Sound Numeric Computations in Abstract Acceleration

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Linear Time Invariant (LTI) Systems.

We verify safety for Linear Loops in Real Domains in the form:

$$\textit{while}\ (k \leq \overline{k})\ \ \boldsymbol{x} = \boldsymbol{A}\boldsymbol{x}$$

- $\boldsymbol{x} \in \mathbb{R}^n$ are state variables
- $\boldsymbol{A} \in R^{n \times n}$ are the dynamics of the system.
- verification is performed using floating point computations.

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- This work applies to General Linear loops with Linear Guards and Inputs. For simplicity we will refer to the case above only.

Using floating points to evaluate reachability

$$\tilde{\boldsymbol{x}}_0 = \boldsymbol{x}_0$$

$$\textit{while } (k \leq \overline{k}) \;\; \tilde{\boldsymbol{x}} = \boldsymbol{A}\tilde{\boldsymbol{x}} + \boldsymbol{e}$$

$$\|\boldsymbol{e}\| \leq n\|\overline{\boldsymbol{\delta}}\| : \overline{\boldsymbol{\delta}} = [\overline{\delta_1} \;\; \cdots \overline{\delta_n}]^T$$

$$\overline{\delta_i} = \sup_j \left(|\delta(\boldsymbol{A}_{ij}\tilde{\boldsymbol{x}}_j : \tilde{\boldsymbol{x}} \in X)|\right)$$

Using eigenvalues to reduce the number of operations

Using the Eigendecomposition $\boldsymbol{A} = \boldsymbol{SJS}^{-1}$, where $\boldsymbol{J}$ is a bidiagonal matrix, we may replace the algorithm for:

Algorithm 1 Reals

1 : $\boldsymbol{x}'_0 = \boldsymbol{S}^{-1}\boldsymbol{x}_0$
2 : *while* $(k \leq \overline{k})$
3 : $\boldsymbol{x}'_k = \boldsymbol{J}\boldsymbol{x}'_{k-1}$
4 : $\boldsymbol{x}_{\overline{k}} = \boldsymbol{S}\boldsymbol{x}'_{\overline{k}}$

Algorithm 2 Floating Point

1 : $\tilde{\boldsymbol{x}}'_0 = \boldsymbol{S}^{-1}\boldsymbol{x}_0 + \boldsymbol{e}_{s1}$
2 : *while* $(k \leq \overline{k})$
3 : $\tilde{\boldsymbol{x}}'_k = \boldsymbol{J}\tilde{\boldsymbol{x}}'_{k-1} + \delta'$
4 : $\tilde{\boldsymbol{x}}_{\overline{k}} = \boldsymbol{S}\tilde{\boldsymbol{x}}'_{\overline{k}} + \boldsymbol{e}_{s2}$

This algorithm is approximately n times faster and has an error of $\frac{1}{n}$ if the eigendecomposition is precise.

Symbolic vs Numerical Eigendecomposition

- Symbolic Eigendecomposition ($n < 10$). $\overline{\boldsymbol{e}_s} \propto n\overline{\delta}$
- Numeric Eigendecomposition ($n < 10000$).
  - Narrow Bounds - Slower($1x$). $\overline{\boldsymbol{e}_s} \propto n^2\overline{\delta}$
  - Wide Bounds - Faster ($100x$). $\overline{\boldsymbol{e}_s} \propto n^n\overline{\delta}$
  - Using higher precision makes for a better trade-off.

