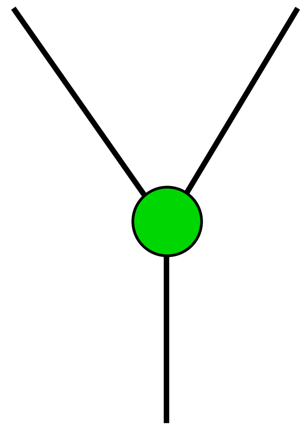
Classical Structures, MUBs, and Pretty Pictures



$|$Bob Coecke$\rangle$ + $|$**Ross Duncan**$\rangle$
Oxford University Computing Laboratory

Motivation

- Quantum observables may be incompatible: position/momentum, polarisation, spin ...
- In traditional quantum logic approaches these observables are simply *incomparable* in the lattice.
- However if one wants to *compute* with quantum mechanics we need know how these observables relate to each other.

No Cloning? No Deleting?

Quanutm physics doesn't like copying or deleting:

Concrete version: There are no quantum operations which can copy or erase non-orthogonal quantum states. [Wooters and Zurek, 1982; Pati and Braunstein, 2000]

Abstract Version: If a †-compact category $\mathcal{C}$ has natural transformations

$$\delta : - \Rightarrow - \otimes -$$

$$\epsilon : - \Rightarrow I$$

then $\mathcal{C}(A, A) \cong \mathcal{C}(I, I)$. [Abramsky, 2005].

Classical Objects

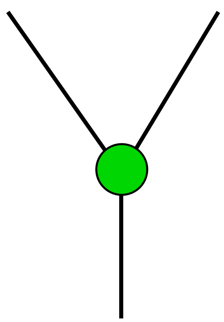
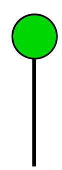
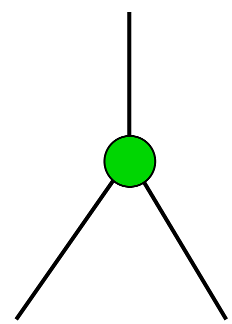
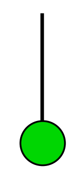
Classical Objects were introduced by Coecke and Pavlovic to axiomatise exactly what is means to be clonable and deletable – these properties are taken to be the definition of classicality.

In a †-category $\mathcal{C}$, a triple (A, δ, ϵ) is called a *classical object* if :

- $\delta : A \to A \otimes A$ and $\epsilon : A \to I$ form a cocommutative comonoid;
- $\delta^\dagger : A \otimes A \to A$ and $\epsilon^\dagger : I \to A$ form a commutative monoid;
- they jointly satisfy the special frobenius condition.

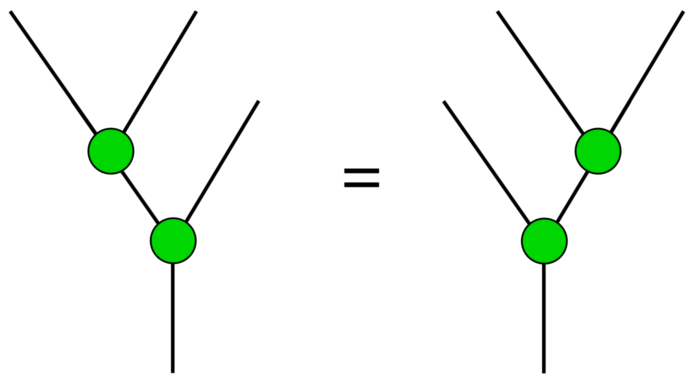
Classical Objects

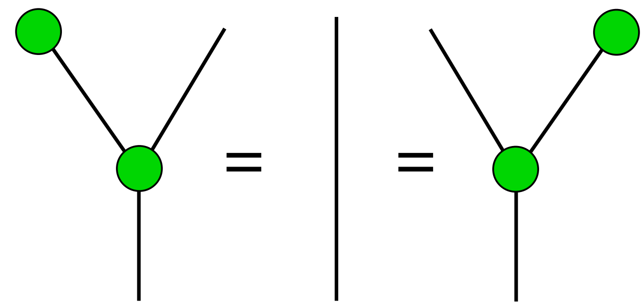
Represent maps constructed from δ and ϵ as graphs built up from:

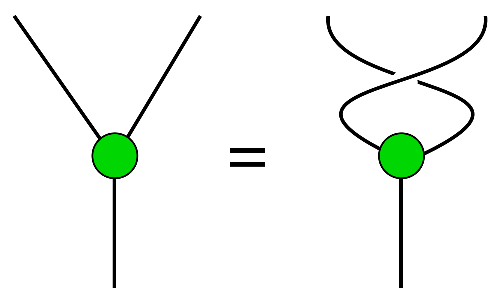
$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

Algebraic Laws

Comonoid laws:



