OXFORD UNIVERSITY

MATHEMATICS, JOINT SCHOOLS AND COMPUTER SCIENCE

WEDNESDAY 5 NOVEMBER 2008

Time allowed: $2\frac{1}{2}$ hours

For candidates applying for Mathematics, Mathematics & Statistics, Computer Science, Mathematics & Computer Science, or Mathematics & Philosophy

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Write your name, test centre (where you are sitting the test), Oxford college (to which you have applied or been assigned) and your proposed course (from the list above) in BLOCK CAPITALS.

NOTE: **Separate sets of instructions for both candidates and test supervisors are provided, which should be read carefully before beginning the test.**

NAME:
TEST CENTRE:
OXFORD COLLEGE (if known):
DEGREE COURSE:
DATE OF BIRTH:

FOR TEST SUPERVISORS USE ONLY:

[   ] **Tick here if special arrangements were made for the test.**

Please either include details of special provisions made for the test and the reasons for these in the space below or securely attach to the test script a letter with the details.

Signature of Invigilator ______________________

FOR OFFICE USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Total

1. For **ALL APPLICANTS**.

For each part of the question on pages 3–7 you will be given four possible answers, just one of which is correct. Indicate for each part **A**–**J** which answer (a), (b), (c), or (d) you think is correct with a tick (✓) in the corresponding column in the table below. *Please show any rough working in the space provided between the parts.*

	(a)	(b)	(c)	(d)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

A. The function

$$y = 2x^3 - 6x^2 + 5x - 7$$

has

(a) no stationary points;
(b) one stationary point;
(c) two stationary points;
(d) three stationary points.

B. Which is the smallest of these values?

(a) $\log_{10} \pi$, (b) $\sqrt{\log_{10}(\pi^2)}$, (c) $\left(\dfrac{1}{\log_{10} \pi}\right)^3$, (d) $\dfrac{1}{\log_{10} \sqrt{\pi}}$.

C. The simultaneous equations in x, y,

$$\begin{aligned}(\cos\theta)\,x - (\sin\theta)\,y &= 2\\ (\sin\theta)\,x + (\cos\theta)\,y &= 1\end{aligned}$$

are solvable

(a) for all values of θ in the range $0 \leqslant \theta < 2\pi$;
(b) except for one value of θ in the range $0 \leqslant \theta < 2\pi$;
(c) except for two values of θ in the range $0 \leqslant \theta < 2\pi$;
(d) except for three values of θ in the range $0 \leqslant \theta < 2\pi$.

D. When

$$1 + 3x + 5x^2 + 7x^3 + \cdots + 99x^{49}$$

is divided by $x - 1$ the remainder is

(a) 2000, (b) 2500, (c) 3000, (d) 3500.

E. The highest power of x in

$$\left\{\left[\left(2x^6+7\right)^3+\left(3x^8-12\right)^4\right]^5+\left[\left(3x^5-12x^2\right)^5+\left(x^7+6\right)^4\right]^6\right\}^3$$

is

(a) x^{424}, (b) x^{450}, (c) x^{500}, (d) x^{504}.

F. If the trapezium rule is used to estimate the integral

$$\int_0^1 f(x)\ \mathrm{d}x,$$

by splitting the interval $0 \leqslant x \leqslant 1$ into 10 intervals then an **overestimate** of the integral is produced. It follows that

(a) the trapezium rule with 10 intervals underestimates $\int_0^1 2f(x)\ \mathrm{d}x$;

