GAME SEMANTICS (DAY 3)

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$$f : \mathsf{int} \to \mathsf{int} \vdash \mathsf{let}\, y = f(0)\, \mathsf{in}\, (\lambda x^{\mathsf{int}}. f(x+y)+1) : \mathsf{int} \to \mathsf{int}$$

$\dagger$	$(\mathsf{c}_f, 0)$	(r_f, j)	$(\mathsf{r}_\downarrow, \dagger)$	(c, i_1)	$(\mathsf{c}_f, i_1 + j)$	(r_f, j_1)	$(\mathsf{r}, j_1 + 1)$	$\cdots$
O	P	O	P	O	P	O	P	

	f :	(int	$\to$	int)	$\vdash$	int	$\to$	int
O			$\dagger$					
P		$(\mathsf{c}_f, 0)$						
O				(r_f, j)				
P							$(\mathsf{r}_\downarrow, \dagger)$	
O						(c, i_1)		
P		$(\mathsf{c}_f, i_1 + j)$						
O				(r_f, j_1)				
P								$(\mathsf{r}_f, j_1 + 1)$

$$\mathsf{gen}' \equiv \mathsf{let}\ x = \mathsf{ref}(0)\ \mathsf{in}\ \lambda z^{\mathsf{int}}.(x := z; x) \quad : \quad \mathsf{int} \to \mathsf{ref\ int}$$

		$\vdash$	int	$\to$	ref int
O	$\star$				
P				$(\mathsf{r}_{\downarrow}, \dagger)$	
O			(c, i_1)		
P					$(\mathsf{r}, a)^{(a, i_1)}$
O			$(\mathsf{c}, i_2)^{(a, i_1')}$		
P					$(\mathsf{r}, a)^{(a, i_2)}$
O			$(\mathsf{c}, i_3)^{(a, i_2')}$		
P					$(\mathsf{r}, a)^{(a, i_3)}$
		$\dots$			

$$\mathsf{gen} \equiv \ \lambda z^{\mathsf{int}}.\, \mathsf{let}\, x = \mathsf{ref}(0)\, \mathsf{in}\, (x := z; x) \ \ : \ \ \mathsf{int} \to \mathsf{ref\,int}$$

	$\vdash$	int	$\to$	$\mathsf{ref\,int}$
O	$\star$			
P			$(\mathsf{r}_\downarrow, \dagger)$	
O		(c, i_1)		
P				$(\mathsf{r}, a_1)^{(a_1,i_1)}$
O		$(\mathsf{c}, i_2)^{(a_1,i_1')}$		
P				$(\mathsf{r}, a_2)^{(a_1,i_1')(a_2,i_2)}$
O		$(\mathsf{c}, i_3)^{(a_1,i_1'')(a_2,i_2')}$		
P				$(\mathsf{r}, a_3)^{(a_1,i_1'')(a_2,i_2')(a_3,i_3)}$
	$\ldots$			

$$a_i \neq a_j$$

Reference equality check $(x = y)$
We take $(\!|\Gamma \vdash x = y : \mathsf{int}|\!)$ to be

$$\{\mathrm{i}^S(\mathsf{r}_\downarrow, 0)^S \mid \mathrm{i}^S \in I_\Gamma^{\mathrm{st}},\ \mathrm{i}_x \neq \mathrm{i}_y\} \cup \{\mathrm{i}^S(\mathsf{r}_\downarrow, 1)^S \mid \mathrm{i}^S \in I_\Gamma^{\mathrm{st}},\ \mathrm{i}_x = \mathrm{i}_y\}.$$

Note that $\mathrm{i}_x, \mathrm{i}_y$ are names from $\mathbb{A}_{\mathsf{int}}$.

Dereferencing $(!x)$
In this instance P simply returns the value provided in the store of the initial move:

$$(\!|\Gamma \vdash !x : \mathsf{int}|\!) = \{\mathrm{i}^S(\mathsf{r}_\downarrow, S(\mathrm{i}_x))^S \mid \mathrm{i}^S \in I_\Gamma^{\mathrm{st}}\}.$$

Note again that $\mathrm{i}_x \in \mathbb{A}_{\mathsf{int}}$.

Reference update $(x := y)$

$$(\!|\, \Gamma \vdash x := y : \mathsf{unit} \,|\!) = \{ \mathrm{i}^S(\mathsf{r}_\downarrow, \star)^{S[\mathrm{i}_x \mapsto \mathrm{i}_y]} \mid \mathrm{i}^S \in I_\Gamma^{\mathrm{st}} \}$$

Here P modifies the store of the initial move using the value for y from the initial move. The types of x, y specify that i_x is a name and i_y is an integer.

Conditionals ($\mathsf{if}\ x\ \mathsf{then}\ N\ \mathsf{else}\ N'$)

In this case we simply borrow plays from $(\!|\, \Gamma \vdash N \,|\!)$ or $(\!|\, \Gamma \vdash N' \,|\!)$ depending on the value of x inside the initial move, i.e. $(\!|\, \Gamma \vdash \mathsf{if}\ x\ \mathsf{then}\ N\ \mathsf{else}\ N' \,|\!)$ is equal to:

$$\{\mathrm{i}^S s \mid \mathrm{i}_x > 0,\ \mathrm{i}^S s \in (\!|\, \Gamma \vdash N \,|\!)\} \cup \{\mathrm{i}^S s \mid \mathrm{i}_x = 0,\ \mathrm{i}^S s \in (\!|\, \Gamma \vdash N' \,|\!)\}.$$

MORE COMPLICATED CASES

- Most terms considered so far were minimalistic (restricted to being performed on variables).
- If it is necessary to translate a term that involves more complex terms, we can follow the relevant recipe and combine it with the way that application will be interpreted, which will be explained shortly.
- This relies on the fact that, for example, the complete-play interpretations of $M := N$ and $\mathsf{let}\, x = M \,\mathsf{in}\, (\mathsf{let}\, y = N \,\mathsf{in}\, x := y)$ are guaranteed to be the same.

$$\lambda x^{\beta}.M \mid \mathsf{while}(M) \mid xy \mid (\lambda x^{\beta}.M)N$$

Lambda abstraction ($\lambda x^\beta.M$)

$\llparenthesis\, \Gamma \vdash \lambda x^\beta.M : \beta \to \beta' \,\rrparenthesis$ is defined to consist of all complete plays of the form

$$\mathrm{i}^S(\mathsf{r}_\downarrow, \dagger)^S X_1 \cdots X_k\,.$$

where $X_1 \cdots X_k$ resembles interleaving plays from

$$\llparenthesis\, \Gamma, x : \beta \vdash M : \beta' \,\rrparenthesis$$

in such a fashion that the names created in each thread by P are disjoint and fresh with respect to the preceding dialogue.