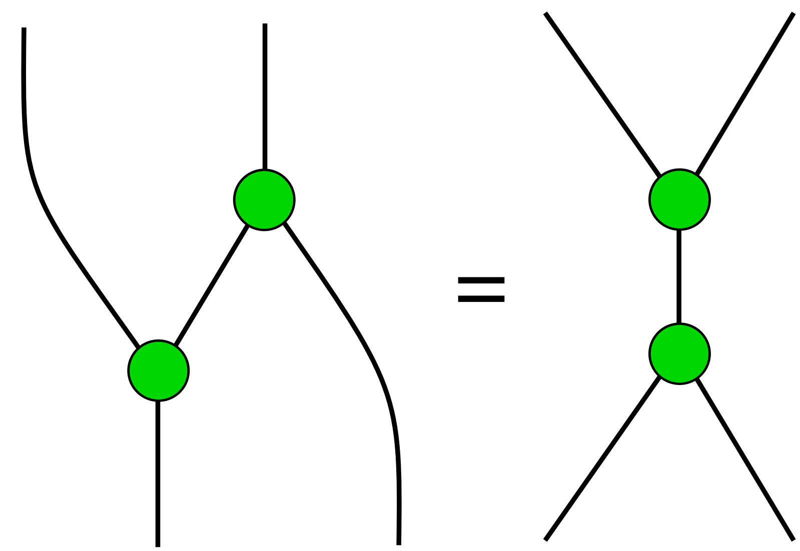


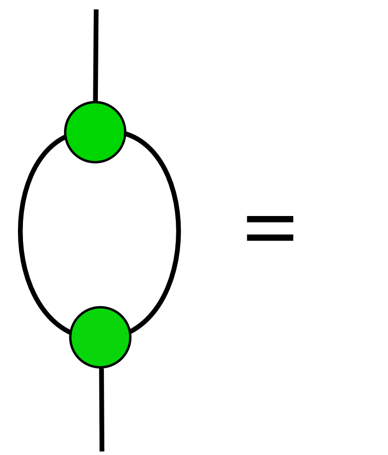


(And their duals, the monoid laws)

Algebraic Laws

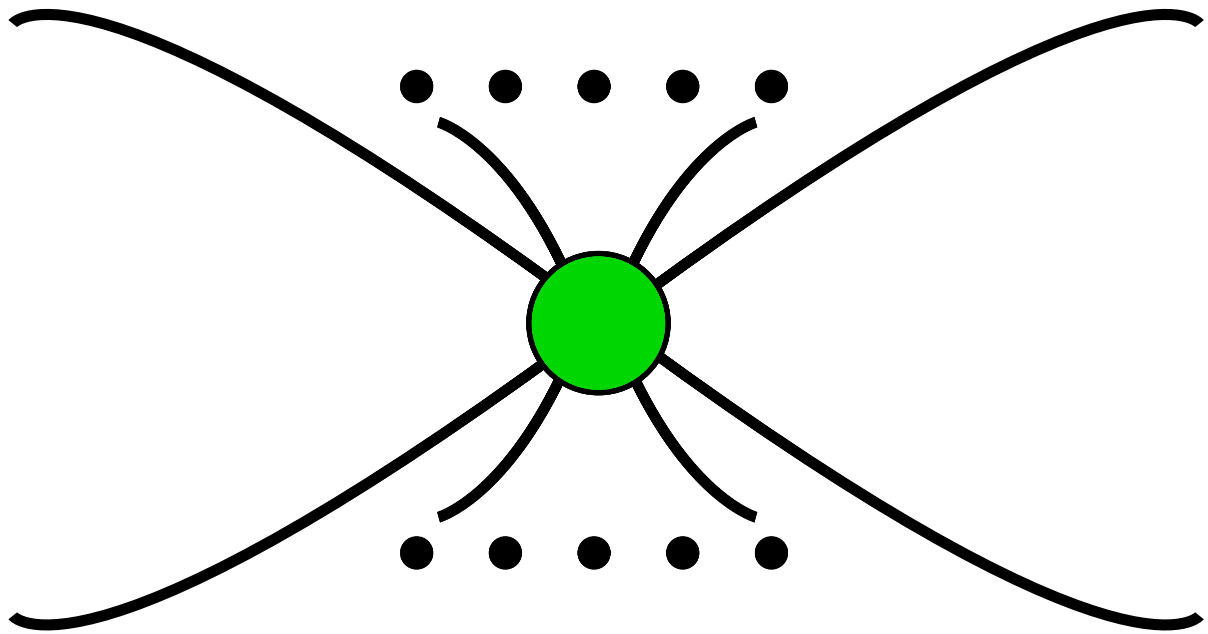
Special Frobenius laws:





Spider Theorem

Theorem 1. *Any map constructed by composing δ and ϵ, and their adjoints, is uniquely determined by the number of inputs and outputs.*



Therefore the graphical calculus for one classical object is rather uninteresting.

Cloning

Consider the map:

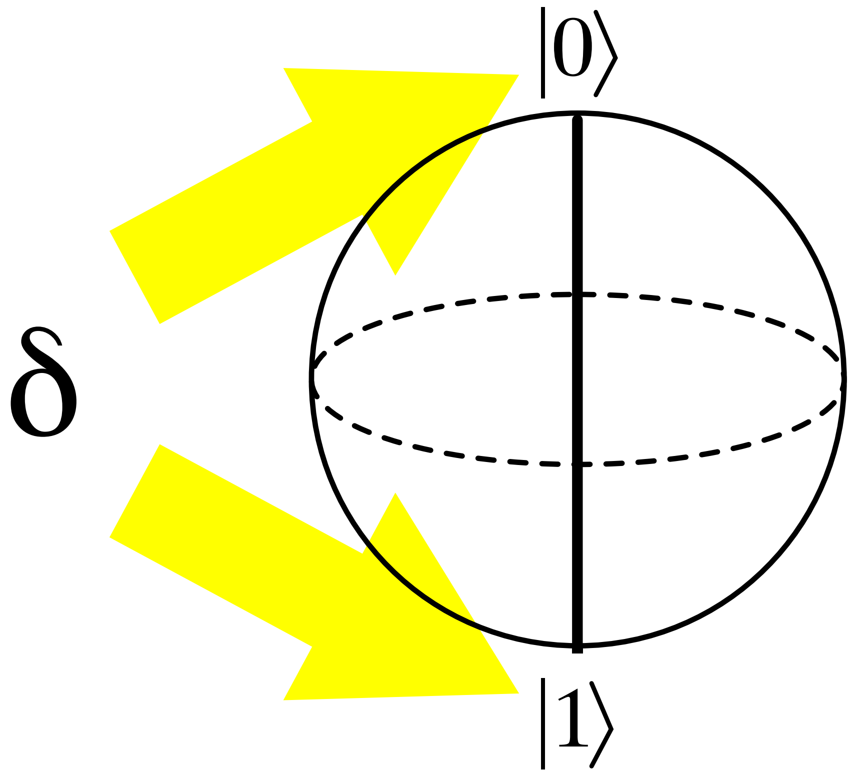
$$\delta_Z : Q \to Q \otimes Q :: |i\rangle \mapsto |ii\rangle$$

