project no 033763

QICS

Foundational Structures for Quantum Information and Computation

Specific Targeted Research Project (STREP)

Thematic priority: Quantum Information Processing and Communications

DELIVERABLE D2

Categorical semantics, logics and diagrammatic methods

report for first reporting period

Feb. 15th 2008

Chancellor, Masters and Scholars of the University of Oxford

Contents

A current account on the objectives of W2 and comparison with the state-of the art. Categorical semantics for quantum mechanics, the corresponding logics and the resulting diagrammatic methods were entirely being carried out by QICS participants at the start of the proposal: the pioneers of this approach are all based at QICS sites or at QICS affiliated sites. This has slightly changed now; several other groups have taken up our approach and in particular several PhD students elsewhere are now being trained in these mathematical methods, which are highly non-standard in quantum information. The interest in this approach has grown substantially since the start of QICS — which is witnessed by the exceptional number of invited talks by its originators. In the light of this growing interest, given the high 'entry cost' of working in this area, a series of volumes is currently being produced by the QICS team, which comprises tutorial chapters on the use of categorical, domain theoretic and other logic tools in quantum informatics [New structures for Physics. Lecture Notes in Physics. Springer-Verlag. (forthcoming, 2008)]. In a separate development, a volume of chapters on computer science methods [Semantic techniques in Quantum computing. Cambridge UP. (forthcoming, 2008)] is also in preparation.

Main developments in W2. Our developments have been specifically focussed on problems and requirements arising from the other workpackages of QICS. This mainly involved refining the axiomatic setting to accommodate all the essential features of the novel quantum computational models and quantum computational paradigms, with a particular focus on measurement-based quantum computing, quantum classical interaction, entanglement as a computational resource, and topology as an (abstract) computational resource. The main objective of this year was to get into sufficiently high gear such that the categorical semantics, the corresponding logics and the resulting diagrammatic methods would provide a powerful vehicle to solve some hard problems of the other workpackages and tasks. The key result obtained are as follows:

CQ We now have a *comprehensive axiomatic account of both quantum and classical data within the diagrammatic language* (**6.1.b**, **6.1.e** in W2.T1). Important insights are starting be gained on the required structural resources which distinguish different measurement-based quantum computational models. E.g. the teleportation model requires less structure than state transfer while it requires more physical qubits to be realised. This seems to point a remarkable trade-off between structure and number of systems. This is crucial for the applications in measurement-based quantum computational models which require classically controlled correction operations. The way this was achieved was by making the ability to copy and uniformly erase classical data (contra quantum data) explicit as a feature.

CO We now have a *axiomatic and diagrammatic account of complementary observables* (**6.1.c** in W2.T1) which enables abstract simulation of elementary gate computations. To put this in some historical perspective, axiomatic approaches to quantum mechanics emphasised *negative* features of quantum theory which followed from the existence of complementary observables: non-distributivity of propositional lattices and non-commutativity of spectral algebras. Categorical axiomatics changed this attitude by making quantum 'features' explicit. The categorical account of mutually unbiassed bases turns the source of the negative quantum structural paradigms now into a posistive and expressive feature. Due to its high-level nature this structure is also promising for automation (**6.1.f** in W2.T1).

Besides the direct developments there were two important strands of study of categorical structures which will be of importance for internal use within this workpackage:

CL We now have a *comprehensive high-level representation for all gates and multipartite entangled states* (**6.1.c**, **6.1.d** in W2.T1 and **6.2.1.a** in W2.T2). this is a further elaboration on **CO**. Interesting results on translations between measurement based quantum computational models (see the picture in **6.1.d**) and a classification of multi-partite entangled states in terms of their informatic capabilities, and which refines SLOCC classification, have started to emerge.

CM *Several new categorical models for particular features of quantum informatics have been produced* (**6.1.1** in W2.T1). Models are very important to guide the development of categorical axiomatics as well as to make computational features such as recursion, cycles, coding-decoding, etc., available within categorical axiomatics.