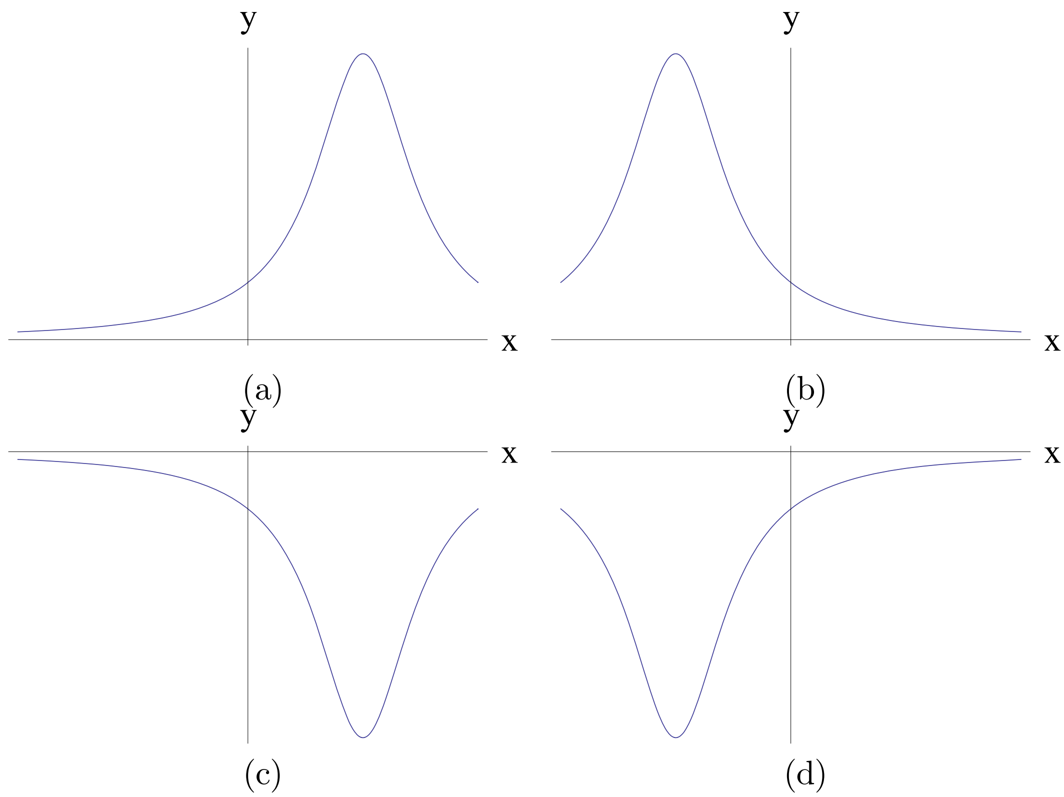
(b) the trapezium rule with 10 intervals underestimates $\int_0^1 (f(x)-1)\ \mathrm{d}x$;

(c) the trapezium rule with 10 intervals underestimates $\int_1^2 f(x-1)\ \mathrm{d}x$;

(d) the trapezium rule with 10 intervals underestimates $\int_0^1 (1-f(x))\ \mathrm{d}x$.

G. Which of the graphs below is a sketch of

$$y = \frac{1}{4x - x^2 - 5} \quad ?$$



H. The equation

$$9^x - 3^{x+1} = k$$

has one or more real solutions precisely when

(a) $k \geqslant -9/4$, (b) $k > 0$, (c) $k \leqslant -1$, (d) $k \geqslant 5/8$.

I. The function $S(n)$ is defined for positive integers n by

$$S(n) = \text{sum of the digits of } n.$$

For example, $S(723) = 7 + 2 + 3 = 12$. The sum

$$S(1) + S(2) + S(3) + \cdots + S(99)$$

equals

(a) 746, (b) 862, (c) 900, (d) 924.

J. In the range $0 \leqslant x < 2\pi$ the equation

$$(3 + \cos x)^2 = 4 - 2\sin^8 x$$

has

(a) 0 solutions, (b) 1 solution, (c) 2 solutions, (d) 3 solutions.

2. For **ALL APPLICANTS.**

(i) Find a pair of positive integers, x_1 and y_1, that solve the equation

$$(x_1)^2 - 2\,(y_1)^2 = 1.$$

(ii) Given integers a, b, we define two sequences $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ by setting

$$x_{n+1} = 3x_n + 4y_n, \qquad y_{n+1} = ax_n + by_n, \qquad \text{for } n \geqslant 1.$$

Find *positive* values for a, b such that

$$(x_{n+1})^2 - 2\,(y_{n+1})^2 = (x_n)^2 - 2\,(y_n)^2\,.$$

(iii) Find a pair of integers X, Y which satisfy $X^2 - 2Y^2 = 1$ such that $X > Y > 50$.

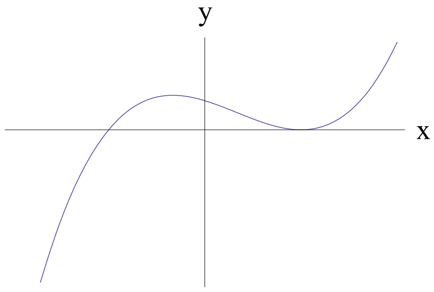
(iv) (Using the values of a and b found in part (ii)) what is the approximate value of x_n/y_n as n increases?

3.

