TOPOS THEORY IN THE FORMULATION OF THEORIES OF PHYSICS

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1. General Relativity:
   - Gravitational field described by the geometrical and, to some extent, topological structure of space-time.
   - The philosophical interpretation is thoroughly 'realist'. GR is the ultimate classical theory!
2. Quantum theory:
   - Normally works *within* a fixed, background space-time.
   - Interpretation is 'instrumentalist' in terms of what would happen *if* a measurement is made.
   - What do such ideas mean if applied to space and time themselves?

The Planck Length

Presumably something dramatic happens to the nature of space and time at $L_P := \sqrt{\frac{G\hbar}{c^3}} \simeq 10^{-35}m \simeq 10^{-42}secs$.

- What?
- Main programmes are string theory and loop quantum gravity. Both suggest a 'discrete' space-time structure.

  The best, simple example of such a theory is *causal sets*.

It is often asserted that classical space and time 'emerge' from the formalism in some limit.

Thus a fundamental theory may have no intrinsic reference at all to spatio-temporal concepts.

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1. The *a priori* use of real numbers:

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2. The interpretational issues: *instrumentalism* versus *realism*.

   We want to talk about 'the way things are' in regard to space and time.

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(i) as the values of *physical quantities*;

(ii) as the values of *probabilities*;

(iii) as a fundamental ingredient in mathematical models of *space and time.*

The use of $\mathbb{R}$ (and $\mathbb{C}$) in standard quantum theory is a reflection of (i) and (ii); and, indirectly, of (iii) too.

2. Why are Physical Quantities Assumed Real-Valued?

Traditionally, quantities are measured with rulers and pointers.

- Thus there is a direct link between the 'quantity-value space' and the assumed structure of *physical* space.

  [Caution: This uses *instrumentalist* interpretation of QT]

- Thus we have a potential 'category error' at L_P: if physical space is not based on $\mathbb{R}$, we should not assume *a priori* that physical quantities are real-valued.

If the quantity-value space is *not* $\mathbb{R}$, then what is the status of the Hilbert-space formalism?

3. Why Are Probabilities Assumed Real Numbers?

Relative-frequency interpretation: $\frac{N_i}{N}$ tends to $r \in [0,1]$ as $N \to \infty$.

- This statement is instrumentalist. It does not work if there is no classical spatio-temporal background in which measurements could be made.

- In 'realist' interpretations, probability is often interpreted as *propensity* (*latency*, *potentiality*).

  – But why should a propensity be a real number in $[0,1]$?

  – Minimal requirement is, presumably, an ordered set, but this need not be *totally* ordered.

The Big Problem

Standard QT is grounded in Newtonian space and time.

How can the formalism be modified, or generalised, so as (i) to be 'realist'; and (ii) not to be dependent *a priori* on real and complex numbers?

- For example, if we have a given causal-set background $\mathcal{C}$, what is the quantum formalism that is *adapted* to $\mathcal{C}$?
- Very difficult: usual Hilbert-space formalism is very rigid.

  There have been some studies using finite fields, but they are rather artificial.

What *are* the basic principles of a 'quantum theory', or beyond?

III. Formulation of Theories of Physics

1. The Realism of Classical Physics:

- A physical quantity A is represented by a function $\tilde{A}: \mathcal{S} \to \mathbb{R}$.

  A state $s \in \mathcal{S}$ specifies 'how things are': i.e., the value of any physical quantity A in that state is $\tilde{A}(s) \in \mathbb{R}$.

- Hence, a proposition "$A \in \Delta$" is represented by the subset $\tilde{A}^{-1}(\Delta) \subseteq \mathcal{S}$.

Thus, because of the structure of set theory, of necessity, the propositions in classical physics form a *Boolean logic*.

The collection of such propositions forms a *deductive system*: i.e., there is a sequent calculus for constructing proofs.

2. The Failure of Realism in Quantum Physics

Kochen-Specker theorem: it is impossible to assign consistent true-false values to all the propositions in quantum theory.

Equivalently: it is not possible to assign consistent values to all the physical quantities in a quantum theory.

Conclusion:

- There is ‘no way things are’.
- Instead an *instrumentalist* interpretation is used.

Theories of a Physical System

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1. *Classical theory* of S:
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   - $A \rightsquigarrow \tilde{A} : \mathcal{S} \to \mathbb{R}$
   - "$A \in \Delta$" $\rightsquigarrow \tilde{A}^{-1}(\Delta) \subseteq \mathcal{S}$; gives Boolean lattice.

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    - $S \rightsquigarrow \mathcal{H}$ — a Hilbert space
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   - "$A \in \Delta$" $\rightsquigarrow \hat{E}[A \in \Delta]$; gives non-distributive lattice.
3. *Category theory* of S in a category τ:
   - $S \rightsquigarrow \Sigma$ — an object in τ
   - $A \rightsquigarrow \breve{A} : \Sigma \to \mathcal{R}$
   - "$A \in \Delta$" $\rightsquigarrow$ a sub-object of Σ?