δ_Z is the *cloning* map for the basis $|0\rangle, |1\rangle$.

Obviously δ_Z is cannot clone all states:

$$\delta_Z |+\rangle = \delta_Z(|0\rangle + |1\rangle) = |00\rangle + |11\rangle$$

However, since quantum states are indistinguishable upto global phase the *vectors* $e^{i\alpha} |0\rangle$ and $e^{i\beta} |1\rangle$, are also cloned, when viewed as quantum states; hence can view δ as fixing an *observable* i.e. an axis of the Bloch sphere.


$|0\rangle$
δ
$|1\rangle$

Deleting

Q: How to "erase" a quantum state $|\psi\rangle$ known to be in some given basis?

A: Use a measurement which gives *no information* about the existing state — i.e measurement in a basis $\{b_i\}$ such that

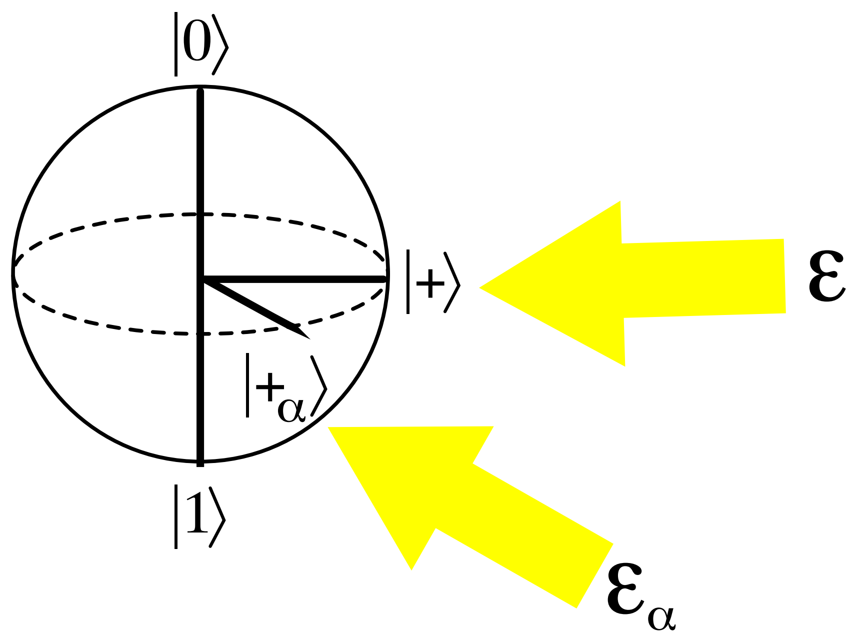
$$\begin{array}{rrcl} & |\langle b_i \mid \psi\rangle| & = & |\langle b_j \mid \psi\rangle| \\ \Rightarrow & |\langle b_i \mid a_k\rangle| & = & |\langle b_j \mid a_k\rangle| \\ \Rightarrow & |\langle b_i \mid a_k\rangle| & = & \frac{1}{\sqrt{d}} \text{ (in finite dim.)} \end{array}$$

Hence the idea of *Mutually Unbiased Bases* arise very naturally from the idea of *deleting* a classical value embedded in a quantum state space.

If we take the basis $|0\rangle, |1\rangle$ as the "classical" basis then the maps

$$\epsilon_Z^\alpha : Q \to I :: |0\rangle + e^{i\alpha}|1\rangle \mapsto 1$$

give a uniform erasing of the Z-basis for every value of α.



However if we compose ϵ_Z^α with δ_Z:

$$(\mathrm{id} \otimes \epsilon_Z^\alpha) \circ \delta_Z = Z_{-\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$$

Hence we need $\alpha = 0$ if $(Q, \delta_Z, \epsilon_Z)$ to be a classical object. (Will come back to this a bit later).

Thus, we have a classical structure:

- δ_Z is the *cloning* map for the basis $|0\rangle, |1\rangle$.
- ϵ_Z is the *uniform deleting* of this basis.

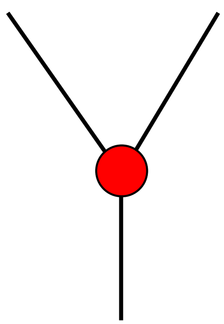
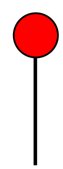
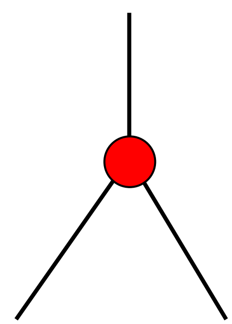
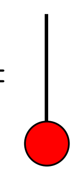
Together these maps describe how to embed classical data into the quantum state space.