CL *Steps toward a categorical axiomatics for topological quantum computing have been made* (**6.3.a** and **6.3.b** in W2.T3). This mainly involved the study of functorial and other structural properties of the topologies involved. The connections with Temperley-Lieb algebra in **6.3.a** bring within reach the powerful results from representation theory

The next steps to take. We are very much progressing according to schedule. We reached milestone W2.M1 within the first year of activity for this workpackage and have made substantial progress towards the other milestones. Due to the available postdocs there will be been a trade-off between activity W2.M2 on topological quantum computing (QICS will hire a postdoc on this topic from September 08 on; currently he is finishing his PhD) and W2.M2 on a logical understanding of multipartite behaviour, on which two QICS postdocs have been active, and hence has seen substantially more progress then indicated in the original plan. The next step on the logic of multi-partite entanglement will be the passage from pure to mixed states; this passage is categorically well-understood (see **6.1.g**). So the only change to the original plan is the fact that W2.M2:(12)$\mapsto$(36)

and W2.M6:(36)↦(24). Due to availability of the appropriate postdoc will we also be able to extend our range of quantum computational models with adiabatic quantum computing; something we indicated as a desirable possibility in the QICS abstract. An important activity will also be automated design and analysis of quantum informatic protocols. Tools are currently under development (**6.1.f** in W2.T1).

Interactions with other workpackages and sites. This workpackage is inherently intertwined with the other ones as is clear from our discussion above. It suffices to observe that the results of W2 figure in each of the reports on the work performed in W1, W3 and W4.

Samson Abramsky and Bob Coecke
Oxford, February 10, 2008.

Workpackage objectives:

W2.O1 Find simple intuitive graphical calculi and more conceptually motivated constructions and proofs to replace the highly non-intuitive definitions and manipulations in terms of matrices.

W2.O2 Expose the foundational structure and axiomatic boundaries of QIC.

W2.O3 Study the structure of multipartite entanglement and distributed quantum systems.

W2.O4 Exploit the above for automated design and verification for algorithms and protocols.

W2.O5 Contribute to the quest of a general model for QIC by studying the topological QC model.

Workpackage milestones:

W2.M1 A comprehensive graphical calculus which captures a substantial fragment of QIC. (12)

W2.M2 Structural insights in the topological quantum computational model. (12)

W2.M3 A logical understanding of distributed quantum systems. (24)

W2.M4 Powerful methods arising from a category-theoretic axiomatic framework. (24)

W2.M5 A simple axiomatic framework which captures the different quantitative quantum-informatic concepts. (36)

W2.M6 A logical understanding of multipartite behavior, including graph states. (36)

Below we discuss the detailed progress for this workpackage which comprises the *workpackage tasks*:

W2.T1 Develop categorical semantics, logics and diagrammatic methods for general QIC; apply these to the problems posed in other workpackages.

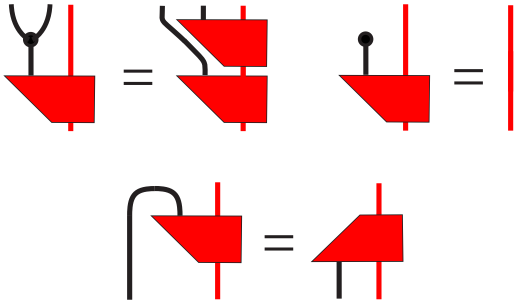
W2.T2 Study the structure of multi-partite entanglement using categorical methods and others; combine quantum structure and spatio-temporal structure.

W2.T3 Study the structure of the topological quantum computational model from the point of view of categorical semantics; build categorical semantics for the knot-theoretic models.

1 Progress towards objectives and performed tasks for W2.T1

6.1.a Ross Duncan's PhD thesis (Objectives: W2.O1, W2.O3, W2.O4; Milestones: W2.M1, W2.M4). In [1] Duncan (QICS postdoc at Ox) developed a general categorical formalism which provides axiomatics for reasoning about entangled systems in the broadest possible generality. Representations of freely constructed abstract quantum theories – precisely, dagger compact categories with certain unitary generators – are constructed. Formal calculi (of an essentially graphical nature) are developed which permit the evolution of quantum systems to be modelled. As an application, the system is used to provide formal rewrites to support reasoning about measurement based quantum computing, and in particular, translations between measurement based and quantum circuit computations.

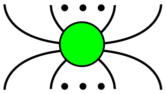
6.1.b A resource-sensitive category theoretic account and diagrammatic calculus for classical data and quantum observables (Objectives: W2.O1, W2.O2; Milestones: W2.M1, W2.M4, W2.M5). In [2] Coecke (Ox) and Pavlovic (Ox) suggest that quantum mechanics can be done without any notion of sum, expressed entirely in terms of the tensor product. The corresponding axioms define classical spaces as objects that allow copying and deleting data. They show that the information exchange between the quantum and the classical worlds is essentially determined by their distinct capabilities to copy and delete data. The sums turn out to be an implicit implementation of this capability. Realizing it through explicit axioms not only dispenses with the unnecessary structural baggage, but also allows a simple and intuitive graphical calculus. In category-theoretic terms, classical data types are dagger-compact Frobenius algebras, of which the data consists of a copying operation $\delta : X \to X \otimes X$ and a deleting operation $\epsilon : X \to \mathrm{I}$, both 'extended by linearity to quantum states'. The algebraic laws for these intuitively capture the nature of copying and deleting. The quantum spectra underlying quantum measurements are exactly dagger Eilenberg-Moore coalgebras induced by these Frobenius algebras. They show that the corresponding three equations, which graphically depict as:



