Kindergarten Quantum Mechanics

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(or Google ‘‘Bob Coecke’’
THE CHALLENGE
Why did discovering quantum teleportation take 60 years?

Claim: bad formalism since ‘too low level’ cf.

\[
\begin{align*}
\text{“GOOD QM”} & \sim \text{HIGH-LEVEL language} \\
\text{von Neumann QM} & \sim \text{low-level language}
\end{align*}
\]

Wouldn’t it be nice to have a such a ‘good’ formalism, in which discovering teleportation would be trivial?

Claim: it exists! And I’ll present it to you.

Isn’t it absurdly abstract coming from you guys?

Claim: It could be taught in kindergarten!
THE APPROACH
1. **Analyse** quantum compoundness.
⇒ A notion of **quantum information-flow** emerges.
   - Physical Traces. Abramsky & Coecke (2003) CTCS’02; cs/0207057
   - Quantum Information-flow, Concretely, and Axiomatically. quant-ph/0506132

2. **Axiomatize** quantum compoundness.
⇒ ... full **quantum mechanics** emerges!
⇒ ... & **quantum logic** ... & open systems/CPM’s!
EXPLICIT OPERATIONALISM

Primitive data are **processes/operations** \( f, g, h, \ldots \) which are **typed** as \( A \rightarrow B, B \rightarrow C, A \rightarrow A, \ldots \) where \( A, B, C, \ldots \) are **kinds/names** of systems.

**Sequential composition** is a primitive connective on processes/operations cf.

\[
f \circ g : A \rightarrow C \quad \text{for} \quad f : A \rightarrow B \; \& \; g : B \rightarrow C
\]

**Parallel composition** is a primitive connective both on systems and processes/operations cf.

\[
f \otimes g : A \otimes C \rightarrow B \otimes D \quad \text{for} \quad f : A \rightarrow B \; \& \; g : C \rightarrow D
\]
Do you want ... 

- states to be ontological or empirical?
- vectorial, projective, POVM-/CPM-/open system-style?
- hidden variables, quantum potential, contextuality, (non-)locality, Bayesianism, ... ?

The bulk of the developments ignores these choices, but, they can be implemented formally since we both have

- great axiomatic freedom
- great expressiveness
Audience: “Seriously, you don’t expect us to learn that?”

Bob: “No! Of course not!”

“We are gonna go far back in time, ...
  to the time you were all still at kindergarten, ...

“We’re gonna draw pictures!”

The sheer magic of the kind of category theory we need here is that it formally justifies its own formal absence.
A NEW FORMALISM

Language and calculus: purely graphical

Behind the scene: categorical algebra

Concrete model: Hilbert space QM, ... and also many others, ...

Not assumed: some number field, any kind of matrix calculus, vectors and sums thereof, elements of objects/types (cf. state space) and corresponding mappings, ...
Primitive data:

Sequential and parallel composition:

Duals, adjoints and EPR-states:
THE SOLE AXIOM
Since

the axiom is equivalent to

\[ \begin{array}{ccc}
\text{Diagram 1} & = & \text{Diagram 2} \\
\text{Diagram 3} & = & \text{Diagram 4}
\end{array} \]
When setting

we obtain
COMPOSITIONALITY
COMPOSITIONALITY bis

\[
\begin{array}{ccc}
  f & g & h \\
\end{array}
\]

\[=\]

\[
\begin{array}{ccc}
  h & o & g & f
\end{array}
\]
We define bipartite projectors as

\[ P_f : A^* \otimes B \rightarrow A^* \otimes B \]
that is, approximately, as

\[ P_f : A \otimes B^* \to A^* \otimes B \]

as
The concepts of bipartite state and of bipartite projector yield the following corrolaries ...
since $\text{id} \circ \text{id} = \text{id}$
since \( \text{id} \circ \text{id} = \text{id} \)
FULL TELEPORTATION

\[ f_i = f_i \]

for \( 1 \leq i \leq 4 \)
\[ f_i = f_{i-1} \]

for \( 1 \leq i \leq 4 \)
since $f \circ \text{id} = f$
ENTANGLEMENT SWAPPING
\( f : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \) is a linear map

\( \psi : \mathbb{C} \rightarrow \mathcal{H} \) cf. \( \psi(1) \in \mathcal{H} \)

\( s : \mathbb{C} \rightarrow \mathbb{C} \) cf. \( s(1) \in \mathbb{C} \)

\( \mathcal{H}^* := \) conjugate Hilbert space of \( \mathcal{H} \)

\( f^\dagger := \) linear adjoint of \( f \)