- Due to multiple calls, more and more names can be generated than those participating in a single call.
- Such names have to be carried along by the play, even though they do not take part in a call. Accordingly, P will not be allowed to modify them.

$$\mathsf{i}^S (\mathsf{r}_\downarrow, \dagger)^S X_1 \cdots X_k$$

We require that each X_i be of the shape

$$(\mathsf{c}, \ell_\mathsf{c})^{S_0 \uplus U_{i,0}}\, m_{i,1}^{S_{i,1} \uplus U_{i,0}} \cdots m_{i,2k}^{S_{i,2k} \uplus U_{i,k}}\, (\mathsf{r}, \ell_\mathsf{r})^{S_{i,2k+1} \uplus U_{i,k}}$$

such that

$$(\mathsf{i}, \ell_\mathsf{c})^{S_0}\, m_{i,1}^{S_{i,1}} \cdots m_{2k}^{S_{i,2k}}\, (\mathsf{r}_\downarrow, \ell_\mathsf{r})^{S_{i,2k+1}}$$

is a complete play from $(\!|\, \Gamma, x : \beta \vdash M \,|\!)$.

While loop ($\mathsf{while}(M)$)

- Suppose $\Gamma \vdash M : \mathsf{int}$. We first calculate $(\!|\, \Gamma \vdash M : \mathsf{int} \,|\!)$ and observe that it must be equal to $(\!|\, \Gamma, x : \mathsf{unit} \vdash M : \mathsf{int} \,|\!)$ except that there is an extra $\star$ in the initial move. Note that the $\star$ has no bearing on names.
- $\mathsf{while}(M)$ will then be interpreted by restricting

$$(\!|\, \Gamma \vdash \lambda x^{\mathsf{unit}}.\, M : \mathsf{int} \,|\!).$$

While loop ($\mathsf{while}(M)$)

- Recall that sequences from $(\!|\, \Gamma \vdash \lambda x^{\mathsf{unit}}.\, M : \mathsf{int} \,|\!)$ match the pattern

  $$X\,(\mathsf{r}_\downarrow, \dagger)(\mathsf{c}, \star)\, X_1\,(\mathsf{r}, \ell_1)(\mathsf{c}, \star) \cdots (\mathsf{r}, \ell_{k-1})(\mathsf{c}, \star)\, X_k\,(\mathsf{r}, \ell_k).$$

- To interpret $\Gamma \vdash \mathsf{while}(M) : \mathsf{unit}$ we select only those sequences above where the induced sequence $\ell_1 \cdots \ell_k$ satisfies $\ell_k = 0$ and $\ell_j > 0$ $(1 \le j \le k)$.

- Subsequently, we erase all moves with tags $\mathsf{r}_\downarrow, \mathsf{r}, \mathsf{c}$ and add the move $(\mathsf{r}_\downarrow, \star)$ at the end. This yields the sequence:

  $$X X_1 \cdots X_k (\mathsf{r}_\downarrow, \star).$$

  In the above we have omitted stores, which simply need to be copied over from one sequence to the other.

Application (xy)

$(\!|\,\Gamma \vdash xy : \beta'\,|\!)$ contains all complete plays of the shape

$$\mathsf{i}^S(\mathsf{c}_x, \mathsf{i}_y)^S(\mathsf{r}_x, \ell)^{S'}(\mathsf{r}_\downarrow, \ell)^{S'}.$$

- P does not change the store in any of the plays, but O can play a different store S'.
- We must have $\mathsf{dom}(S) \subseteq \mathsf{dom}(S')$ and the inclusion can be proper if $\ell \in \mathbb{A} \setminus \mathsf{dom}(S)$.

Application $((\lambda x^{\beta}.M)N)$

In the store-free case, given

$$\begin{aligned} s &= \mathrm{i}\, u\, (\mathsf{r}_{\downarrow}, \ell) \in (\!|\, \Gamma \vdash N : \beta \,|\!) \\ t &= (\mathrm{i}, \ell)\, m_1 \cdots m_{2k+1} \in (\!|\, \Gamma, x : \beta \vdash M : \beta' \,|\!) \end{aligned}$$

take

$$\mathrm{i}\, u\, m_1 \cdots m_{2k+1}.$$

Application $((\lambda x^{\beta}.M)N)$

In the general case, consider

$$\begin{aligned} s &= \mathrm{i}^{S_0}\, u\, (\mathsf{r}_{\downarrow}, \ell)^{S} \in (\!|\, \Gamma \vdash N : \beta \,|\!)\,, \\ t &= (\mathrm{i}, \ell)^{T_0}\, m_1^{T_1} \cdots m_{2k+1}^{T_{2k+1}} \in (\!|\, \Gamma, x : \beta \vdash M : \beta' \,|\!)\,. \end{aligned}$$

Not all such s, t are allowed to contribute to $(\!|\, \Gamma \vdash \mathsf{let}\, x = N \,\mathsf{in}\, M \,|\!)$.

SELECTION CRITERIA

$$\begin{array}{rcl} s & = & \mathrm{i}^{S_0}\, u\, (\mathsf{r}_{\downarrow}, \ell)^S \in (\!|\, \Gamma \vdash N : \beta \,|\!) \\ t & = & (\mathrm{i}, \ell)^{T_0}\, m_1^{T_1} \cdots m_{2k+1}^{T_{2k+1}} \in (\!|\, \Gamma, x : \beta \vdash M : \beta' \,|\!) \end{array}$$

1. $T_0 \subseteq S$
2. $\nu(s) \cap P(t) = \emptyset$, where we let $P(t)$ be the set of P-names of t.
3. If $\ell \in \mathbb{A}$ and ℓ is fresh (does not occur in $\mathrm{i}^{S_0}u$) then the first move m_i that contains ℓ must be a P-move and until that moment O cannot change the stored values of ℓ. If ℓ does not occur in any m_i, O cannot change ℓ at all.