IV. Introducing Topos Theory

Does such ‘categorification’ work?

1. Not in general: usually, sub-objects of an object do not have a logical structure. However, they *do* in a *topos*!
2. A topos is a category that ‘behaves much like **Sets**’. In particular there are:
   - $0, 1$; pull-backs & push-outs (hence, products & co-products)
   - Exponentiation:
   $$\mathrm{Hom}(C, A^B) \simeq \mathrm{Hom}(C \times B, A)$$
   - A ‘sub-object classifier’, Ω: to any sub-object A of B, $\exists \chi_A : B \to \Omega$ such that $A = \chi_A^{-1}(1)$.

The Logical Structure of Sub-objects

In a topos:

1. The collection, $\mathrm{Sub}(A)$, of sub-objects of an object A forms a *Heyting algebra*.
2. The same applies to $\Gamma\Omega := \mathrm{Hom}(1, \Omega)$, 'global elements'

A Heyting algebra is a distributive lattice, $\mathfrak{H}$, with 0 and 1, and such that to each $\alpha, \beta \in \mathfrak{H}$ there exists $\alpha \Rightarrow \beta \in \mathfrak{H}$ such that

$$\gamma \preceq (\alpha \Rightarrow \beta) \text{ iff } \gamma \wedge \alpha \preceq \beta.$$

- Negation is defined as $\neg\alpha := (\alpha \Rightarrow 0)$.
- *Excluded middle* may not hold: there may exist $\alpha \in \mathfrak{H}$ such that $\alpha \vee \neg\alpha \prec 1$.

  Equivalently there may be β such that $\beta \prec \neg\neg\beta$.

The Mathematics of 'Neo-Realism'

- **In set theory:** let $K \subseteq X$ and $x \in X$. Consider the proposition "$x \in K$". The truth value is

$$\nu(x \in K) = \begin{cases} 1 & \text{if } x \text{ belongs to } K; \\ 0 & \text{otherwise.} \end{cases}$$

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  Thus "$x \in K$" is true if, and only if, x belongs to K.

- **In a topos:** a proposition can be only 'partly true':

  Let $K \in \mathrm{Sub}(X)$ with $\chi_K : X \to \Omega$ and let $x \in X$, i.e., $\ulcorner x \urcorner : 1 \to X$ is a global element of X. Then

$$\nu(x \in K) := \chi_K \circ \ulcorner x \urcorner$$

  where $\chi_K \circ \ulcorner x \urcorner : 1 \to \Omega$. Thus the 'generalised truth value' of "$x \in K$" belongs to the Heyting algebra $\Gamma\Omega$.

  This represents a type of 'neo-realism'.

Our Main Contention

For a given theory-type, each system S to which the theory is applicable can be formulated and interpreted within the framework of a particular topos $\tau_\phi(S)$.

Conceptually, this structure is 'neo-realist' in the sense:

1. A physical quantity, A, is represented by an arrow $A_{\phi,S} : \Sigma_{\phi,S} \to \mathcal{R}_{\phi,S}$ where $\Sigma_{\phi,S}$ and $\mathcal{R}_{\phi,S}$ are two special objects in the topos $\tau_\phi(S)$.
2. Propositions about S are represented by sub-objects of $\Sigma_{\phi,S}$. These form a Heyting algebra.
3. The topos analogue of a state is a 'truth object'. Propositions are assigned truth values in $\Gamma\Omega_{\tau_\phi(S)}$.

- Thus a theory expressed in this way *looks* like classical physics except that classical physics always employs the topos **Sets**, whereas other theories—including quantum theory—use a different topos.

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- A topos can be used as a *foundation* for mathematics itself, just as set theory is used in the foundations of 'normal' (or 'classical') mathematics.

- In fact, any topos has an 'internal language' that is similar to the formal language on which set theory is based.

  This internal language is used to *interpret* the theory in a 'neo-realist' way.

The Idea of a Truth Object

In classical physics, a truth value is assigned to propositions by specifying a micro-state, $s \in \mathcal{S}$. Then, the truth value of "$A \in \Delta$" is

$$\nu(A \in \Delta; s) = \begin{cases} 1 & \text{if } \tilde{A}(s) \in \Delta; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

- But: in a topos, the state object $\Sigma_{\phi,S}$ may have no global elements.

  For example, this is the case for the 'spectral presheaf' in quantum theory.

- So, what is the analogue of a state in a general topos?