\( \psi = |\psi\rangle \quad \pi = \langle \phi | \quad \text{for} \quad \pi := \phi^\dagger \quad \frac{\pi}{\psi} = \langle \phi | \psi \rangle \)
\[ f : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \text{ is a linear map} \]

\[ \psi : \mathbb{C} \rightarrow \mathcal{H} \quad \text{cf. } \psi(1) \in \mathcal{H} \]

\[ s : \mathbb{C} \rightarrow \mathbb{C} \quad \text{cf. } s(1) \in \mathbb{C} \]

\[ \mathcal{H}^* := \text{conjugate Hilbert space of } \mathcal{H} \]

\[ f^\dagger := \text{linear adjoint of } f \]

\[ \begin{align*}
\psi &= |\psi\rangle \\
\pi &= \langle \phi | \\
\text{for } \pi &:= \phi^\dagger
\end{align*} \]

\[ \begin{align*}
\psi &= |\psi\rangle \\
\pi &= \langle \phi | \\
\text{for } \pi &:= \phi^\dagger
\end{align*} \]
EPR-states and their adjoints:

\[\mathbb{C} \rightarrow \mathcal{H}^* \otimes \mathcal{H}:: 1 \mapsto \sum e_i \otimes e_i\]

\[\mathcal{H}^* \otimes \mathcal{H} \rightarrow \mathbb{C}:: \Phi \mapsto \langle \sum e_i \otimes e_i | \Phi \rangle\]

\[:: \phi_1 \otimes \phi_2 \mapsto \langle \phi_1 | \phi_2 \rangle\]

We verify the axiom:

\[= \langle - | \sum e_i \otimes e_i \rangle = \sum (- \otimes e_i) \otimes e_i\]

\[= \sum \langle - | e_i \rangle \cdot e_i = \text{id}\]
**Exercise.** Verify that in Hilbert space bipartite projectors on one-dimensional subspaces indeed factor as $f f^\dagger$. 
A key role is played by
\[ \mathcal{H}_1^* \otimes \mathcal{H}_2 \cong \mathcal{H}_1 \rightarrow \mathcal{H}_2 \]
i.e. bipartite states \( \Psi \in \mathcal{H}_1^* \otimes \mathcal{H}_2 \) are representable by linear functions \( f : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \) and vice versa. Indeed

\[
\Psi = \sum_{ij} m_{ij} |ij\rangle \cong \begin{pmatrix} m_{11} \cdots m_{1n} \\ \vdots \quad \vdots \\ m_{k1} \cdots m_{kn} \end{pmatrix} \cong \sum_{ij} m_{ij} |j\rangle \langle i|
\]
e.g.

\[
|00\rangle + |11\rangle \cong \text{id} = |0\rangle \langle 0| + |1\rangle \langle 1|
\]
for the bijection $f \mapsto \lceil f \rceil$ i.e.
Proof of injectivity.

\[ f \circ g = g \circ f \]

\[ \Leftrightarrow \]

\[ f = g \]
Proof of injectivity.
The inner-product of $\psi, \phi : I \to A$ is

$$\langle \phi \mid \psi \rangle := \begin{array}{c}
\pi \\
\psi
\end{array} = \phi^\dagger \circ \psi : I \to I$$

where $\pi := \phi^\dagger$ cf.

$$\text{bra} := \langle \phi \mid \quad \text{ket} := \mid \psi \rangle \quad \text{bra-ket} := \langle \phi \mid \psi \rangle$$

e.g. for $f : A \to B$ we have

$$\mid f \circ \psi \rangle = \begin{array}{c}
f \\
\psi
\end{array} = f \circ \psi \quad \langle f \circ \phi \mid = \begin{array}{c}
\pi \\
f^\dagger
\end{array} = \phi^\dagger \circ f^\dagger$$
**Adjointness** implies

\[
\langle f \circ \phi \mid \psi \rangle = \pi f^\dagger = \langle \phi \mid f^\dagger \circ \psi \rangle
\]