OUTCOME 1

ℓ not fresh or not a name

$$\begin{array}{rcl} s & = & \mathrm{i}^{S_0}\, u\, (\mathsf{r}_{\downarrow}, \ell)^{S} \in (\!|\, \Gamma \vdash N : \beta \,|\!) \\ t & = & (\mathrm{i}, \ell)^{T_0}\, m_1^{T_1} \cdots m_{2k+1}^{T_{2k+1}} \in (\!|\, \Gamma, x : \beta \vdash M : \beta' \,|\!) \end{array}$$

$$\mathrm{i}^{S_0}\, u\, m_1^{T_1 \uplus (S \setminus T_0)}\, m_2^{T_2 \uplus U_1}\, m_3^{T_3 \uplus U_1} \ldots m_{2k}^{T_{2k} \uplus U_l}\, m_{2k+1}^{T_{2k+1} \uplus U_l}$$

OUTCOME II

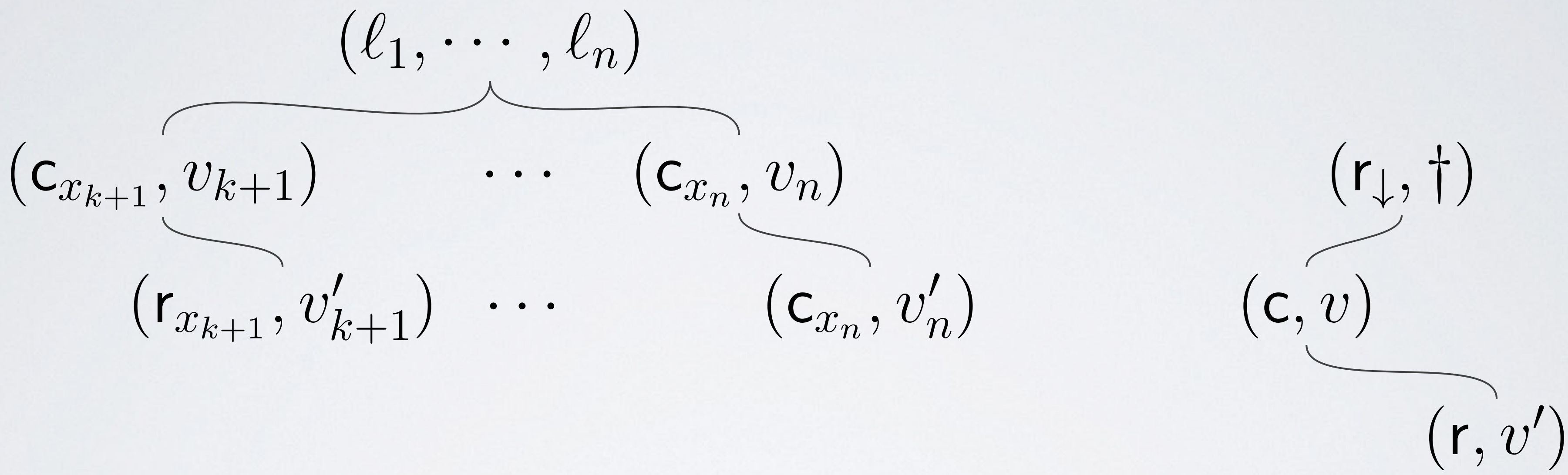
ℓ fresh

$$
\begin{aligned}
s &= \mathrm{i}^{S_0}\, u\, (\mathsf{r}_{\downarrow}, \ell)^{S} \in (\!|\, \Gamma \vdash N : \beta \,|\!) \\
t &= (\mathrm{i}, \ell)^{T_0}\, m_1^{T_1} \cdots m_{2k+1}^{T_{2k+1}} \in (\!|\, \Gamma, x : \beta \vdash M : \beta' \,|\!)
\end{aligned}
$$

$$
\mathrm{i}^{S_0}\, u\, m_1^{T_1^{\ell} \uplus (S \setminus T_0)}\, m_2^{T_2^{\ell} \uplus U_1} \cdots m_{2k'}^{T_{2k'}^{\ell} \uplus U_{k'}}\, m_{2k'+1}^{T_{2k'+1} \uplus U_{k'}} \cdots m_{2k}^{T_{2k} \uplus U_k}\, m_{2k+1}^{T_{2k+1} \uplus U_k}
$$

(ℓ cannot feature in a play until P reveals it to O)

$T_i^{\ell} = T_i \restriction (\mathsf{dom}(T_i) \setminus \ell)$


$(\ell_1, \cdots, \ell_n)$
$(\mathsf{c}_{x_{k+1}}, v_{k+1})$
$\cdots$
(c_{x_n}, v_n)
$(\mathsf{r}_{x_{k+1}}, v'_{k+1})$
$\cdots$
(c_{x_n}, v'_n)
$(\mathsf{r}_{\downarrow}, \dagger)$
(c, v)
(r, v')

ARENAS

An ***arena*** $A = (M_A, I_A, \vdash_A, \lambda_A)$ is given by:

- a set M_A of moves,
- a subset $I_A \subseteq M_A$ of initial moves,
- a relation $\vdash_A \subseteq M_A \times (M_A \setminus I_A)$,
- a function $\lambda_A : M_A \to \{O, P\} \times \{Q, A\}$,

satisfying, for each $m, m' \in M_A$, the conditions:

- $m \in I_A \implies \lambda_A(m) = (P, A)$,
- $m \vdash_A m' \wedge \lambda_A^{QA}(m) = A \implies \lambda_A^{QA}(m') = Q$,
- $m \vdash_A m' \implies \lambda_A^{OP}(m) \neq \lambda_A^{OP}(m')$.

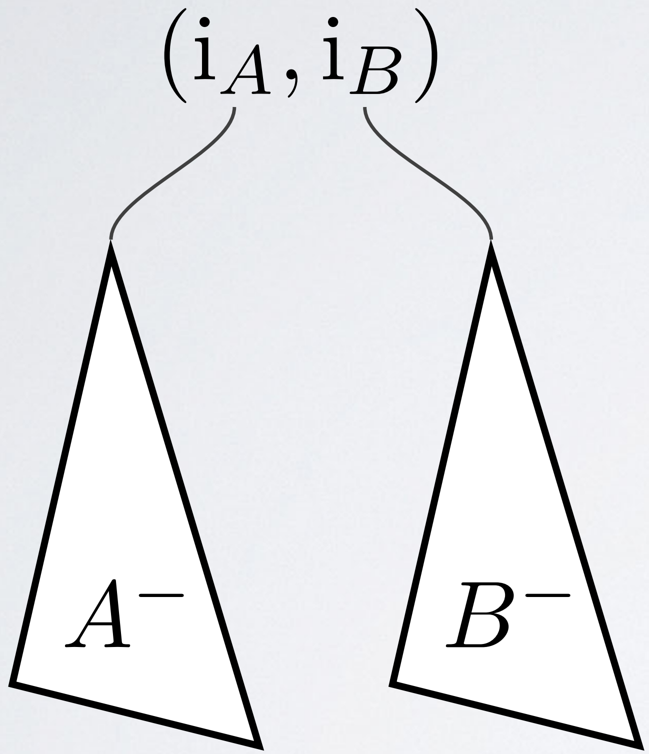
We call $\vdash_A$ the *justification relation* of A, and λ_A its *labelling function.*

