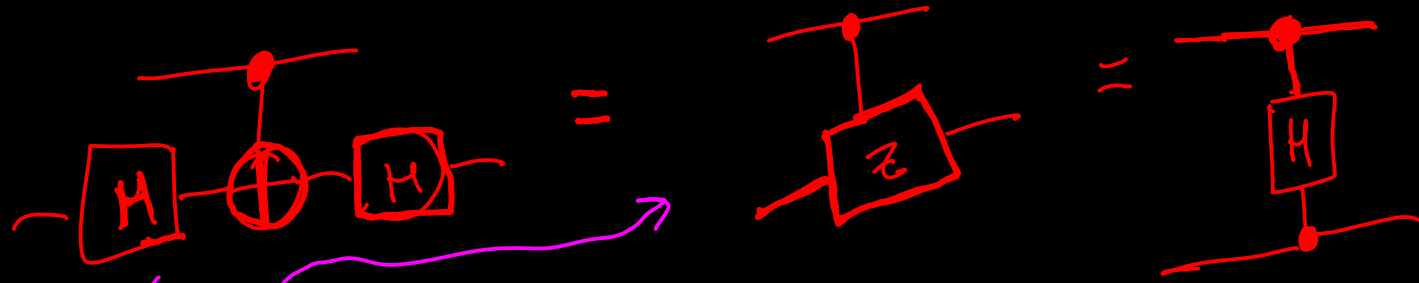
$$N+1 \equiv 1$$



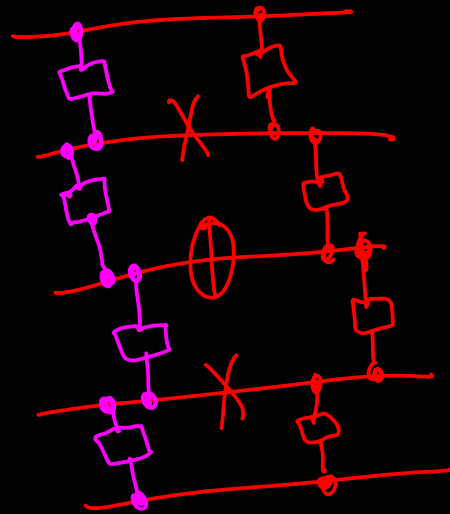
acts on single spins

cluster model on a ring of spins

6-spins



$$Z_i X_{i+1} Z_{i+2}$$

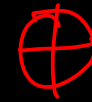


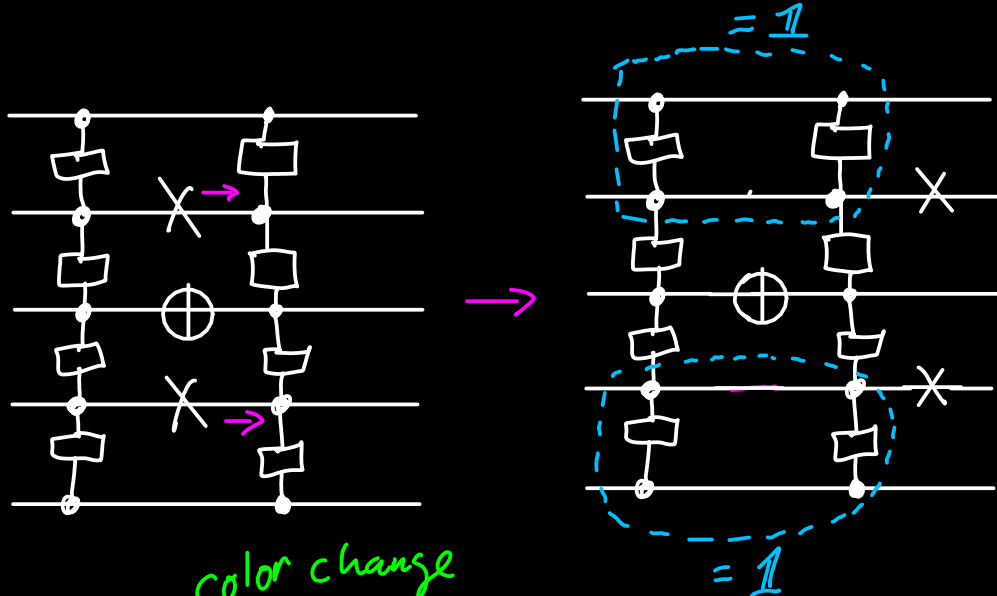
$$\sum_{i=1}^N (-J X_i + B Z_i)$$

$$\begin{pmatrix} B & -J \\ -J & B \end{pmatrix}$$

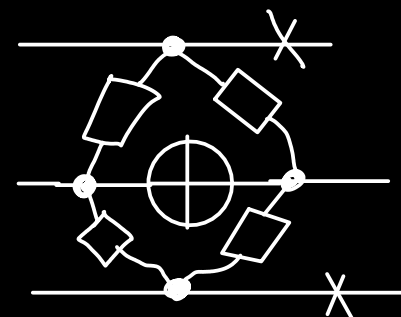
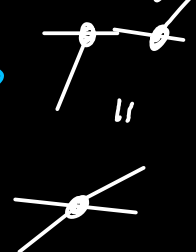


$$-\lambda Z$$

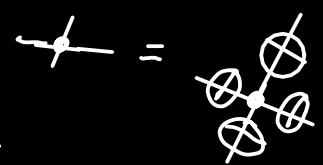




we group dots, eg

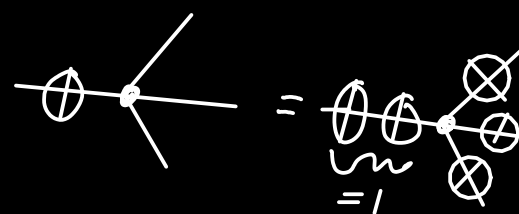


using

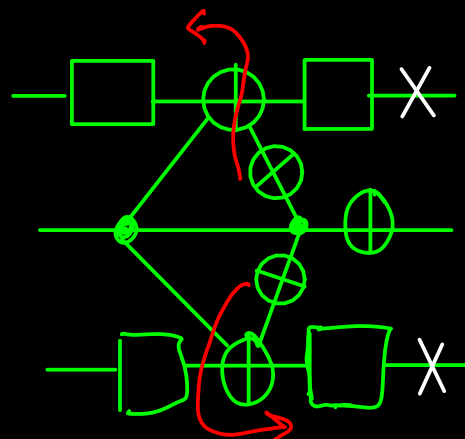
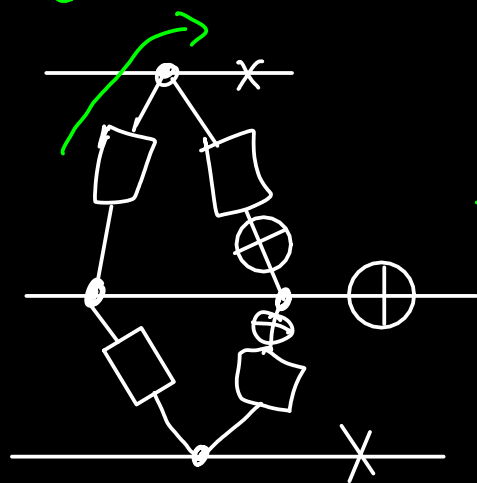


as $XXXX$ is a stab for

for $|0000\rangle + |1111\rangle$



color change



dots of same color move through each other.