Another Classical Structure

Can equally well use the X basis to define a classical structure:

$$\delta_X : \begin{cases} |+\rangle \mapsto |{+}{+}\rangle \\ |-\rangle \mapsto |{-}{-}\rangle \end{cases} \qquad\qquad \epsilon_X : \sqrt{2}\,|0\rangle \mapsto 1$$

These obey all the same algebraic laws as δ_Z, ϵ_Z.

$\delta_X =$  $\epsilon_X =$  $\delta_X^\dagger =$  $\epsilon_X^\dagger =$ 

Relating the X-Structure and the Z-Structure

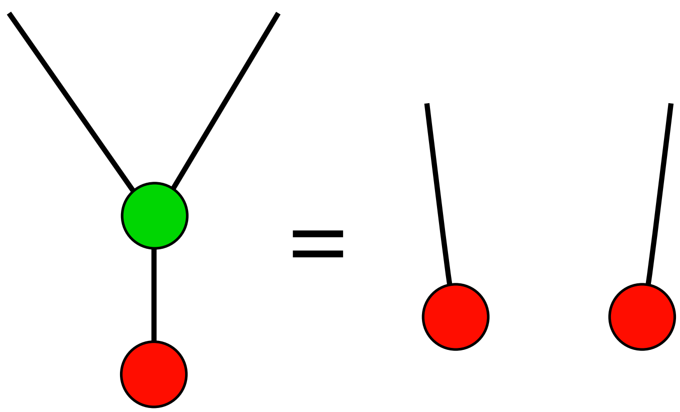
These two structures enjoy a very special relationship:

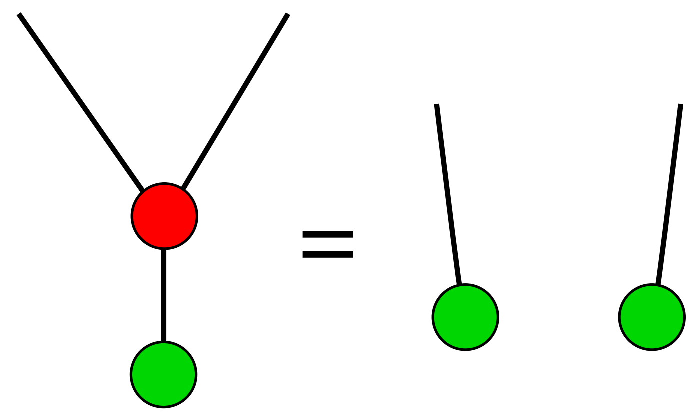
- $\sqrt{2}\,|0\rangle = \epsilon_X^\dagger$;
- $\delta_Z \epsilon_X^\dagger = \delta_Z\,|0\rangle = |00\rangle = \epsilon_X^\dagger \otimes \epsilon_X^\dagger$;
- $\sqrt{2}\,|+\rangle = \epsilon_Z^\dagger$
- $\delta_X \epsilon_Z^\dagger = \delta_X\,|+\rangle = |{+}{+}\rangle = \epsilon_Z^\dagger \otimes \epsilon_Z^\dagger$

Don't read this: In fact, by choosing a different ϵ one could have the same relationships between any pair from X, Y, or Z bases.

Bialgebraic Laws for Mutually Unbiased Observables

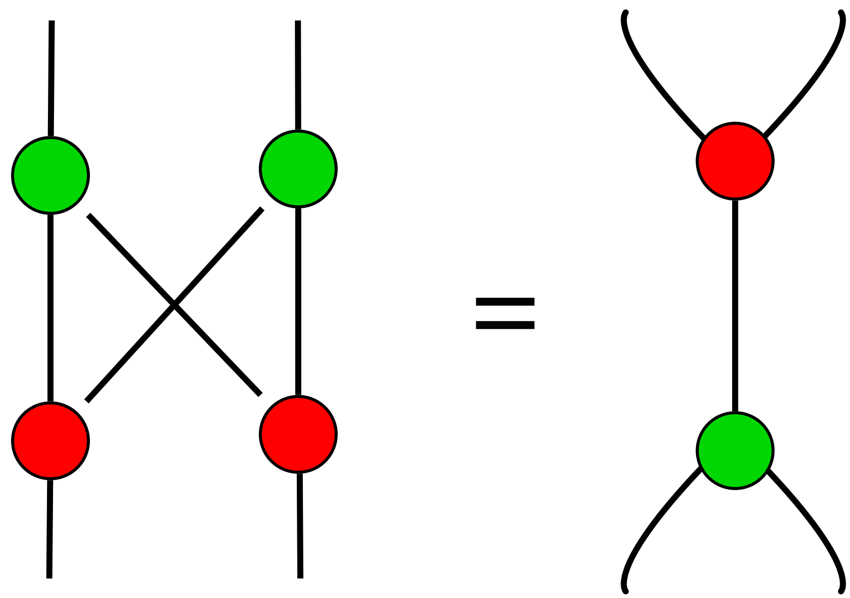
Cloning Laws:





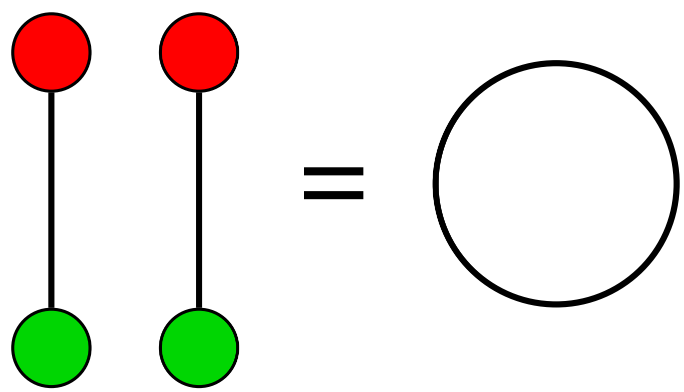
Bialgebraic Laws for Mutually Unbiased Observables

Bialgebra Law:



Bialgebraic Laws for Mutually Unbiased Observables

Dimension Law:



The pair of non-commuting observables fails to be a true bialgebra: every equation has a (hidden) scalar factor. Call this structure a *scaled bialgebra*.