1, $\mathbb{Z}$ and $\mathbb{A}_\zeta$

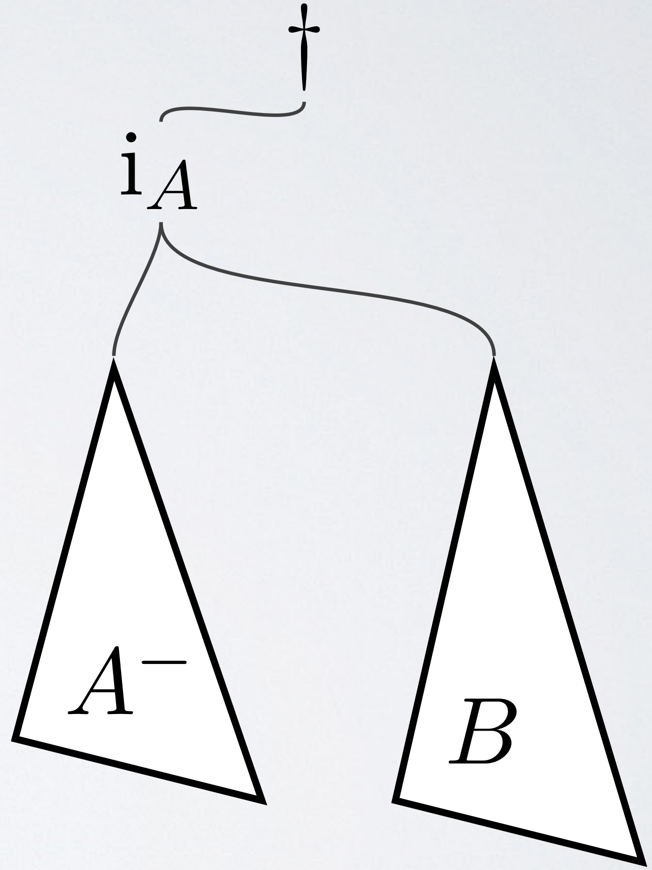
$$M_1 = I_1 = \{\star\} \qquad M_{\mathbb{Z}} = I_{\mathbb{Z}} = \mathbb{Z}$$

$$M_{\mathbb{A}_\zeta} = I_{\mathbb{A}_\zeta} = \mathbb{A}_\zeta$$

CONSTRUCTIONS



$A \otimes B$



$A \Rightarrow B$

TYPE INTERPRETATION

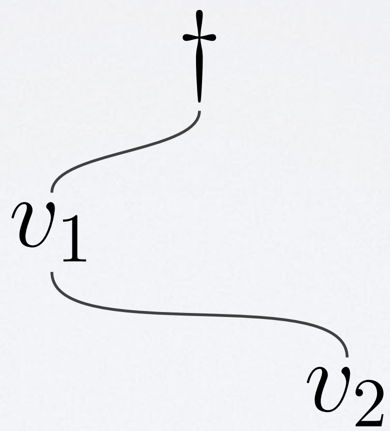
The types of GroundML are interpreted into arenas by

$$
\begin{aligned}
[\![\mathsf{unit}]\!] &= 1 \\
[\![\mathsf{int}]\!] &= \mathbb{Z} \\
[\![\mathsf{ref}\,\zeta]\!] &= \mathbb{A}_\zeta
\end{aligned}
$$

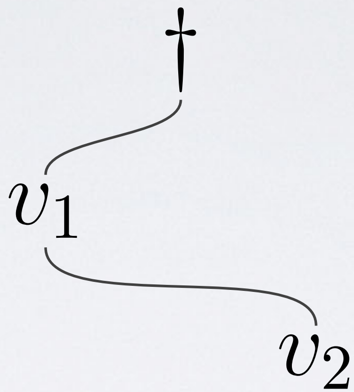
$$
\begin{aligned}
[\![\theta_1 \times \theta_2]\!] &= [\![\theta_1]\!] \otimes [\![\theta_2]\!] \\
[\![\theta_1 \to \theta_2]\!] &= [\![\theta_1]\!] \Rightarrow [\![\theta_2]\!]
\end{aligned}
$$

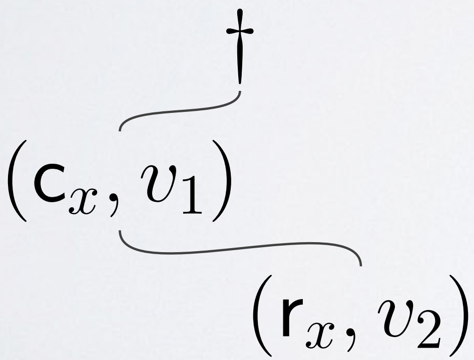
$$[\![\zeta_1 \to \zeta_2]\!] = [\![\zeta_1]\!] \Rightarrow [\![\zeta_2]\!]$$

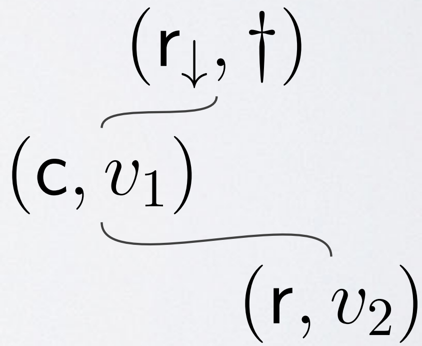
$$\zeta ::= \mathsf{unit} \mid \mathsf{int} \mid \mathsf{ref}\,\zeta$$