For **APPLICANTS IN** { **MATHEMATICS** / **MATHEMATICS & STATISTICS** / **MATHEMATICS & PHILOSOPHY** / **MATHEMATICS & COMPUTER SCIENCE** } **ONLY.**

Computer Science **applicants should turn to page 14.**

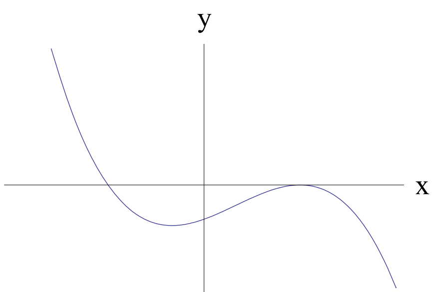
(i) The graph $y = f(x)$ of a certain function has been plotted below.



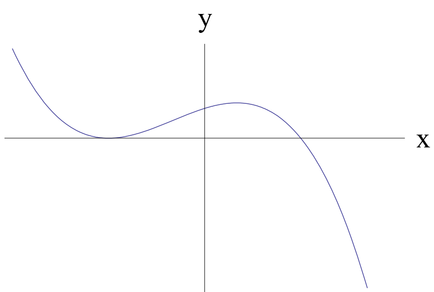
On the next three pairs of axes (A), (B), (C) are graphs of

$$y = f(-x), \qquad f(x-1), \qquad -f(x)$$

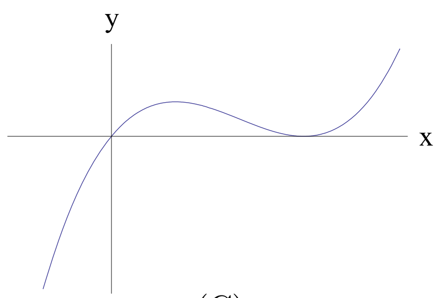
in some order. Say which axes correspond to which graphs.



(A)



(B)



(C)

(ii) Sketch, on the axes opposite, graphs of *both* of the following functions

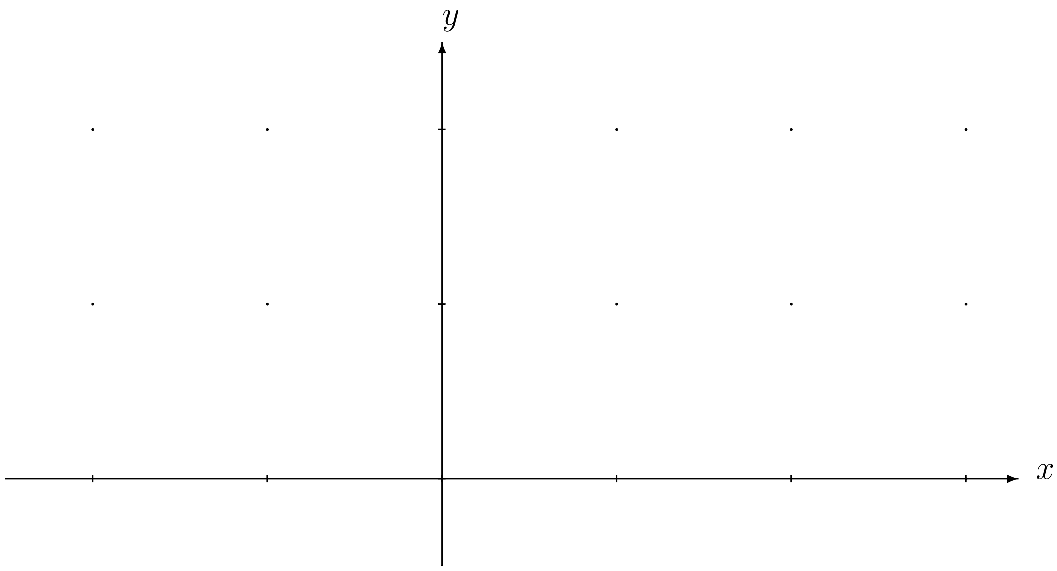
$$y = 2^{-x^2} \qquad \text{and} \qquad y = 2^{2x-x^2}.$$

Carefully label any stationary points.

(iii) Let c be a real number and define the following integral

$$I(c) = \int_0^1 2^{-(x-c)^2}\,\mathrm{d}x.$$

State the value(s) of c for which $I(c)$ is largest. Briefly explain your reasoning.
[Note you are not being asked to calculate this maximum value.]