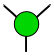
are necessary and sufficient to characterise all projective quantum measurements within the category of $\mathbf{FdHilb}$ of finite dimensional Hilbert spaces, linear maps and tensor product. Moreover, each of these rules admits a clear operational interpretation.

Each base in a Hilbert space canonically induces a dagger-compact Frobenius algebra ($|i\rangle \mapsto |ii\rangle, |i\rangle \mapsto 1$). Recently, Coecke (Ox), Pavlovic (Ox) and Vicary showed that the coverse is also true: each dagger-compact Frobenius algebra arises in this way. From this one can derive that the dagger-compact Frobenius algebras in the category $\mathbf{FdHilb}$ together with comonoid homomorphisms and inherited monoidal structure, is equivalent to the category of sets, functions and the cartesian product. This procedure establishes a new kind of classical limit: classical process structure is a restriction of the quantum process structure in the sense that classical processes have enhanced capabilities relative to copying and deleting. A paper on this is in preparation.

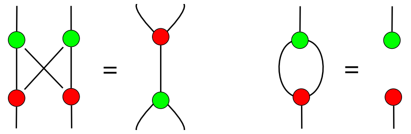
In [3] Coecke (Ox) and Paquette (McGi affiliated & forthcoming QICS postdoc at Ox) build further on the work in [2] by providing an abstract purely compositional characterisation of POVMs. They show that at this highly level Naimark's dilation theorem still holds, and provide a purely graphical proof of this. Also in [3] Coecke (Ox) and Paquette (McGi affiliated & forthcoming QICS postdoc at Ox) prove that all morphism generated from dagger-compact Frobenius algebra structure on an object X and the dagger symmetric monoidal structure, which have the same domain and codomain, and for which the graphical representation is connected, must coincide. Hence, when taking all but the symmetry natural isomorphism of the symmetric monoidal structure to be strict, such a morphism only depends on the object X and it's number of inputs and outputs. This result admits a very convenient diagrammatic interpretation in the sense that each such connected network obtains a 'spider'-normal form



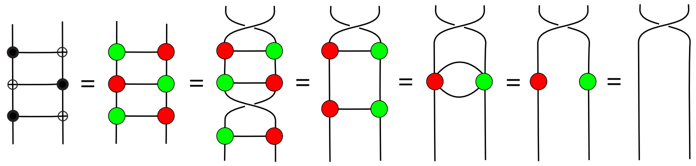
which is obtained by fusing dots which diagrammatically represent δ and ϵ:

 $:= \delta_Z :: |i\rangle \mapsto |ii\rangle$  $:= \epsilon_Z :: |i\rangle \mapsto 1$

6.1.c Complementary observables as bialgebras (Objectives: W2.O1, W2.O2, W2.O3; Milestones: W2.M1, W2.M4, W2.M6). In [4] Coecke (Ox) and Duncan (QICS postdoc at Ox) formalise the constructive content of an essential feature of quantum mechanics: the interaction of incompatible quantum observables. Axiomatic approaches to quantum mechanics have historically focussed on the failure of various properties: non-commutative algebras, non-distributive lattices, non-Kolmogorovian probabilities, etc. More recent attempts originating in quantum computer science, concentrate mainly on accommodating no-cloning and no-deleting. Coecke (Ox) and Duncan (QICS postdoc at Ox) take a positive attitude: they identify the flow of information between incompatible observables. Using a general categorical formulation, and in particular relying on the structure introduced in [2], they show that a pair mutually unbiassed quantum observables forms a bialgebra-like structure; additional structure enables them to derive all observables and define a computationally universal system. The resulting equations suffice to perform intuitive computations with elementary quantum gates, translate between distinct quantum computational models, establish the equivalence of entangled quantum states, and simulate quantum algorithms such as the quantum Fourier transform. This formalism moreover admits a purely diagrammatic calculus in which the basic laws depict as