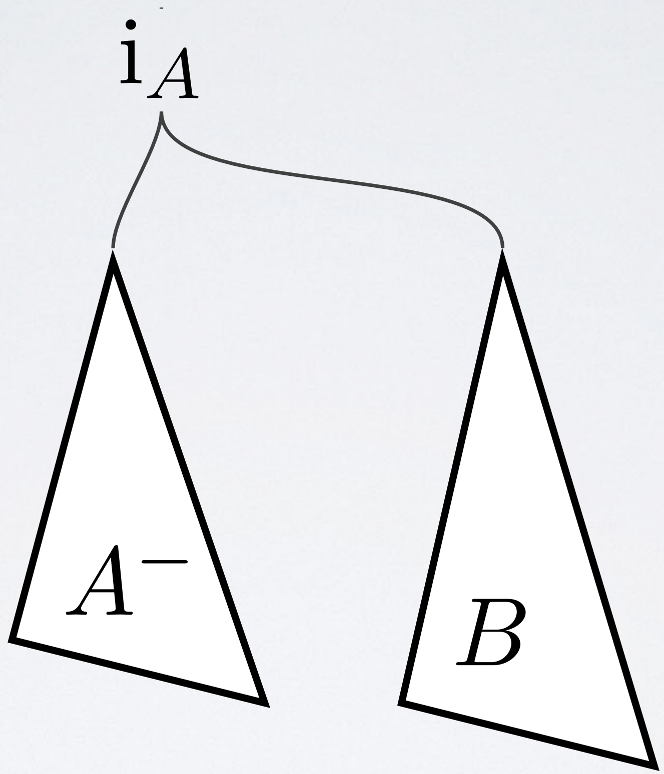
TAGGING


†
v_1
v_2


†
(c_x, v_1)
(r_x, v_2)


$(\mathsf{r}_\downarrow, \dagger)$
(c, v_1)
(r, v_2)

PLAYGROUND (PREARENA)



$$A \to B$$

TYPING JUDGMENTS

Type signatures

$$\Gamma \vdash \theta$$

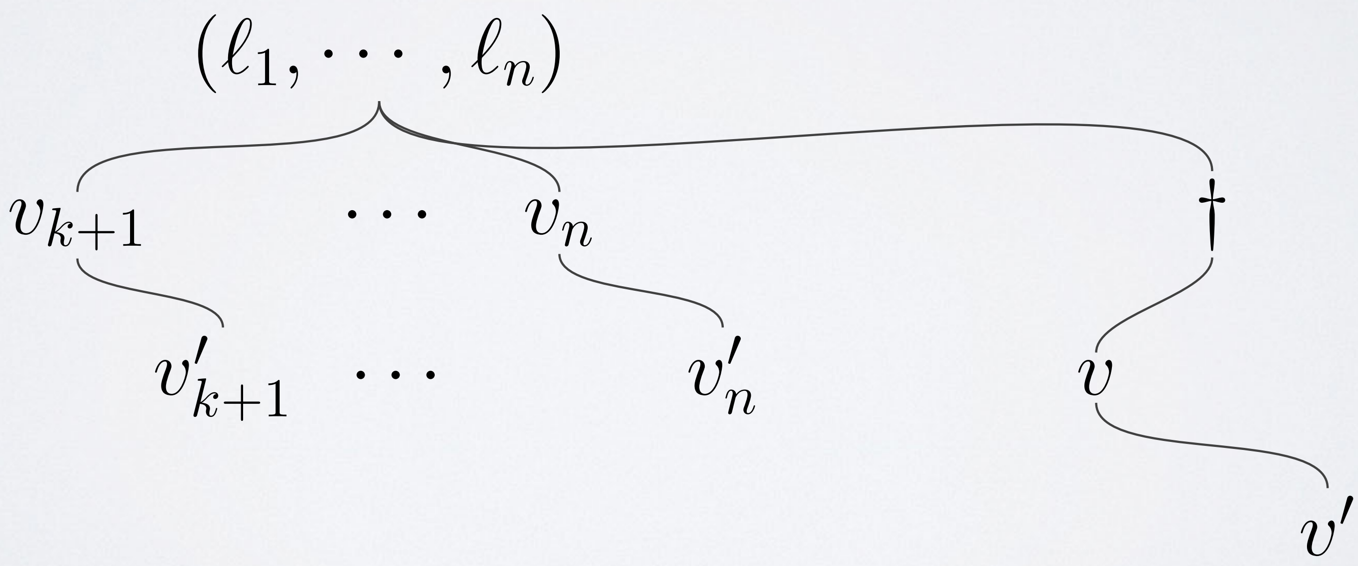
with $\Gamma = \{ x_1 : \theta_1, \cdots, x_k : \theta_k \}$ are interpreted as (the prearena)

$$[\![\theta_1]\!] \otimes \cdots \otimes [\![\theta_k]\!] \rightarrow [\![\theta]\!]$$

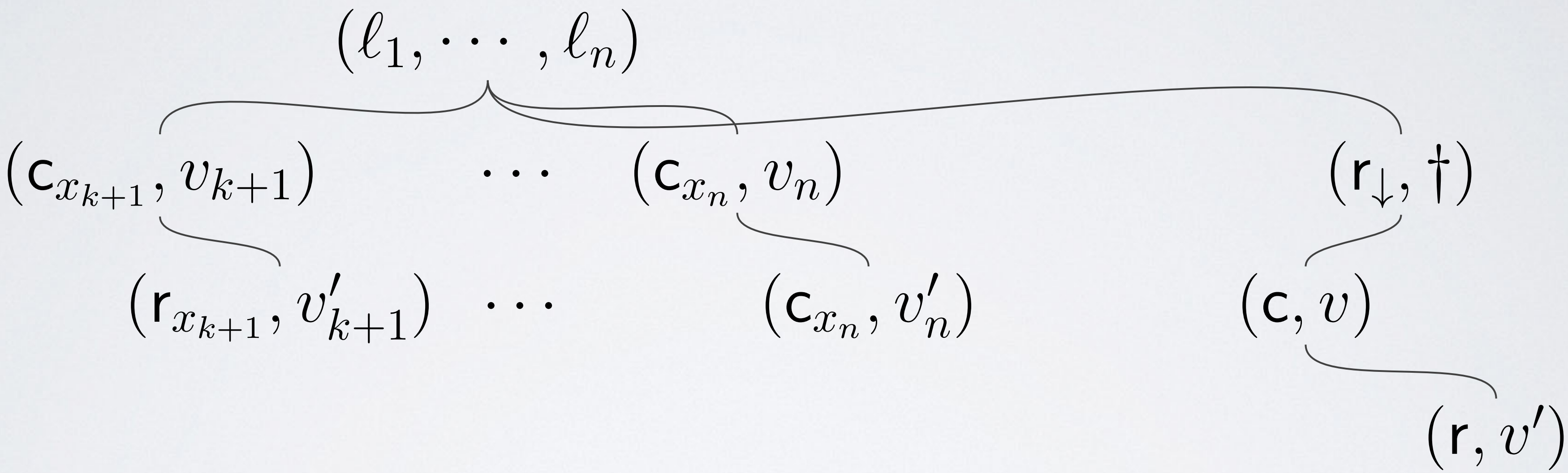
which we shall denote by $[\![\Gamma \vdash \theta]\!]$.