**Unitarity** means \( U^{-1} = U^\dagger \) i.e.

\[
\begin{array}{c}
U \quad U^\dagger \\
\hline
U^\dagger \\
U
\end{array} = \begin{array}{c}
U^\dagger \\
U
\end{array} = \begin{array}{c}
\pi \\
\pi
\end{array}
\]

hence

\[
\langle U \circ \phi \mid U \circ \psi \rangle = \pi U^\dagger = \pi = \pi = \langle \phi \mid \psi \rangle
\]
A “contravariant” Barr-Kelly-Laplaza involution

\[ f : A \to B \quad \iff \quad f^* : B^* \to A^* \]
called upper star arises as
A “covariant” involution

\[ f : A \to B \quad \mapsto \quad f^* : A^* \to B^* \]

called lower star arises as

\[
\begin{align*}
\begin{array}{c}
f_* \\
\end{array}
\end{align*}
\]
From

\[
\begin{align*}
(f^\ast)_\ast &: = \quad \begin{array}{c}
\downarrow \\
\ast \\
\uparrow
\end{array}
\begin{array}{c}
\ast \\
\uparrow
\end{array}
\begin{array}{c}
\downarrow \\
\ast \\
\uparrow
\end{array}
\begin{array}{c}
\downarrow
\end{array} \\
\end{align*}
\]

and

\[
\begin{align*}
(f^\ast)_\ast &: = \quad \begin{array}{c}
\downarrow \\
\ast \\
\uparrow
\end{array}
\begin{array}{c}
\ast \\
\uparrow
\end{array}
\begin{array}{c}
\downarrow \\
\ast \\
\uparrow
\end{array}
\begin{array}{c}
\uparrow
\end{array} \\
\end{align*}
\]

follows

\[
\begin{align*}
((f^\ast)_\ast)^\ast &= \quad \begin{array}{c}
\downarrow \\
\ast \\
\uparrow
\end{array}
\begin{array}{c}
\ast \\
\uparrow
\end{array}
\begin{array}{c}
\ast \\
\uparrow
\end{array}
\begin{array}{c}
\uparrow
\end{array} \\
\end{align*}
\]

and analogous we can prove that \((f^\ast)_\ast = f^\dagger\)
Hence the star operations

\[ f^* := f \quad \text{and} \quad f^* := f^\dagger \]

provide a decomposition of the adjoint:

\[ f^\dagger = (f^*)^* = (f^*)^* \]

In particular, for the Hilbert space model we have

\((-)^* := \text{transposition} \]

\((-)^* := \text{complex conjugation} \]
A Joyal-Street-Verity partial trace

\[ f : C \otimes A \to C \otimes B \quad \mapsto \quad \text{Tr}_C(f) : A \to B \]

arises as

\[
\begin{array}{c}
\text{Tr}_C(f) \\
\end{array}
\quad := 
\begin{array}{c}
f
\end{array}
\]
A corresponding **full trace**

\[ h : A \rightarrow A \quad \iff \quad \text{Tr}(h) : I \rightarrow I \]

arises as

\[ \text{Tr}(h) \quad := \quad \begin{array}{c}
\end{array} \]

\[ \Rightarrow h \quad \text{“carries a diamond”} \quad \text{cf. probabilistic weight} \]
From

follows

and hence
\( \text{Tr}(\rho_{\phi} \circ P) \equiv \langle \phi \mid P \circ \phi \rangle \) for \( \rho_{\phi} := |\phi\rangle\langle \phi| \)
Symmetric monoidal bifunctor $\dashv \otimes \dashv : C \times C \to C$ and

- $\otimes$-involution **dual** $A \mapsto A^*$;
- contravariant $\otimes$-involution **adjoint** $f_{A \to B} \mapsto f_{B \to A}^\dagger$;
- **Units** $\eta_A : I \to A^* \otimes A$ with $\eta_{A^*} = \sigma_{A^*,A} \circ \eta_A$;

\[
\begin{array}{ccc}
A & \xrightarrow{\sim} & I \otimes A \\
1_A & & \eta_{A^*}^\dagger \otimes 1_A \\
& (A \otimes A^*) \otimes A & \\
& & \sim
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{\sim} & A \otimes I \\
\sim & & 1_A \otimes \eta_A \\
& A \otimes (A^* \otimes A) & \\
\sim \end{array}
\]
Symmetric monoidal bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and

- $\otimes$-involution **dual** $A \mapsto A^*$;
- contravariant $\otimes$-involution **adjoint** $f_{A \to B} \mapsto f_{B \to A}^\dagger$;
- **Units** $\eta_A : \mathbb{I} \to A^* \otimes A$ with $\eta_{A^*} = \sigma_{A,A^*} \circ \eta_A$;
BIFUNCTORIALITY OF \( \otimes \)

\[
\begin{array}{c}
A_1 \otimes A_2 \xrightarrow{f_1 \otimes \text{id}} B_1 \otimes A_2 \\
\downarrow \text{id} \otimes f_2 \\
A_1 \otimes B_2 \xrightarrow{f_1 \otimes \text{id}} B_1 \otimes B_2 \\
\downarrow \text{id} \otimes f_2 \\
\end{array}
\]

\[
gf = \text{id} \otimes f_2
\]

\[
f \otimes g = \text{id} \otimes f_2
\]
NATURAL SYMMETRY

\[ A_1 \otimes A_2 \xrightarrow{f_1 \otimes f_2} B_1 \otimes B_2 \]

\[ \sigma_{A_1,A_2} \]

\[ A_2 \otimes A_1 \xrightarrow{f_2 \otimes f_1} B_2 \otimes B_1 \]

\[ \sigma_{B_1,B_2} \]

\[ f \quad g = g \quad f \]
We use the unit $I$ for $\otimes$ i.e.