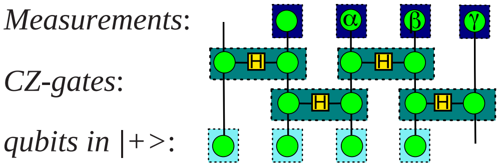
which lead to computations such the reduction of three cnot gates to symmetry:



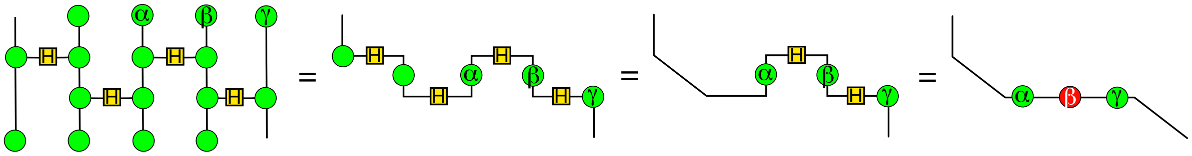
We can also introduce phase data which still obeys a ‘generalised spider theorem’:

resulting in a language which is rich enough to represent all states, operations and measurements which can be described in the Hilbert space quantum formalism, and diagrammatically reason about them.

6.1.d Diagrammatic accounts on measurement-based quantum computing I: the conditional case (Objectives: W1.O5, W2.O1, W2.O2; Milestones: W1.M4, W2.M1, W2.M4). In [4] Coecke (Ox) and Duncan (QICS postdoc at Ox) provided an elegant diagrammatic schemes to reason about measurement based quantum computational schemes and illustrated this on Perdrix’ state transfer and the one-way model. For example, that the network

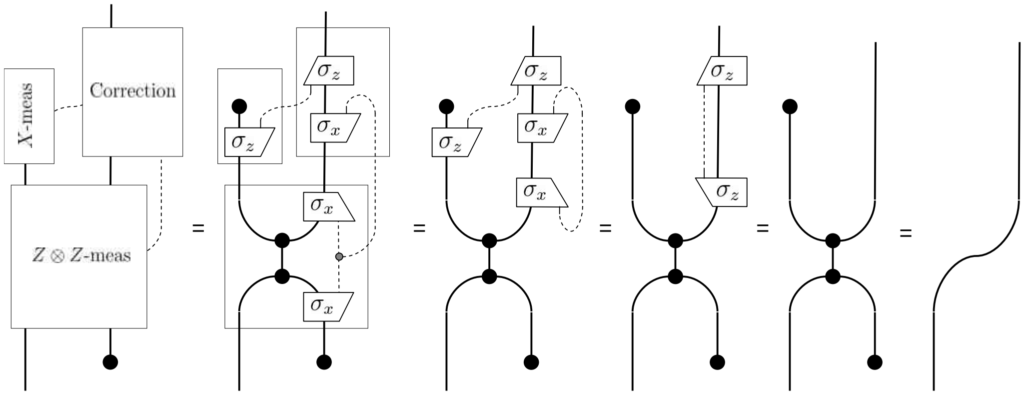


simulate an arbitrary qubit unitary can be proved like this



A paper specifically focussing on the applications of this language to measurement-based quantum computing is in preparation.

6.1.e Diagrammatic accounts on measurement-based quantum computing II: classical control (Objectives: W1.O5, W2.O1, W2.O2, W3.O4, W3.O5; Milestones: W1.M4, W2.M1, W2.M4, W3.M4, W3.M5). In a complementary strand of research Coecke (Ox), Paquette (McGi affiliated & forthcoming QICS postdoc at Ox) and Perdrix (Gren & Paris & QICS postdoc at Ox) focus on the classical-quantum interaction and the minimal (diagrammatic) structural requirements to prove correctness for measurement based quantum computational schemes including classical control. For Perdrix’ state transfer we have



A paper is in preparation; some basic cases were already considered in [5] by Coecke (Ox), Pavlovic (Ox) and Paquette (McGi affiliated & forthcoming QICS postdoc at Ox).

6.1.f An automated tool to reason about complementary observables, (Objectives: W2.O4; Milestones: W2.M4). Duncan (QICS postdoc at Ox), Kissinger (Ox) and Dixon (Ox affiliated) have partially implemented an automated tool for reasoning about quantum processes and entangled states. This is a semi-automatic rewriting engine with a GUI based on the graphical formalism of Coecke (Ox) and Duncan (QICS postdoc at Ox). A paper is preparation; development of the tool continues.