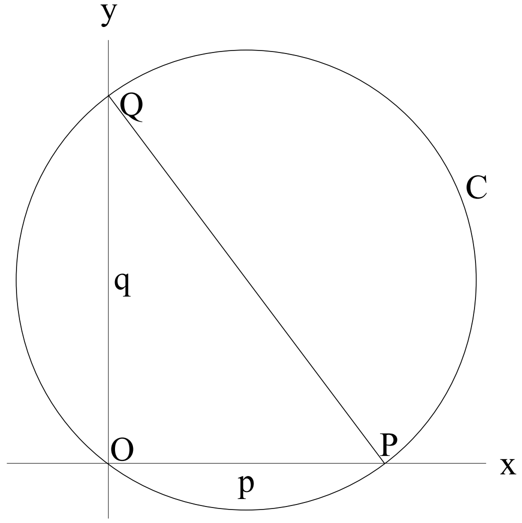


y
x

4.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

Mathematics & Computer Science **and** *Computer Science* **applicants should turn to page 14.**



Let p and q be positive real numbers. Let P denote the point $(p, 0)$ and Q denote the point $(0, q)$.

(i) Show that the equation of the circle C which passes through P, Q and the origin O is

$$x^2 - px + y^2 - qy = 0.$$

Find the centre and area of C.

(ii) Show that

$$\frac{\text{area of circle } C}{\text{area of triangle } OPQ} \geqslant \pi.$$

(iii) Find the angles OPQ and OQP if

$$\frac{\text{area of circle } C}{\text{area of triangle } OPQ} = 2\pi.$$

5. For **ALL APPLICANTS.**

The Millennium school has 1000 students and 1000 student lockers. The lockers are in a line in a long corridor and are numbered from 1 to 1000.

Initially all the lockers are closed (but unlocked).

The first student walks along the corridor and opens every locker.

The second student then walks along the corridor and closes every second locker, i.e. closes lockers 2, 4, 6, etc. At that point there are 500 lockers that are open and 500 that are closed.

The third student then walks along the corridor, changing the state of every third locker. Thus s/he closes locker 3 (which had been left open by the first student), opens locker 6 (closed by the second student), closes locker 9, etc.

All the remaining students now walk by in order, with the kth student changing the state of every kth locker, and this continues until all 1000 students have walked along the corridor.

(i) How many lockers are closed immediately after the third student has walked along the corridor? Explain your reasoning.

(ii) How many lockers are closed immediately after the fourth student has walked along the corridor? Explain your reasoning.

(iii) At the end (after all 1000 students have passed), what is the state of locker 100? Explain your reasoning.

(iv) After the *hundredth* student has walked along the corridor, what is the state of locker 1000? Explain your reasoning.

6.

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{COMPUTER SCIENCE} \\ \textbf{MATHEMATICS \& COMPUTER SCIENCE} \end{array} \right\}$ **ONLY.**

(i) A, B and C are three people. One of them is a *liar* who always tells lies, another is a *saint* who always tells the truth, and the third is a *switcher* who sometimes tells the truth and sometimes lies. They make the following statements:

A: I am the liar.
B: A is the liar.
C: I am not the liar.

Who is the liar among A, B and C? Briefly explain your answer.

(ii) P, Q and R are three more people, one a liar, one a saint, and the third a *contrarian* who tells a lie if he is the first to speak or if the preceding speaker told the truth, but otherwise tells the truth. They make the following statements:

P: Q is the liar.
Q: R is the liar.
R: P is the liar.

Who is the liar among P, Q and R? Briefly explain your answer.

(iii) X, Y and Z are three more people, one a liar, one a switcher and one a contrarian. They make the following statements:

X: Y is the liar.
Y: Z is the liar.
Z: X is the liar.
X: Y is the liar.
Y: X is the liar.

Who is the liar among X, Y and Z? Briefly explain your answer.

7. For **APPLICANTS IN COMPUTER SCIENCE ONLY.**

Ox-words are sequences of letters a and b that are constructed according to the following rules:

I. The sequence consisting of no letters is an Ox-word.

II. If the sequence W is an Ox-word, then the sequence that begins with a, followed by W and ending in b, written aWb, is an Ox-word.

III. If the sequences U and V are Ox-words, then the sequence U followed by V, written UV, is an Ox-word.

All Ox-words are constructed using these rules. The *length* of an Ox-word is the number of letters that occur in it. For example $aabb$ and $abab$ are Ox-words of length 4.

(i) Show that every Ox-word has an even length.

(ii) List all Ox-words of length 6.

(iii) Let W be an Ox-word. Is the number of occurrences of a in W necessarily equal to the number of occurrences of b in W? Justify your answer.

You may now assume that every Ox-word (of positive length) can be written *uniquely* in the form $aWbW'$ where W and W' are Ox-words.

(iv) For $n \geqslant 0$, let C_n be the number of Ox-words of length $2n$. Find an expression for C_{n+1} in terms of $C_0, C_1, \cdots, C_n$. Explain your reasoning.

End of Last Question