Provably efficient algorithms for Hybrid Systems

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Models of complex systems

Modelling power

Solvability/efficiency

\[ x(k + 1) = \begin{cases} A_0 x(k) \\ A_1 x(k) \end{cases} \]

\[ x_{t+1} = A_{\sigma(t)} x_t \]
\[ x_0 \in \mathbb{R}^n \]
\[ \sigma(0), \sigma(1), \ldots \in G \]

\[ A_{\sigma(t)} \in M \subset \mathbb{R}^{n \times n} \]
If I have seen further it is by standing on the shoulders of Giants.
Outline

• Two motivations
  – Consensus
  – Wireless control networks

• Three techniques
  – Observability/controllability of hybrid systems
  – Guaranteed accuracy for stability analysis
  – Data-driven/blackbox control

• Discussion
Outline

• Two motivations
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    – Wireless control networks

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• Discussion
Autonomous robots
Gossip algorithms

S. Boyd, A. Ghosh, B. Prabhakar, D. Shah
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S. Boyd, A. Ghosh, B. Prabhakar, D. Shah
Consensus of multi-agent systems

Our setting: We are given a set of stochastic matrices, representing different connectivity topologies.

\[
\begin{pmatrix}
 x_1 \\
 x_2 \\
 \vdots \\
 x_n
\end{pmatrix} \leftarrow \begin{pmatrix}
 0.5 & 0.5 & 0 \\
 0.5 & 0.5 & 0 \\
 \vdots \\
 0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
 x_1 \\
 x_2 \\
 \vdots \\
 x_n
\end{pmatrix}
\]

\[x(k + 1) = \begin{cases} A_0 x(k) \\ A_1 x(k) \end{cases}\]

\[
A_0 = \begin{pmatrix}
 0.5 & 0.5 & 0 & 0 \\
 1/3 & 1/3 & 1/3 & 0 \\
 0 & 1/3 & 1/3 & 1/3 \\
 0 & 0 & 0.5 & 0.5
\end{pmatrix}
\]

\[
A_1 = \begin{pmatrix}
 0.5 & 0 & 0 & 0.5 \\
 0 & 1/3 & 1/3 & 1/3 \\
 0 & 0.5 & 0.5 & 0 \\
 1/3 & 1/3 & 0 & 1/3
\end{pmatrix}
\]

Problem: Do all products of these matrices converge to consensus? (that is, all rows are equal)
This is actually a stability problem

- Property: \( A1 = 1 \) with \( 1 = (1, 1, \ldots, 1) \)

- Proposition [Jadbabaie 03]: After projection along the \((1,1,\ldots,1)\) vector, convergence to consensus becomes convergence to zero

\[
x(k + 1) = \begin{cases} 
  A_0 x(k) \\
  A_1 x(k)
\end{cases}
\]

Do all the products converge to zero?

Paz, Wolfovitz, Blondel-Olshevsky, Chevalier,..
Switching systems

\[ x(k + 1) = \begin{cases} A_0 x(k) \\ A_1 x(k) \end{cases} \]

**Point-to-point** Given \( x_0 \) and \( x_* \), is there a product (say, \( A_0 A_0 A_1 A_0 \ldots A_1 \)) for which \( x_* = A_0 A_0 A_1 A_0 \ldots A_1 x_0 \)?

**Mortality** Is there a product that gives the zero matrix?

**Boundedness** Is the set of all products \( \{A_0, A_1, A_0A_0, A_0A_1, \ldots \} \) bounded?
Global convergence to the origin Do all products of the type $A_0 A_0 A_1 A_0 \ldots A_1$ converge to zero? (GUAS)

The spectral radius of a matrix $A$ controls the growth or decay of powers of $A$

$$\rho(A) = \lim_{t \to \infty} \left\| A^t \right\|^{1/t}$$

The powers of $A$ converge to zero iff $\rho(A) < 1$

The joint spectral radius of a set of matrices $\Sigma$ is given by

$$\rho(\Sigma) = \lim_{t \to \infty} \max_{A_1 \ldots A_t \in \Sigma} \left\| A_1 A_2 \ldots A_t \right\|^{1/t}$$

All products of matrices in $\Sigma$ converge to zero iff $\rho(\Sigma) < 1$

[Rota, Strang, 1960]
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Wireless Control Networks

Industrial automation

Physical Security and Control

Supply Chain and Asset Management

Environmental Monitoring, Disaster Recovery and Preventive Conservation
Networked Control Systems With Communication Constraints: Tradeoffs Between Transmission Intervals, Delays and Performance