Bialgebraic Laws for Mutually Unbiased Observables

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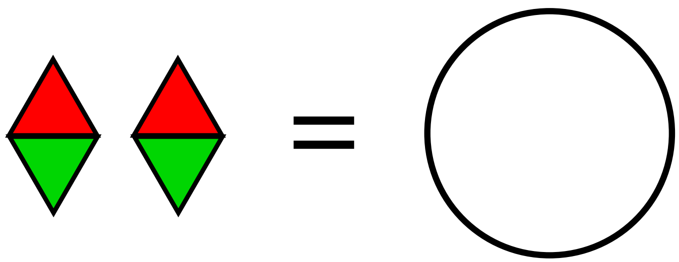

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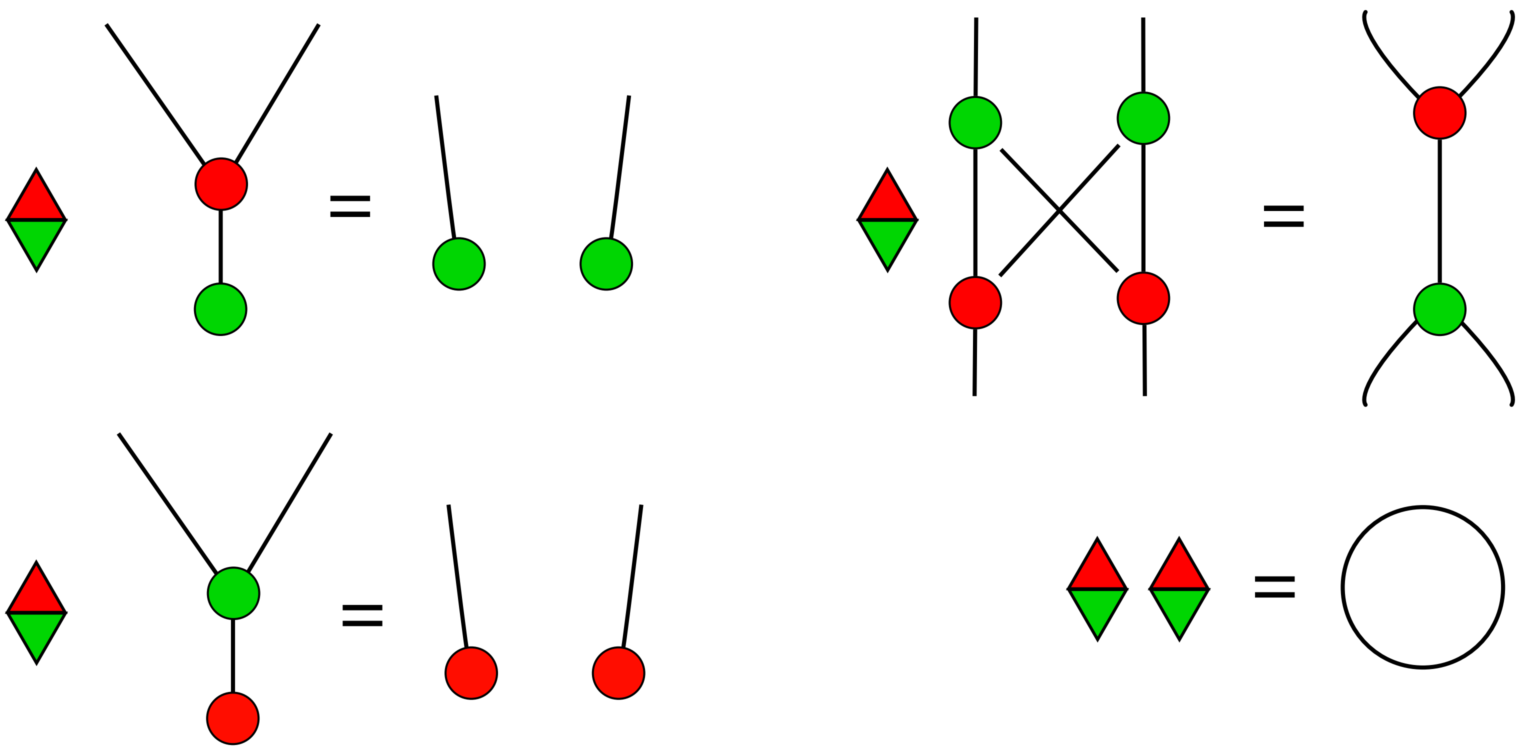
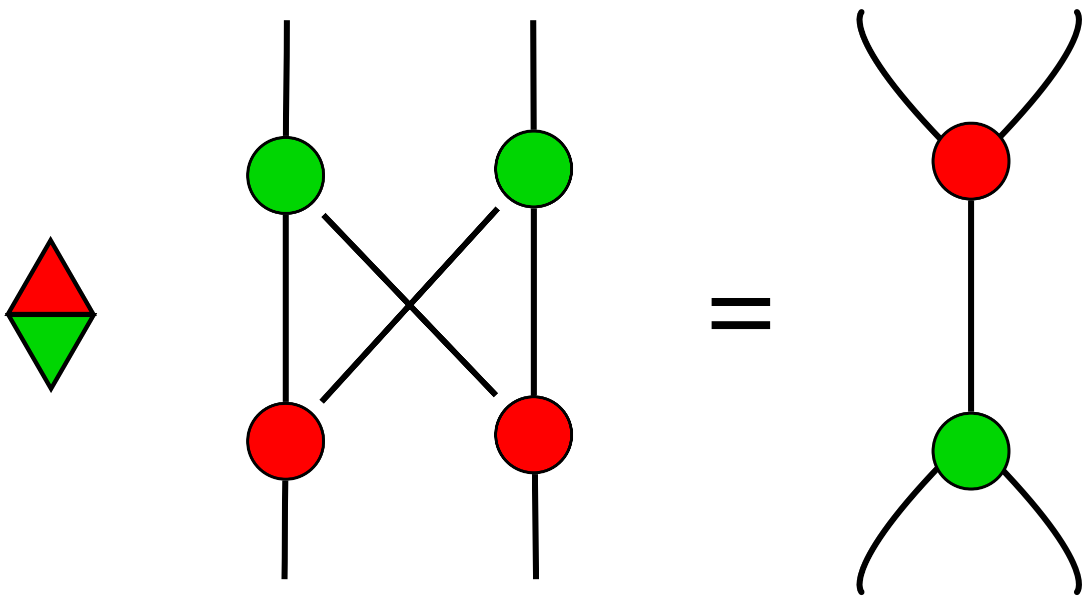
Scaled Bialgebra Laws