$$x_1 : \zeta_1, \cdots, x_k : \zeta_k,\ x_{k+1} : \zeta_{k+1} \to \zeta'_{k+1}, \cdots, x_n : \zeta_n \to \zeta'_n \vdash \ \zeta \to \zeta'$$

$$(\,[\![\zeta_1]\!] \otimes \cdots \otimes [\![\zeta_k]\!] \otimes ([\![\zeta_{k+1}]\!] \Rightarrow [\![\zeta'_{k+1}]\!]) \otimes \cdots \otimes ([\![\zeta_n]\!] \Rightarrow [\![\zeta'_n]\!])) \to \ [\![\zeta]\!] \Rightarrow [\![\zeta']\!]$$



COMPARISON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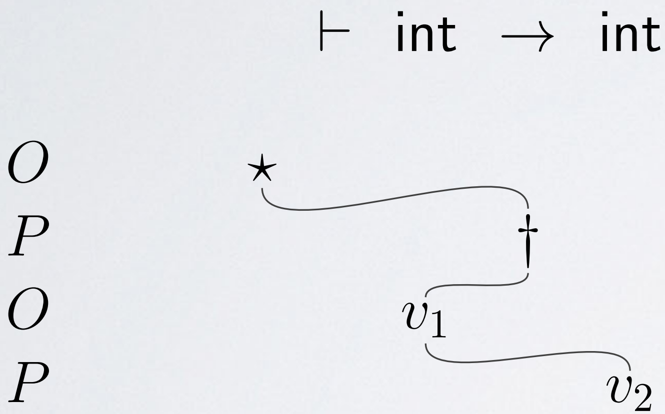
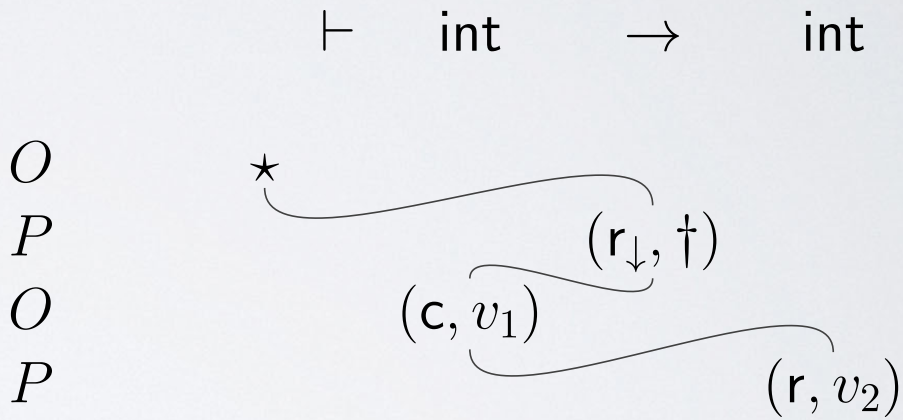
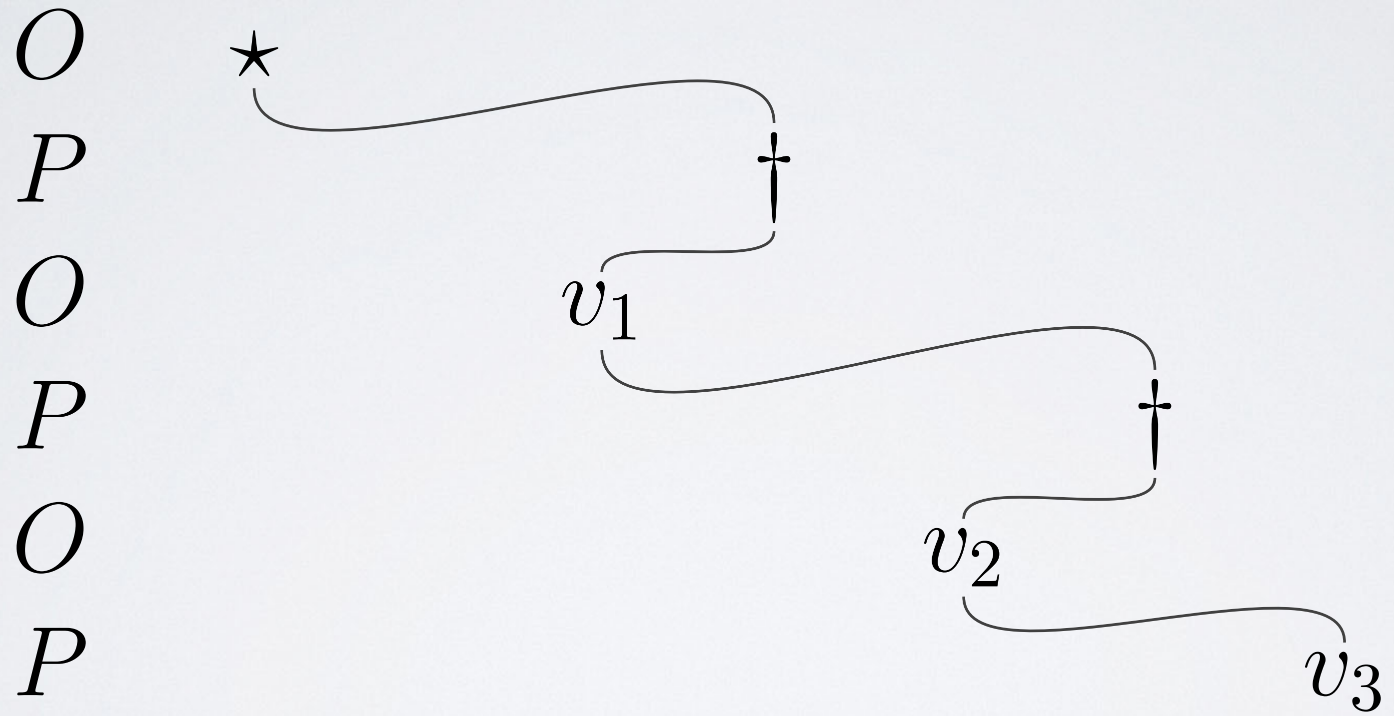
(c_{x_n}, v'_n)
(c, v)
(r, v')