$$A \simeq I \otimes A \simeq A \otimes I$$

to define states and numbers respectively as

$$\psi : I \rightarrow A$$

and

$$s : I \rightarrow I$$
**NATURAL SCALAR MULTIPLES**

**Scalars** satisfy

\[ s \circ t = I \xrightarrow{\sim} I \otimes I \xrightarrow{s \otimes t} I \otimes I \xrightarrow{\sim} I \]

and we define **scalar multiplication** as

\[ s \bullet f := A \xrightarrow{\sim} A \otimes I \xrightarrow{f \otimes s} B \otimes I \xrightarrow{\sim} B \]

for which we can then prove

\[ (s \bullet f) \circ (t \bullet g) = (s \circ t) \bullet (f \circ g) \]
\[ (s \bullet f) \otimes (t \bullet g) = (s \circ t) \bullet (f \otimes g) \]

i.e. **diamonds can move around freely in ‘time’ and ‘space’**
and similarly

\[
\psi \circ \pi = A \xrightarrow{\sim} I \otimes A \xrightarrow{\psi \otimes \pi} B \otimes I \xrightarrow{\sim} B
\]
NO-CLONING NO-DELETING


Obviously we do not want to be a categorical (co-)product since that would imply existence of

\[ A \xrightarrow{\Delta} A \otimes A \quad A \otimes B \xrightarrow{p} A \]

i.e. there are no logical rules

\[ A \vdash A \land A \quad A \land B \vdash A \]
The **squared Hilbert-Schmidt norm**

\[ \|f\| = \sum_i \langle f(e_i) | f(e_i) \rangle \]

exists in the picture formalism as

\[ \|f\| := (\neg f \neg)^{\dagger} \circ \neg f \neg \]

i.e.
The **squared Hilbert-Schmidt norm**

\[ \|f\| = \sum_i \langle f(e_i) \mid f(e_i) \rangle \]

exists in the picture formalism as

\[ \|f\| := (\dashv f \lrcorner) \circ (\rightharpoonup f) \]

**Proof.**

\[ \|f\|(1) = (\eta^\dagger \circ (1 \otimes f)^\dagger \circ (1 \otimes f) \circ \eta) (1) \]

\[ = (\eta^\dagger \circ (1 \otimes (f^\dagger \circ f))) \left( \sum e_i \otimes e_i \right) \]

\[ = \eta^\dagger \left( \sum e_i \otimes f^\dagger(f(e_i)) \right) \]

\[ = \sum \langle e_i \mid f^\dagger(f(e_i)) \rangle \]

\[ = \sum \langle f(e_i) \mid f(e_i) \rangle. \]
The corresponding **Hilbert-Schmidt inner-product** also exists in the picture formalism as

\[ \langle f \mid g \rangle := (\downarrow f \downarrow)^\dagger \circ \downarrow g \downarrow \]

i.e.

\[
\begin{align*}
(\downarrow \psi \downarrow)^\dagger \circ \downarrow \phi \downarrow &= \psi^\dagger \circ \phi 
\end{align*}
\]
ALL IS QUANTITATIVE!

The squared Hilbert-Schmidt norm yields:

- a canonical norm on processes

The Hilbert-Schmidt inner-product yields:

- an inner-product on processes
Abstract Global Phases

\[ f \otimes f^\dagger = e^{i\theta} \cdot g \otimes (e^{i\theta} \cdot g)^\dagger = e^{i\theta} \cdot g \otimes e^{-i\theta} \cdot g^\dagger = g \otimes g^\dagger \]

Proposition 1.

\[ s \bullet f = t \bullet g , s \circ s^\dagger = t \circ t^\dagger = 1_I \implies f \otimes f^\dagger = g \otimes g^\dagger \]

Proposition 2.

\[ f \otimes f^\dagger = g \otimes g^\dagger \implies \exists s, t : s \bullet f = t \bullet g, s \circ s^\dagger = t \circ t^\dagger \]

E.g.

\[ s := (\lnot f \lnot)^\dagger \circ \lnot f \lnot \quad \text{and} \quad t := (\lnot g \lnot)^\dagger \circ \lnot f \lnot \]
Proof.