A Useful Lemma



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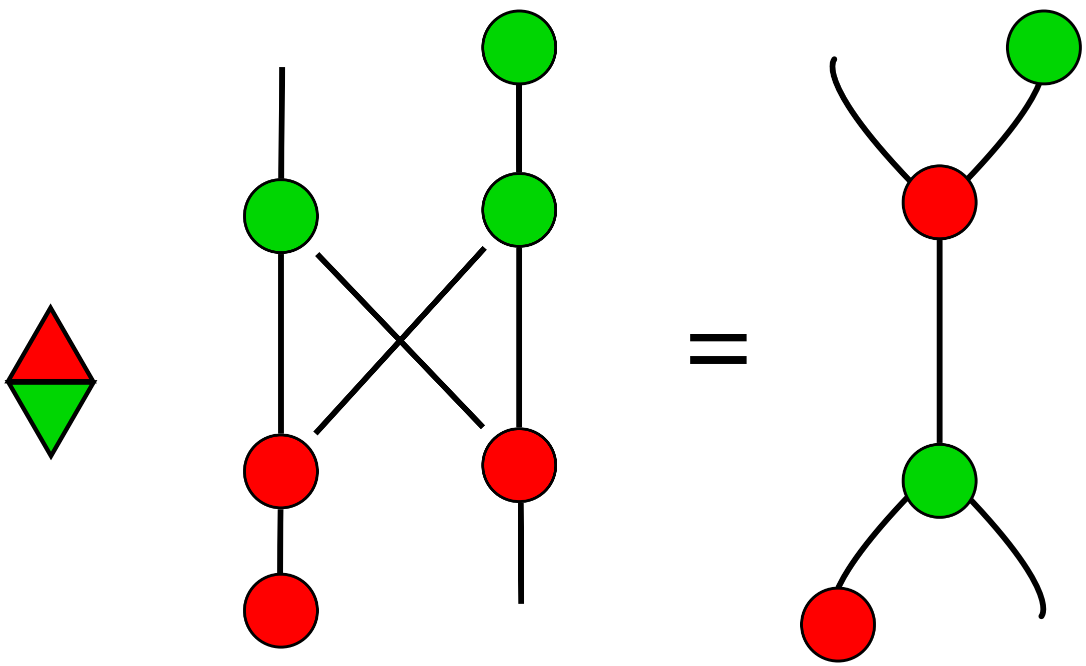

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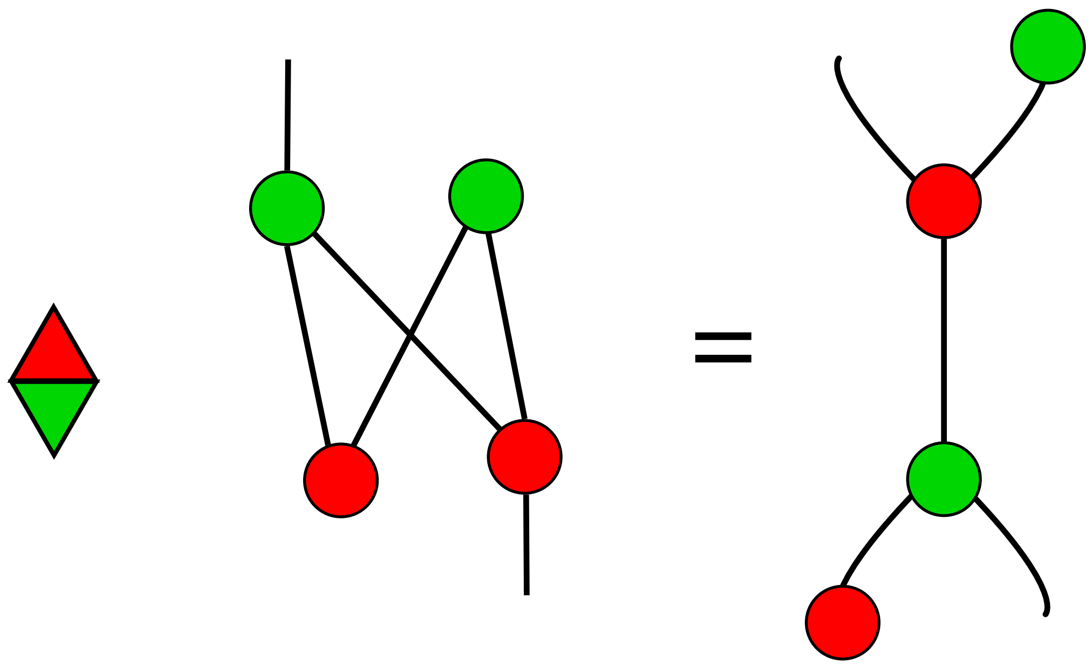