6.1.g Axiomatics for complete positivity (Objectives: W2.O2; Milestones: W2.M4, W2.M5). In [7] Coecke (Ox) shows that given any dagger symmetric monoidal category $\mathbf{C}$ we can construct a new category $\mathbf{Mix}(\mathbf{C})$, which, in the case that $\mathbf{C}$ is a †-compact category, is isomorphic to Selinger's $\mathbf{CPM}(\mathbf{C})$. Hence, if $\mathbf{C}$ is the category $\mathbf{FdHilb}$ we exactly obtain completely positive maps as morphisms. This means that mixedness of states and operations is a concept which can exist independently of compactness. Moreover, since our construction does not require †-compactness, it can be applied to categories which have infinite dimensional Hilbert spaces as objects. Finally, in general $\mathbf{Mix}(\mathbf{C})$ is not a †-category, so does not admit a notion of positivity. This means that, in the abstract, the notion of 'complete positivity' can exist independently of a notion of 'positivity', which points at a very unfortunately terminology.

1.1 Categorical models

6.2.a A categorical account on Rob Spekkens' toy model (Objectives: W1.O1, W2.O2; Milestones: W2.M5). Coecke (Ox) and Edwards (Ox) provided a very simple category-theoretic presentation of Rob Spekkens' toy model. It suffices to specify the appropriate category, which can be done by specifying only 3 generators, to obtain the model, which is substantially simpler than Spekkens'. All quantum phenomena follow by purely abstract categorical reasons. We expect to get useful insights on the required concrete categories for discrete models of quantum reasoning. A paper is in preparation.

6.2.b Categories and domains of partial isometries (Objectives: W2.O2, W2.O4; Milestones: W2.M4). In [8] Hines (QICS postdoc at York) and Braunstein (York) study of partial isometries in order-theoretic and domain-theoretic terms. In order-theoretic terms, they consider a partial order on partial isometries introduced by Halmos-McLaughlin, and show that this is a natural generalisation of the orthomodular lattice ordering used in Birkhoff - von Neumann quantum logic. From a categorical point of view, it is well-known that the composite of two partial isometries is not itself a partial isometry. However, we introduce an alternative composition (based on Girard's execution formula) that allows us to define a category of partial isometries. We show that this composite is given by a supremum in the Halmos-McLaughlin partial ordering. The resulting category is shown to be an inverse category, and hence has a 'natural partial order' on homsets derived from the inverse structure. This natural partial order is shown to be exactly the Halmos-McLaughlin ordering, making hom-sets are directed-complete partial orders. This category, and the partial ordering on hom-sets, are studied from the point of view of teleportation protocols and compact closure, in order to establish both physical interpretations and connections with the Abramsky-Coecke categorical approach to foundations of quantum mechanics. The logical interpretations of the category theoretic, the orthomodular lattices, approach to quantum logic are contrasted from this point of view.

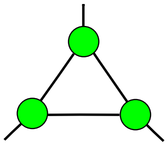
6.2.c A categorical view on code and data (Objectives: W2.O2, W2.O4; Milestones: W2.M4). In [9] Hines (QICS postdoc at York) considers the differences between quantum and classical information that mean that a quantum computer cannot run the von Neumann architecture, and the implications for quantum information and computation. A case is made that the computational utility of the von Neumann architecture arises from the 'decode a byte and apply the appropriate operation' step in the fetch-execute cycle. The properties of such 'evaluation' operations are considered from the point of view of code /

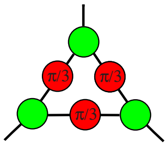
data correspondences (i.e. the categorical property of ‘naming arrows’), and the no-cloning and no-deleting theorems. These are studied from the point of view of Nielsen-Chuang’s ‘Encoding unitary maps on an orthonormal basis’, Abramsky-Coecke’s categorical foundations program, and the Jamiolkowski-Choi correspondence between density matrices and completely positive maps.

2 Progress towards objectives and performed tasks for W2.T2

2.1 Graphical calculus for multipartite entanglement

6.2.1a A diagrammatic notation and classification for arbitrary multi-partite entangled states (Objectives: W2.O3; Milestones: W2.M3, W2.M6). The language introduced in [4] is sufficiently expressive to give diagrammatic presentations of multi-partite entangled states: we showed that we can simulate any unitary, hence we can simulate any entangled state as the image of some reference state given the appropriate unitary. For example, the GHZ and W state can be presented as follows





(the three green dots of the GHZ state can in fact be fused into one). Coecke (Ox) and Edwards (Ox) have used this fact to define classes of multipartite states which further refine the SLOCC classes. For example, there are subclasses of the W-SLOCC class which depict differently, and it turns out that they indeed have distinct behavioural properties. A paper is in preparation.