TOWARDS HIGHER TYPES

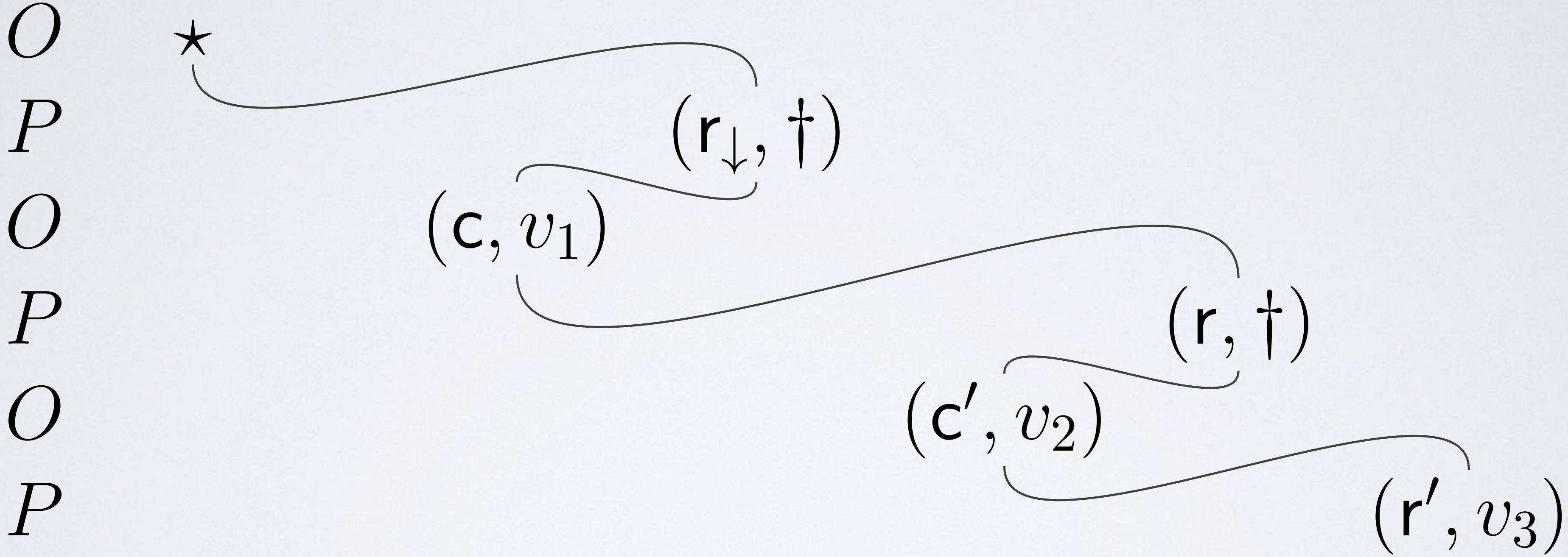




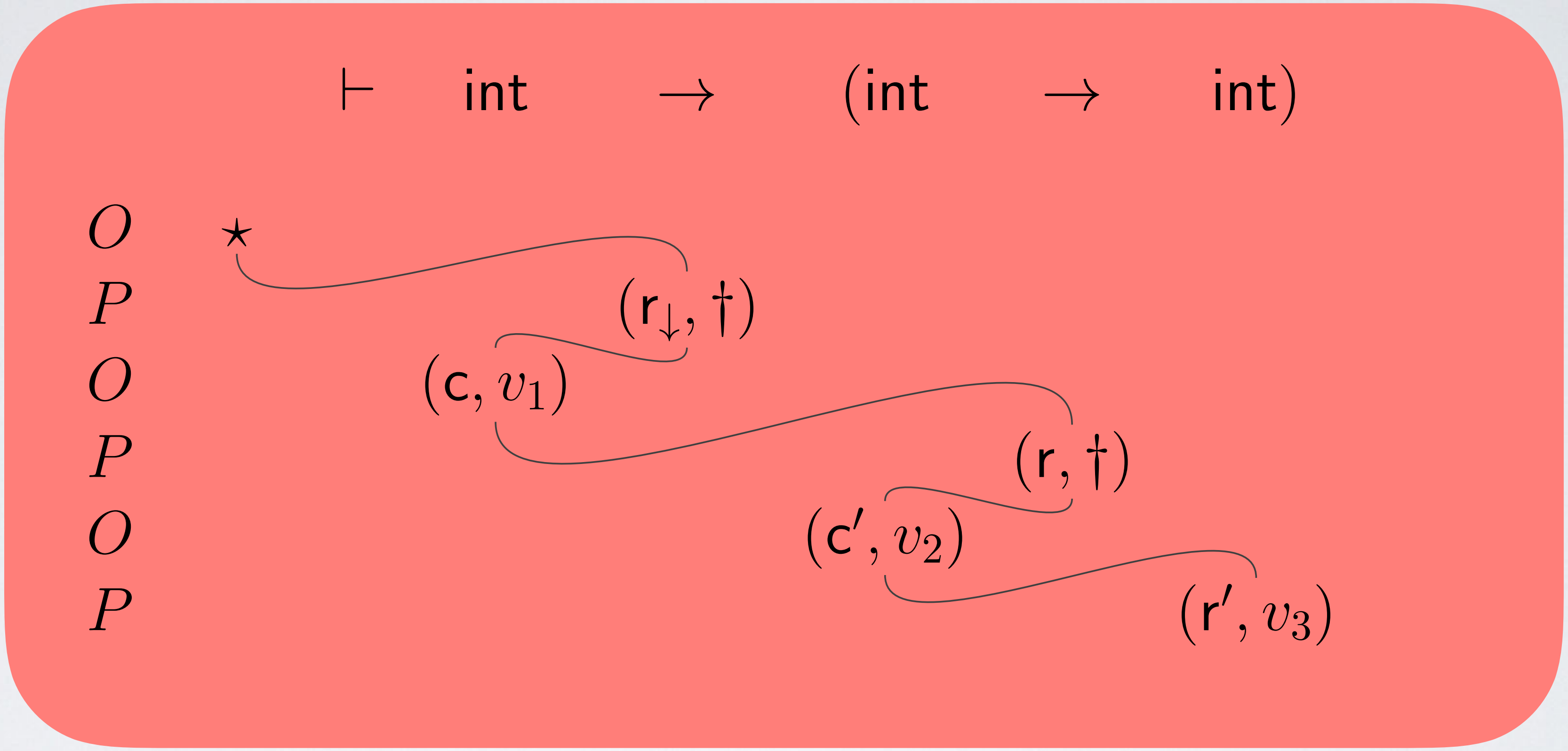
$$\vdash \mathsf{int} \to (\mathsf{int} \to \mathsf{int})$$



$$\vdash \quad \mathsf{int} \quad \rightarrow \quad (\mathsf{int} \quad \rightarrow \quad \mathsf{int})$$