- **In classical physics:** Let T be a collection of sub-sets of $\mathcal{S}$; i.e., $T \subseteq P\mathcal{S}$, or, equivalently, $T \in PP\mathcal{S}$. Then

$$
\begin{aligned}
\nu(A \in \Delta; T) &= \begin{cases} 1 & \text{if } \{s \in \mathcal{S} \mid \tilde{A}(s) \in \Delta\} \in T; \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \tilde{A}^{-1}(\Delta) \in T; \\ 0 & \text{otherwise} \end{cases}
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- The two notions of a truth object coincide if we define

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- **For a general topos**: a truth object is $T \in PP\Sigma_{\phi,S}$. Then, if $K \in \mathrm{Sub}(\Sigma_{\phi,S})$, $\ulcorner K \urcorner : 1 \to P\Sigma_{\phi,S}$, we have $\nu(K; T) \in \Gamma\Omega_{\phi,S}$.

V. Formal Languages

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- We want to allow for a logic that is not Boolean, but still gives a deductive system. We choose *intuitionistic* axioms for the language.
- Equivalently, we construct a *translation* of $\mathcal{L}(S)$ into the internal language of the topos.

The Language $\mathcal{L}(S)$

The language $\mathcal{L}(S)$ of a system S is *typed*. It includes:

- A symbol Σ: the linguistic precursor of the state object.
- A symbol $\mathcal{R}$: the linguistic precursor of the quantity-value object.
- A set, $F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$ of 'function symbols' $A : \Sigma \to \mathcal{R}$: the linguistic precursors of physical quantities.
- A symbol Ω: the linguistic precursor of the sub-object classifier.
- A 'set builder' $\{\tilde{x} \mid \omega\}$. This is a term of type PT, where $\tilde{x}$ is a variable of type T, and ω is a term of type Ω.

Representing the Language $\mathcal{L}(S)$

Next step: find a representation of $\mathcal{L}(S)$ in a suitable topos.

A classical theory of S: The representation σ is:

- The topos $\tau_\sigma(S)$ is **Sets**.
- Σ is represented by a symplectic manifold $\Sigma_{\sigma,S}$ (was $\mathcal{S}$).
- $\mathcal{R}$ is represented by the real numbers $\mathbb{R}$; i.e., $\mathcal{R}_{\sigma,S} := \mathbb{R}$.
- The function symbols $A : \Sigma \to \mathcal{R}$ become functions $A_{\sigma,S} : \Sigma_{\sigma,S} \to \mathbb{R}$ (was $\tilde{A}$)
- Ω is represented by the set $\{0, 1\}$ of truth values.

The Topos of Quantum Theory

- The key ingredient of normal quantum theory on which we focus is the intrinsic *contextuality* implied by the Kocken-Specher theorem.

- In standard theory, we can potentially assign 'actual values' only to members of a commuting set of operators.

  We think of such a set as a *context* or 'classical snapshot' of the system.

- This motivates considering the topos of presheaves over the category of abelian subalgebras of $\mathcal{B}(\mathcal{H})$. This category is a partially-ordered set under the operation of sub-algebra inclusion.

- The state object that represents the symbol Σ is the '*spectral presheaf* $\underline{\Sigma}$.

  1. For each abelian subalgebra V, $\underline{\Sigma}(V)$ is spectrum of V.
  2. The K-S theorem is equivalent to the statement that $\underline{\Sigma}$ has no global elements.
  3. $\underline{\Sigma}$ replaces the (non-existent) state space.
  4. A proposition represented by a projector $\hat{P}$ in QT is mapped to a sub-object $\delta(\hat{P})$ of $\underline{\Sigma}$. We call this '*daseinisation*'.

- The quantity-value symbol $\mathcal{R}$ is represented by a presheaf $\underline{\mathbb{R}^{\succeq}}$. This is *not* the real-number object in the topos.

- Physical quantities represented by arrows $\breve{A} : \underline{\Sigma} \to \underline{\mathbb{R}^{\succeq}}$.

  They are constructed from the Gel'fand transforms of the spectra in $\underline{\Sigma}$

VI. Conclusions

1. General considerations of quantum gravity suggest the need to go 'beyond' standard quantum theory:
   1.1 Must escape from *a priori* use of $\mathbb{R}$ and $\mathbb{C}$.
   1.2 Need a 'realist' interpretation (K-S not withstanding)
2. Main idea: construct theories in a topos other than **Sets**.
   2.1 A physical quantity, A, is represented by an arrow $A_{\phi,S} : \Sigma_{\phi,S} \to \mathcal{R}_{\phi,S}$ where $\Sigma_{\phi,S}$ and $\mathcal{R}_{\phi,S}$ are special objects in the topos $\tau_\phi(S)$.
   2.2 The interpretation is 'neo-realist' with truth values that lie in the Heyting algebra $\Gamma\Sigma_{\phi,S}$. Propositions are represented by elements of Heyting algebra $\mathrm{Sub}(\Sigma_{\phi,S})$
3. Our scheme involves representing language $\mathcal{L}(S)$ in $\tau_\phi(S)$.