2.2 General results on entanglement

6.2.2a (Objectives: W2.O2, W2.O3; Milestones: W2.M3, W2.M6). In [10] Winter (Bris) and co-authors show that genuine multipartite quantum correlations can exist for states which have no genuine multipartite classical correlations, even in macroscopic systems. They construct such states for an arbitrary odd number of qubits. Such possibilities can have important implications in the physics of quantum information and phase transitions.

6.2.2b (Objectives: W1.O1, W2.O2, W2.O3; Milestones: W1.M4, W2.M3, W2.M6). Motivated by the recent discovery of a quantum Chernoff theorem for asymptotic state discrimination, Matthews (Bris) and Winter (Bris) investigate in [11] the distinguishability of two bipartite mixed states under the constraint of local operations and classical communication (LOCC), in the limit of many copies. Surprisingly, the single-copy optimal measurement remains optimal for n copies, in the sense that the best strategy is measuring each copy separately, followed by a simple classical decision rule.

6.2.2c (Objectives: W1.O4, W2.O3; Milestones: W1.M5, W2.M3). In [12] Markham (QICS postdoc at Paris) has developed techniques for calculating entanglement for large classes of states important in many body physics and quantum information, particularly in particular for measurement-based quantum computing (W-states and stabilizer states). Those results imply bounds on local discrimination for these states (important in many quantum information protocols), and the optimality of associated entanglement witnesses (important in the realisable verification of entanglement).

2.3 Distributed quantum computing

6.2.3a (Objectives: W1.O1, W1.O4, W3,O2, W4.O2, W4.O4; Milestones: W1.M4, W1.M5). In [13] Linden (Bris), Popescu (Bris), Short (Bris) and Winter (Bris) investigate the problem of ”nonlocal” computation, in which separated parties must compute a function with nonlocally encoded inputs and output, such that each party individually learns nothing, yet together they compute the correct function output. They show that the best that can be done classically is a trivial linear approximation. Surprisingly, they also show that quantum entanglement provides no advantage over the classical case. On the other hand, generalized (i.e. super-quantum) nonlocal correlations allow perfect nonlocal computation.

3 Progress towards objectives and performed tasks for W2.T3

6.3.a Temperley-Lieb algebra and quantum computing (Objectives: W2.O2, W2.O5; Milestones: W2.M5). In [14, 15] Abramsky (Ox) shows how Abramsky-Coecke categorical quantum axiomatics can be connected in a very direct way with diagram algebras, in particular the Temperley-Lieb algebra, which plays a central rôle in the Jones polynomial and ensuing developments. We find that the Temperley-Lieb algebra is the ‘planar’ version of our quantum setting; and we find new connections between logic and geometry. For example, we can give a simple, direct (no quotients) description of the Temperley-Lieb algebra, which leads in turn to full completeness results for various non-commutative logics. Moreover, we show that planarity is an invariant of the infomation flow analysis of cut elimination. This leads to a number of interesting new kinds of questions: (i) What is the computational significance of planarity as a constraint on expressiveness or complexity? (ii) Most quantum protocols appear to live on the plane; which do not? (iii) What is the computational or logical significance of braiding? There are obvious direct connections between these notions and topological quantum computing.

6.3.b Modular functors for topological quantum computing (Objectives: W2.O5; Milestones: W2.M5). In [16] Panangaden (McGi) and Paquette (McGi affiliated & forthcoming QICS postdoc at Ox) fill the gap between physics, mathematics and computer science in order to give a simple yet thorough introduction to the cluster of ideas involving category theory, topology, condensed matter physics – specifically anyons – and quantum computation. There is a need for such an introduction as the technical introductions to the subject available are either written for physicists or computer scientists but the categorical ideas are usually not described. For a researcher with a background in category theory the subject can be approached with a simple algebraic language, that is, the language of modular tensor categories and modular functors. The exposition is intended to explain the peculiar physical properties of anyons and two-dimensional physics and to introduce the notion of modular tensor categories and finally to show how such a context is intended to be used in robust quantum computation. Currently ongoing research in the area is: (i) To develop a categorical semantics for topological quantum computation; (ii) Investigate the algorithmic properties of topological computation; (iii) To analyze the various connections with other recent developments in categorical quantum computation i.e., dagger compact categories, classical objects etc.

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