1. \[ s := (f\downarrow) \circ \downarrow f \quad \text{and} \quad t := (g\downarrow) \circ \downarrow f \]

\[ s \quad := \quad \begin{array}{c}
\text{boxed}\, f^\dagger \\
\text{boxed}\, f
\end{array} \]

\[ t \quad := \quad \begin{array}{c}
\text{boxed}\, g^\dagger \\
\text{boxed}\, f
\end{array} \]

2. \[ f \otimes f^\dagger = g \otimes g^\dagger \]

\[ \begin{array}{c}
f \\
\text{boxed}\, f^\dagger \end{array} \quad = \quad \begin{array}{c}
g \\
\text{boxed}\, g^\dagger
\end{array} \]
Proof.

\[ \#3 \quad s \cdot f = t \cdot g \quad \text{with} \quad s/t := (\lceil f/g \rceil)\dagger \circ \lceil f \rceil \]
Proof.

\[ s \circ s^\dagger = t \circ t^\dagger \text{ with } s/t := (\lceil f/g \rceil)^\dagger \circ \lceil f \rceil \]
Hilbert space $\mathcal{H}$

kill redundant global scalars

lattice of subspaces $\mathcal{L}(\mathcal{H})$

Birkhoff & von Neumann (1936)

abstract lattices

$\text{go abstract}$

FdHilb

'vectorial' strong compact closure

our approach

'projective' strong compact closure

kill redundant global scalars
**ABSENCE OF GLOBAL PHASES**

**Proposition.** \(WProj(C) \simeq C\) (canonically) iff

\[f \otimes f^\dagger = g \otimes g^\dagger \implies f = g\]

iff

\[P_f = P_g \implies \mathcal{R} f = \mathcal{R} g\]

iff

\[\psi \circ \psi^\dagger = \phi \circ \phi^\dagger \implies \psi = \phi\]

iff

**Equal Preparations Produce Equal States**
OPEN SYSTEMS AND CPMs

⇒ projective process
OPEN SYSTEMS AND CPMs

⇒ projective process with ancila
⇒ projective process with hidden ancila

= open process on open system
In the case of Hilbert spaces and linear maps we exactly obtain completely positive maps (Selinger 2005)!
System of type $A := \text{ Object } A$

Composite of $A$ and $B := \text{ Tensor } A \otimes B$

Process of type $A \rightarrow B := \text{ Morphism } f : A \rightarrow B$

State of $A := \text{ Element } \psi : I \rightarrow A$

Evolution of $A := \text{ Unitary } U : A \rightarrow A$

Measurement on $A := \text{ “Projectors” } \{P_i : A \rightarrow A\}_i$

- Data $:= \nu \in \{i\}_i$
- Dynamics $:= \psi \mapsto P_\nu \circ \psi$
- Probability $:= \psi^\dagger \circ P_\nu \circ \psi = \text{Tr}(P_\nu \circ \rho_\psi) : I \rightarrow I$
Some extra structure is required both for

- Specification of the families \( \{ P_i : A \to A \}_i \)
- Combining \( \{ P_i \}_i \) into a single \( M : A \to \ldots \)

But, you can pick your favorite!

For each unitary morphism \( U : A \to \bigoplus_i A_i \) we have

\[
\{ P_j := \pi_j^\dagger \circ \pi_j \}_j \quad M := \left( \bigoplus_i \pi_i^\dagger \right) \circ U : A \to \bigoplus_i A
\]

where \( \pi_j := p_j \circ U \). Alternatively, \( \{ f_i \}_i \) has to satisfy
\[
\sum_i f_i = 1_A
\]
and the corresponding measurement is

\[
M := \langle f_i \rangle_i : A \to \bigoplus_i A.
\]
... first full formal description of protocols
  ... types reflect kinds
  ... classical data-flow is included
  ... quantum info-flow is explicit

... kindergarten description/correctness proofs

... space for formal/conceptual choices

... the thing people call QM-relationalism?
APPLICATIONS
— “why computer scientists care about this stuff” —

Quantum programing language design
Quantum program logics for verification
Quantum protocol specification
Quantum protocol verification
Appropriate semantics for new quantum computational paradigms e.g. one-way (Briegel), teleportation based (Gottesman-Chuang), measurement based in general, topological quantum computing (Kitaev et al.) etc.