AMBIGUITY



$$\dagger\ (\mathsf{r}_\downarrow, \dagger)\ (\mathsf{c}, 0)\ (\mathsf{r}, \dagger)\ (\mathsf{c}, 1)\ (\mathsf{r}, \dagger)\ (\mathsf{c}', 2)$$

DISAMBIGUATION

$$\dagger\ (\mathsf{r}_{\downarrow}, \dagger)\ (\mathsf{c}, 0)\ (\mathsf{r}, \dagger)\ (\mathsf{c}, 1)\ (\mathsf{r}, \dagger)\ (\mathsf{c}', 2)$$

$$\dagger\ (\mathsf{r}_{\downarrow}, \dagger)\ (\mathsf{c}, 0)\ (\mathsf{r}, \dagger)\ (\mathsf{c}, 1)\ (\mathsf{r}, \dagger)\ (\mathsf{c}', 2)$$

JUSTIFICATION POINTERS

$$\dagger\ (\mathsf{r}_{\downarrow}, \dagger)\ (\mathsf{c}, 0)\ (\mathsf{r}, \dagger)\ (\mathsf{c}, 1)\ (\mathsf{r}, \dagger)\ (\mathsf{c}', 2)$$

$$\dagger\ (\mathsf{r}_{\downarrow}, \dagger)\ (\mathsf{c}, 0)\ (\mathsf{r}, \dagger)\ (\mathsf{c}, 1)\ (\mathsf{r}, \dagger)\ (\mathsf{c}', 2)$$

JUSTIFIED SEQUENCES

A ***justified sequence*** on a prearena A is a sequence s of moves-with-store on A such that:

- the first move is of the form i^S with $\mathrm{i} \in I_A$;
- every other (i.e. not first) move $n^{S'}$ in s is equipped with a pointer to an earlier move m^S such that $m \vdash_A n$.

In the latter case, m is called the *justifier* of n; if n is an answer, we also say that n *answers* m.

CONSTRAINTS

- **Alternation**: players take turns, O begins.
- **Bracketing**: questions/calls and answers/returns are well-nested.
- **Visibility**: only visible functions can be called.

VISIBILITY

$$f : \mathsf{int} \to \mathsf{int} \vdash \lambda x^{\mathsf{int}}.f(1) + x : \mathsf{int} \to \mathsf{int}$$

$$s = \begin{array}{cccc} \dagger & (\mathsf{r}_{\downarrow}, \dagger) & (\mathsf{c}, 3) & (\mathsf{c}_f, 1) \\ O & P & O & P \end{array}$$

O can play (r_f, i) next but cannot play (c, i).

VIEW

The ***view*** $\ulcorner s \urcorner$ of a justified sequence s is defined by:

$$\ulcorner \varepsilon \urcorner = \varepsilon$$
$$\ulcorner s\, m^{S}\, t\, n^{S'} \urcorner = \ulcorner s \urcorner\, m^{S} n^{S'}$$

$$s = \underset{O}{\dagger} \quad \underset{P}{(\mathsf{r}_{\downarrow}, \dagger)} \quad \underset{O}{(\mathsf{c}, 3)} \quad \underset{P}{(\mathsf{c}_f, 1)}$$

$$\ulcorner s \urcorner = \dagger \quad (\mathsf{c}_f, 1)$$

FRUGALITY

Finally, the *frugality* condition controls the flow of names and in particular restricts the store to its public/available part. For each $X \subseteq \mathbb{A}$ and store S we define $S^*(X) = \bigcup_{i\in\omega} S_i(X)$, where

$$S_0(X) = X \quad \text{and} \quad S_{i+1}(X) = S(S_i(X)) \cap \mathbb{A},$$

to be the set of names that can be reached from X through S. Then, the set of ***available*** names of a justified sequence is defined inductively by:

$$\mathsf{Av}(\varepsilon) = \emptyset \qquad \text{and} \qquad \mathsf{Av}(sm^S) = S^*(\mathsf{Av}(s) \cup \nu(m)).$$

Note below that we write $s' \sqsubseteq s$ if s' is a prefix of s.

PLAY

A justified sequence s is a ***play*** if it satisfies the following conditions.

- No two adjacent moves belong to the same player (*Alternation*).
- For all $tm^S \sqsubseteq s$ with m an answer, the justifier of m is the pending question of t (*Bracketing*).
- For all $tm^S \sqsubseteq s$ with non-empty t, the justifier of m is in $\ulcorner t \urcorner$ (*Visibility*).
- For all $tm^S \sqsubseteq s$, $\mathsf{dom}(S) = \mathsf{Av}(tm^S)$ (*Frugality*).

The set of plays on A is denoted by P_A.

STRATEGIES

A ***strategy*** σ on a prearena A is a non-empty set of even-length plays of A satisfying:

- If $so^{S}p^{S'} \in \sigma$ then $s \in \sigma$ (*Even-prefix closure*).
- If $s \in \sigma$ then, for all permutations π, $\pi \cdot s \in \sigma$ (*Equivariance*).
- If $sp_1^{S_1}, sp_2^{S_2} \in \sigma$ then $sp_1^{S_1} = \pi \cdot sp_2^{S_2}$ for some permutation π (*Determinacy*).

GAME SEMANTICS

Terms

$$x_1 : \theta_1, \cdots, x_k : \theta_k \vdash M : \theta$$

are interpreted by strategies on

$$[\![\theta_1]\!] \otimes \cdots \otimes [\![\theta_k]\!] \;\to\; [\![\theta]\!]$$