Method	Dimension	Precision	Speed	Error Scale
Narrow	$n = 20$	128 bit	$2x$	10^{-37}
Wide	$n = 20$	128 bit	$1.01x$	10^{-15}
Narrow	$n = 20$	256 bit	$4x$	10^{-77}
Wide	$n = 20$	256 bit	$2.02x$	10^{-55}

Using Abstract Acceleration to reduce the number of operations

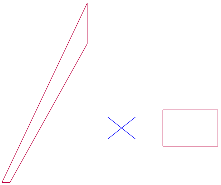
The set of points evaluated at each iteration is:

$$X'_k = \boldsymbol{J} X_{k-1} = \boldsymbol{J}^k X'_0$$

$$X' = \bigcup_{k \le \overline{k}} X'_k$$

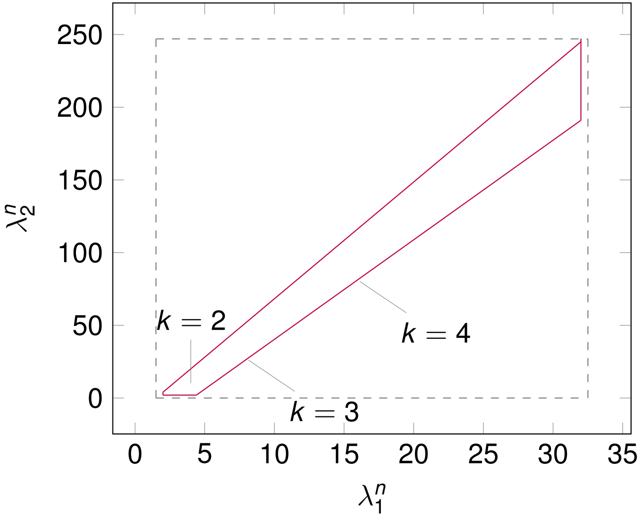
Which can be overapproximated by a single multiplication using an Abstract Matrix

$$X' \subseteq X^\sharp = \mathcal{J} X'_0, \text{ such that } \bigcup_{k \le \overline{k}} \boldsymbol{J}^k \subseteq \mathcal{J},$$



$\mathcal{J}$ X'_0

Abstract Matrices for positive real eigenvalues



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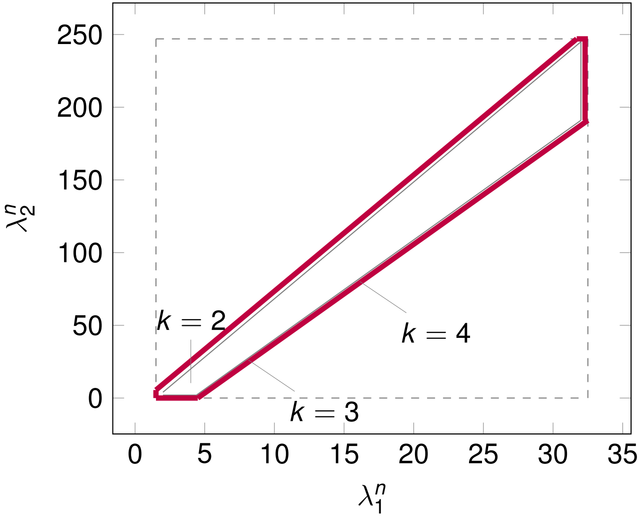
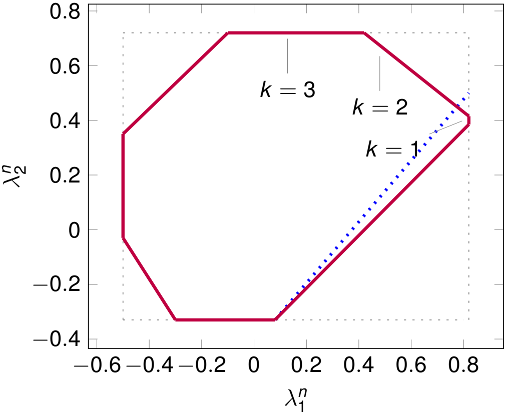
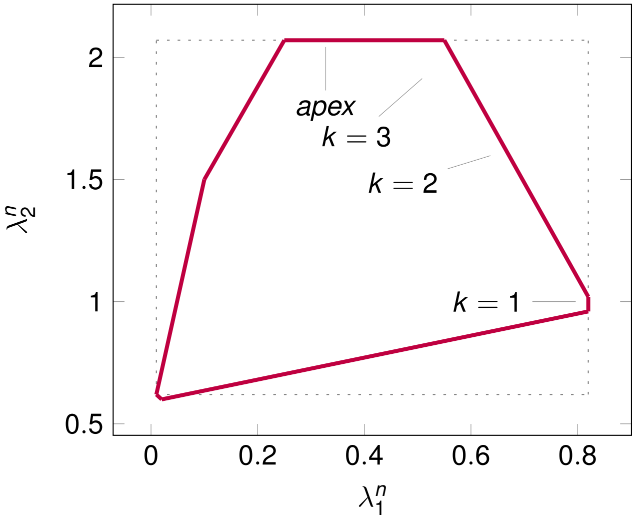