W. P. Maurice H. Heemels, Member, IEEE, Andrew R. Teel, Fellow, IEEE, Nathan van de Wouw, Member, IEEE, and Dragan Nešić, Fellow, IEEE

Roughly speaking, the network-induced imperfections and constraints can be categorized in five types:

(i) Quantization errors in the signals transmitted over the network due to the finite word length of the packets;
(ii) Packet dropouts caused by the unreliability of the network;
(iii) Variable sampling/transmission intervals;
(iv) Variable communication delays;
(v) Communication constraints caused by the sharing of the network by multiple nodes and the fact that only one node is allowed to transmit its packet per transmission.
Networked Control Systems With Communication Constraints: Tradeoffs Between Transmission Intervals, Delays and Performance

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[Jungers D’Innocenzo Di Benedetto, TAC 2015]
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[Jungers Kundu Heemels, 2016]
Controllability with packet dropouts

Wireless Control Networks are subject to packet dropouts

\[ x(1) = Ax(0) + Bu(0) \]

\[ \sigma(0) = 1 \]

A data loss signal determines the packet dropouts

\[ \sigma(t) = 1 \text{ or } 0 \]

...this is a switching system!

\[ x(t + 1) = \begin{cases} 
Ax(t) + bu(t), & \text{if } \sigma(t) = 1, \\
Ax(t), & \text{if } \sigma(t) = 0 
\end{cases} \]
Controllability with packet dropouts

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\[
\begin{align*}
\sigma(0) &= 1 \\
\sigma(1) &= 0
\end{align*}
\]

\[
x(1) &= Ax(0) + Bu(0) \\
x(2) &= A^2x(0) + ABu(0)
\]

\[
\sigma = 1001 \ldots
\]

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\end{cases}
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Controllability with packet dropouts

Wireless Control Networks are subject to packet dropouts

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\begin{align*}
\sigma(0) &= 1 \\
\sigma(1) &= 0 \\
\sigma(2) &= 0 \\
x(1) &= Ax(0) + Bu(0) \\
x(2) &= A^2x(0) + ABu(0) \\
x(3) &= A^3x(0) + A^2Bu(0) \\
x(4) &= A^4x(0) + A^3Bu(0) + Bu(3)
\end{align*}
\]

A data loss signal determines the packet dropouts

\[
\sigma(t) = 1 \text{ or } 0
\]

...this is a switching system!

\[
x(t + 1) = \begin{cases} 
Ax(t) + bu(t), & \text{if } \sigma(t) = 1, \\
Ax(t), & \text{if } \sigma(t) = 0 
\end{cases}
\]
The switching signal

We are interested in the controllability of such a system

\[
\begin{align*}
\sigma(0) &= 1 \\
\sigma(1) &= 0 \\
\sigma(2) &= 0
\end{align*}
\]

\[
\begin{align*}
x(1) &= Ax(0) + Bu(0) \\
x(2) &= A^2 x(0) + ABu(0) \\
x(3) &= A^3 x(0) + A^2 Bu(0) \\
x(4) &= A^4 x(0) + A^3 Bu(0) + Bu(3)
\end{align*}
\]

Of course we need an assumption on the switching signal

The switching signal is constrained by an automaton

Example:
Constrained switching systems
A more general model

The switching sequence is constrained by a graph (AKA an automaton)

Many natural applications
- Communication networks
- Markov Chains
- Supervisory control

The dynamics is subject to switchings:

\[ x_{t+1} = A_{\sigma(t)}x_t + B_{\sigma(t)}u(t) \]

\[ x_0 \in \mathbb{R}^n \]

\[ \sigma(0), \sigma(1), \ldots \in \mathcal{L}(\theta) \]

\[ A_{\sigma(t)} \in M \subset \mathbb{R}^{n \times n} \]
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• Discussion

Joint work with
A. Kundu, M. Heemels
Controllability with Packet Dropouts

We are given a pair \((A, b)\) and an automaton

\[
\begin{align*}
\sigma(0) &= 1 \\
\sigma(1) &= 0 \\
\sigma(2) &= 0
\end{align*}
\]

\[
\begin{align*}
x(1) &= Ax(0) + Bu(0) \\
x(2) &= A^2x(0) + ABu(0) \\
x(3) &= A^3x(0) + A^2Bu(0) \\
x(4) &= A^4x(0) + A^3Bu(0) + Bu(3)
\end{align*}
\]

The controllability problem: for any starting point \(x(0)\), and any target \(x^*\), does there exist, for any admissible switching signal, a control signal \(u(.)\) and a time \(T\) such that \(x(T) = x^*\) ?