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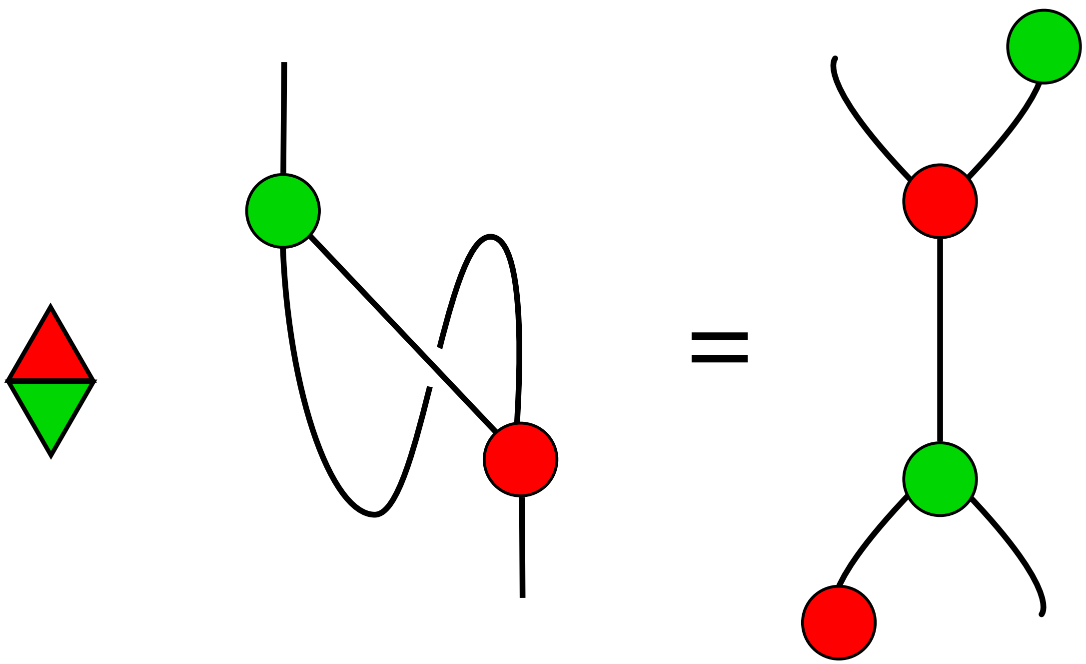

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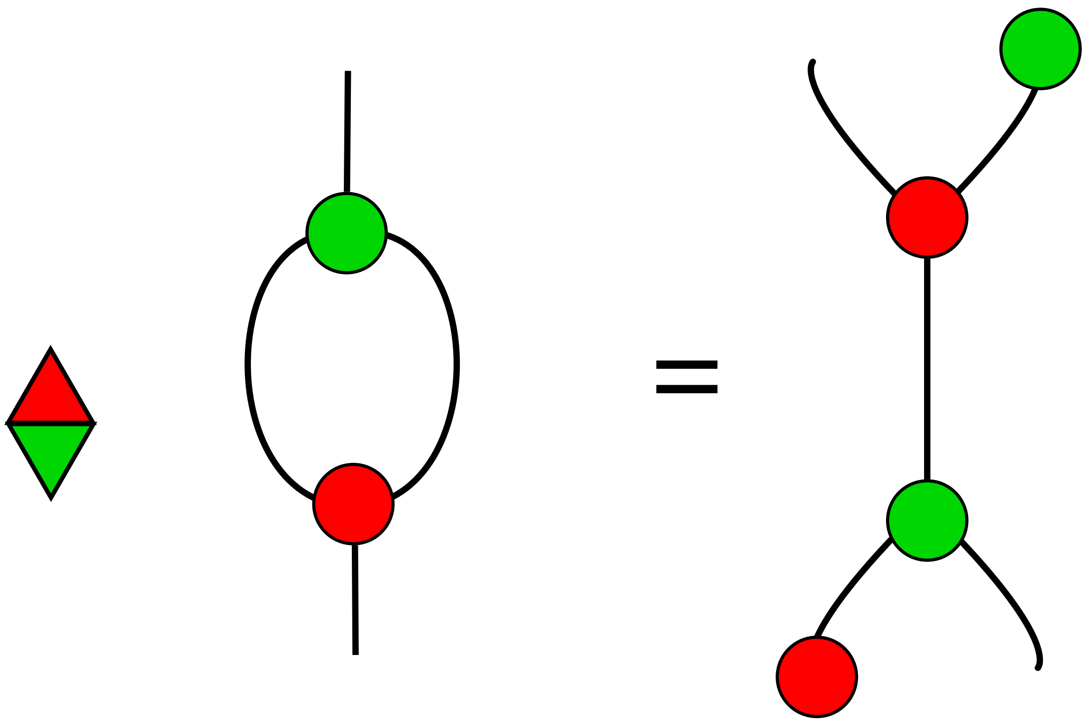