Abstract Matrices for Complex Eigenvalues



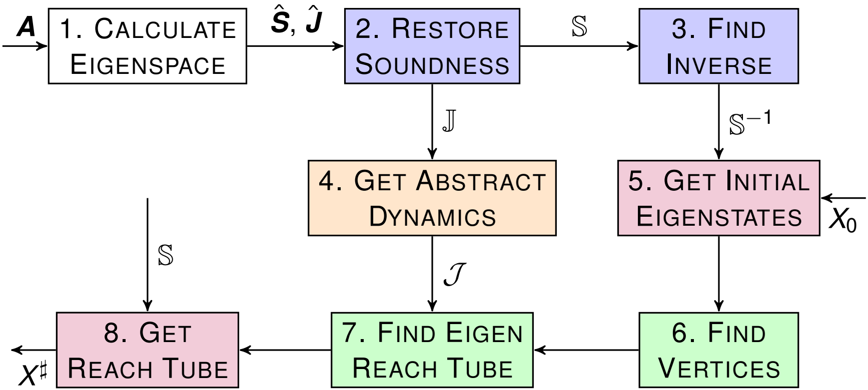
Polyhedral faces from an $\mathbb{R}^2$ complex conjugate subspace .

Abstract Matrices for Jordan blocks

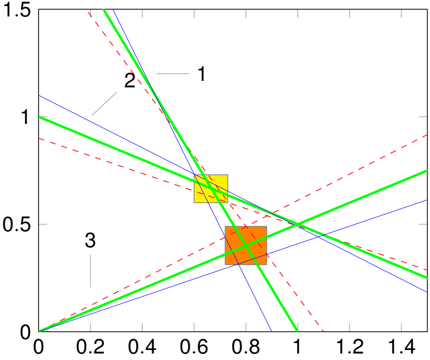


Polyhedral faces from an $\mathbb{R}^2$ Jordan block subspace .

Calculation of Abstract Reach Tube Using Numeric Abstract Acceleration



Pivot operations



Three interval half-planes with negative (dashed red), zero (thick green) and positive (thin blue) angular error representations. The yellow and orange areas (hypercubes) over-approximate all possible vertices of the resulting polyhedron at the given location. If these hypercubes partially intersect, the abstract vertex ι^k must necessarily contain all intersecting hypercubes.

Performance results

Benchmark	Dimension	Unsound		Sound	
		long double	mp	mpi	exact
Building	48	18.10s	185.03s	558.15s	$t.o.$
issr10	10	2.02s	23.46s	41.23s	$t.o.$
Convoy Car 3	6	0.30s	1.31s	3.60s	24.60s
Convoy Car 2	3	0.013s	0.033s	0.07s	5.46s
Parabola	4	0.012s	0.012s	0.05s	2.50s

Table: Axelerator[1] time performance on various benchmarks.
mp is the required precision for the algorithm using non-interval arithmetic
mpi is the sound algorithm

[1] www.cprover.org/LTI

Conclusions

- We have shown a numerical method for performing abstract acceleration using interval analysis
- It significantly improves the speed of the algorithm, allowing for user defined compromises that ensure scalability and soundness.
- The use of eigendecomposition and interval simplex can be applied to a number of approaches in order to achieve fast sound results.

The tool can be found at www.cprover.org/LTI