**Theorem:** Deciding controllability of switching systems is undecidable in general (consequence of [Blondel Tsitsiklis, 97])
We are given a pair \((A,b)\) and an automaton

\[
\begin{align*}
\sigma(0) &= 1 \\
\sigma(1) &= 0 \\
\sigma(2) &= 0 \\
x(1) &= Ax(0) + Bu(0) \\
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x(4) &= A^4x(0) + A^3Bu(0) + Bu(3)
\end{align*}
\]

The controllability problem: for any starting point \(x(0)\), and any target \(x^*\), does there exist, for any admissible switching signal, a control signal \(u(.)\) and a time \(T\) such that \(x(T)=x^*\)?

**Proposition:** The system is controllable iff the **generalized controllability matrix**

\[
C_\sigma(t) = [A^{(t-1)}b\sigma(0) \mid A^{(t-2)}b\sigma(1) \mid \ldots \mid Ab\sigma(t-2) \mid b\sigma(t-1)]
\]

is bound to **become full rank** at some time \(t\).
Standing on Giants shoulders...

Thoralf Skolem
1887-1963
Axioms, set theory, lattices, first order logic

\[ C_\sigma(t) = [A^{(t-1)}b\sigma(0)\left| A^{(t-2)}b\sigma(1) \right| \ldots \left| A\sigma(t-2)\left| b\sigma(t-1) \right] \]

Theorem ([Skolem 34]): Given a matrix A and two vectors b,c, the set of values n such that
\[ c^T A^n b = 0 \]
is eventually periodic. Example: 1 0 0 1 0 1 1 1 0 0 1 0 0 1 0 0 1 0 0 ...

Algorithm for deciding controllability

Now, how to optimally chose the control signal, if one does not know the switching signal in advance?

Jungers, Kundu, Heemels, TAC 2017
The dual observability problem

Observability under intermittent outputs is algebraically equivalent

\[ x(t + 1) = Ax(t), \]
\[ y(t) = \sigma(t)Cx(t) \]
Proton therapy
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• Discussion
Switching systems stability criteria

The joint spectral radius of a set of matrices $\Sigma$ is given by

$$
\rho(\Sigma) = \lim_{t \to \infty} \max_{A_1, A_2, \ldots, A_t \in \Sigma} \|A_1 A_2 \ldots A_t\|^{1/t}
$$

All products of matrices in $\Sigma$ converge to zero iff $\rho(\Sigma) < 1$
Theorem: The LMIs are a sufficient condition for stability IFF their representation $G$ is path-complete.

Results valid beyond the LMI framework

[Ahmadi J. Parrilo Roozbehani 14]
[J. Ahmadi Parrilo Roozbehani 17]
Path-complete stability criteria

Theorem: We provide a hierarchy of criteria that reaches arbitrary accuracy.

Results valid beyond the LMI framework.

[Ahmadi J. Parrilo Roozbehani 14]
[J. Ahmadi Parrilo Roozbehani 17]
Some examples

- Examples:
  - CQLF

- Example 1

  This type of graph gives a max-of-quadratics Lyapunov function (i.e. intersection of ellipsoids)

- Example 2
Standing on Giants shoulders

Symbolic dynamics

Stephen Smale
1930-
Fields Medal 1966
Wolf prize 2007

Gustav Hedlund
1904-1993

Marston Morse
1892-1977

Emil Artin
1898-1962

Symbolic dynamics

M Morse, GA Hedlund - American Journal of Mathematics, 1938 - JSTOR
Path-complete techniques

A. Ahmadi (Princeton), P. Parrilo, M. Roozbehani (MIT)  
SICON’14, TAC’16  
Path-complete characterization

Geir Dullerud  
And Ray Essick (UIUC)  
Automatica’16  
Generalization to Constrained switching systems

David Angeli (Imperial)  
TAC’18  
Nikos Athanasopoulos  
Equivalent Common Lyapunov Function

F. Forni and R. Sepulchre (Cambridge)  
IFAC WC’16  
Generalization to delta-ISS certificates

Paulo Tabuada (UCLA)  
ADHS’18  
Generalization to Invariant sets computation

Guillaume Berger

Benoit Legat
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Joint work with
J. Kenanian, A. Balkan,
P. Tabuada
Control in the industry

Often, models can be nonlinear, complex, hybrid, heterogeneous, with look-up tables, pieces of code, proprietary softwares, old legacy components...

Termination of Computer programs

\[ \text{while } (Bx > 0) \text{ do } \]
\[ \{ x := Ax \}; \]
\[ \text{end while} \]
Control in the industry
Data-driven stability analysis of SS

\[ x_{t+1} = A_0 x_t \]
\[ A_1 x_t \]

Global convergence to the origin Do all products of the type \(A_0 A_0 A_1 A_0 \ldots A_1\) converge to zero? (GUAS)

The joint spectral radius of a set of matrices \(\Sigma\) is given by

\[ \rho(\Sigma) = \lim_{t \to \infty} \max_{A_2 \in \Sigma} \|A_1 A_2 \ldots A_t\|^{1/t} \]

All products of matrices in \(\Sigma\) converge to zero iff \(\rho(\Sigma) < 1\)

[Rota, Strang, 1960]
Data-driven stability analysis of SS

**Setting:** we sample N points at random in the state space, and observe their image by one (unknown) of the modes

**Question:** Is the system stable?
Data-driven stability analysis of SS

**Observation:** One thing we can do is to check for the existence of a Common Lyapunov function for the $N$ sampled points

$$\begin{align*}
\min_P & \quad \lambda_{\max}(P) \\
\text{s.t.} & \quad (A_j x)^T P (A_j x) \leq \gamma^2 x^T P x, \quad \forall (x, j) \in \omega_N \subset Z \\
& \quad P \succeq I
\end{align*}$$
Standing on giants shoulders...

**Theorem [adapted from Campi, Calafiore]:** Consider the optimization problem below, where $\omega_N$ is a random homogeneous $N$-sampling of the infinite number of constraints $Z$. For any desired correctness level $\varepsilon$ one can guarantee

$$
\mu^N\{\omega_N \in Z^N : \mu(V(\omega_N)) \leq \varepsilon \} \geq 1 - \sum_{i=0}^{d} \binom{N}{i} \varepsilon^i (1-\varepsilon)^{N-i}
$$

Measure of ‘good’ N-Samplings

Measure of violated constraints

Lower bound on the measure of good samplings (Tends to 1 when N grows)

$$
\begin{align*}
\min_P & \quad \lambda_{\max}(P) \\
\text{s.t.} & \quad (A_jx)^T P (A_jx) \leq \gamma^2 x^T P x, \quad \forall (x, j) \in \omega_N \subset Z \\
& \quad P \succeq I
\end{align*}
$$
From a partial guarantee to a formal upper bound

With some level of confidence, we have:

Not much!
Already for a linear system, this does not imply much:

The challenge: translate this guarantee on a subset of the statespace into a global guarantee (on the whole statespace)
From a partial guarantee to a formal upper bound

**Theorem:** Consider a $n$-dimensional linear switched system and a uniform random sampling $\omega_N \subseteq \mathbb{Z}$, where $N \geq \frac{m(n+1)}{2} + 1$. Let $\gamma^*(\omega_N)$ be the optimal solution to $\text{Opt}(\omega_N)$. Given a desired level of certainty $\beta \in (0, 1)$, given a size of sample $N$ and $\alpha > 1$, we can compute $\delta(\beta, \omega_N)$, such that with probability at least $\beta$ we have:

$$\rho \leq \frac{\alpha \gamma^*(\omega_N)}{\delta(\beta, \omega_N)},$$

where $\lim_{N \to \infty} \delta(\beta, \omega_N) = 1$ with probability 1.

**Figure:** Evolution of $\delta$ (left) and of the upper and lower bounds on the JSR (right) with increasing $N$, for $\beta = 0.92$.
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Solvability/efficiency

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The JSR Toolbox:
[Van Keerberghen, Hendrickx, J. HSCC 2014]
The CSS toolbox, 2015

Several open positions:
raphael.jungers@uclouvain.be

References:
http://perso.uclouvain.be/raphael.jungers/

Joint work with
A.A. Ahmadi (Princeton), D. Angeli (Imperial), N. Athanasopoulos (UCLouvain), V. Blondel (UCL), G. Dullerud (UIUC), F. Forni (Cambridge), M. Heemels (TU/e), B. Legat (UCLouvain), P. Parrilo (MIT), M. Philippe (UCLouvain), V. Protasov (Moscow), M. Roozbehani (MIT), R. Sepulchre (Cambridge), P. Tabuada (UCLA)…