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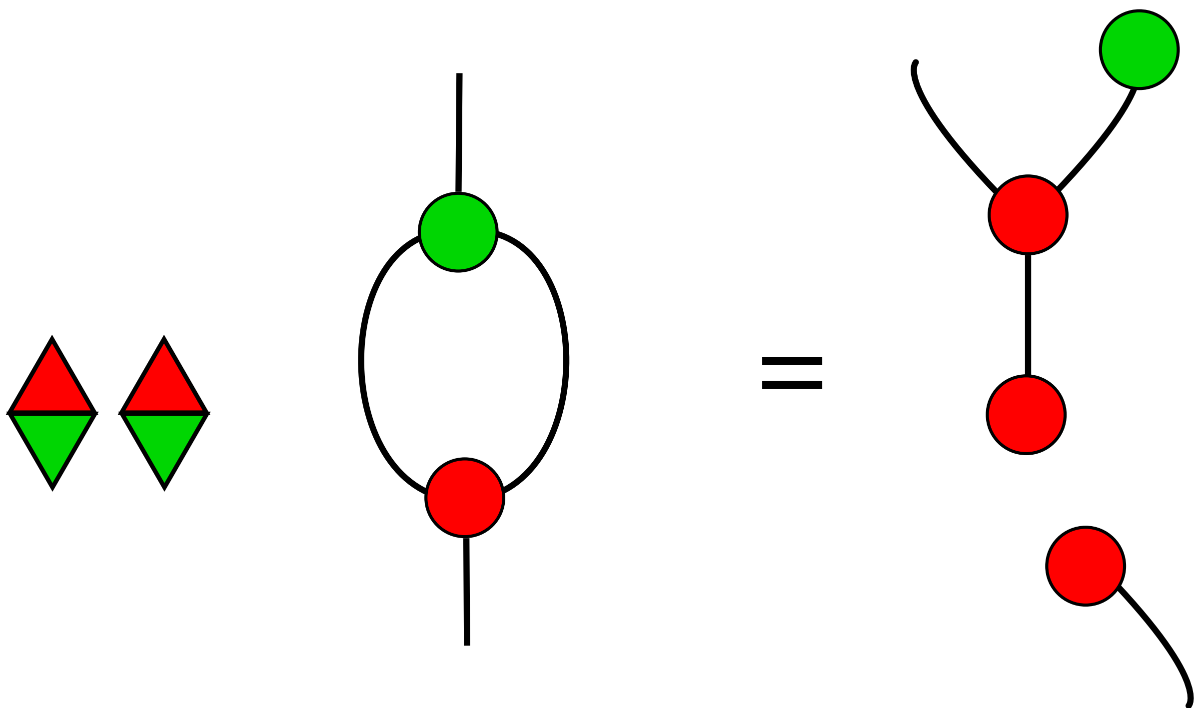

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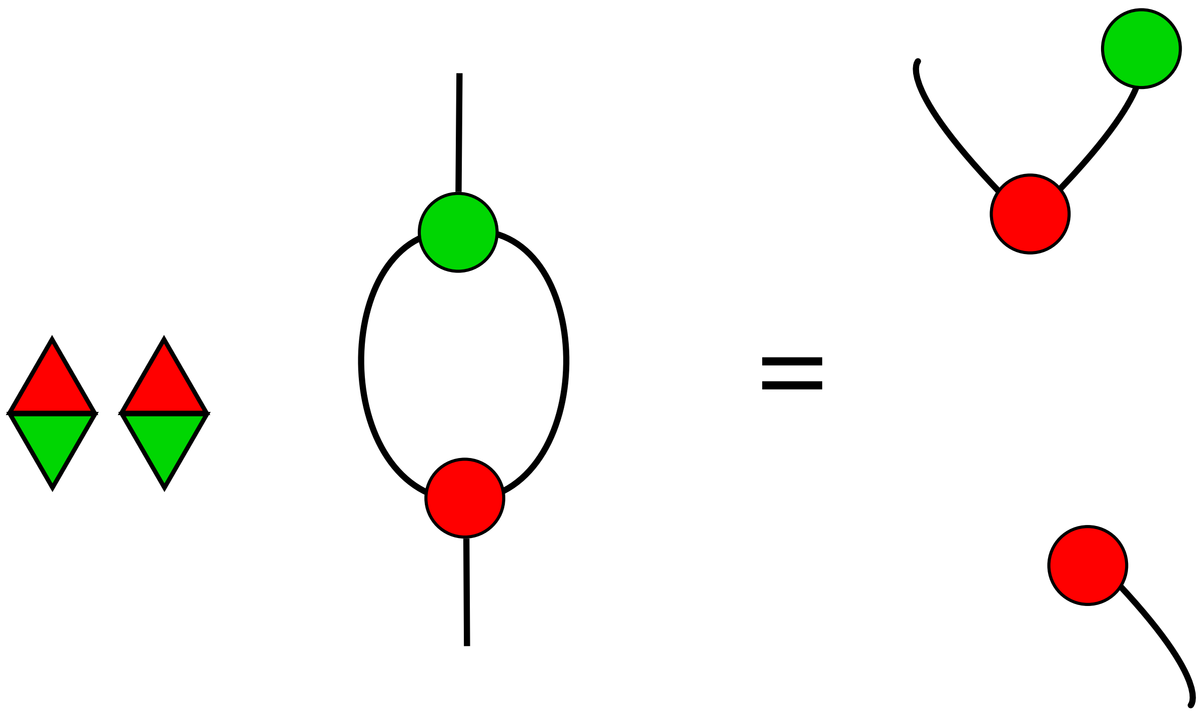

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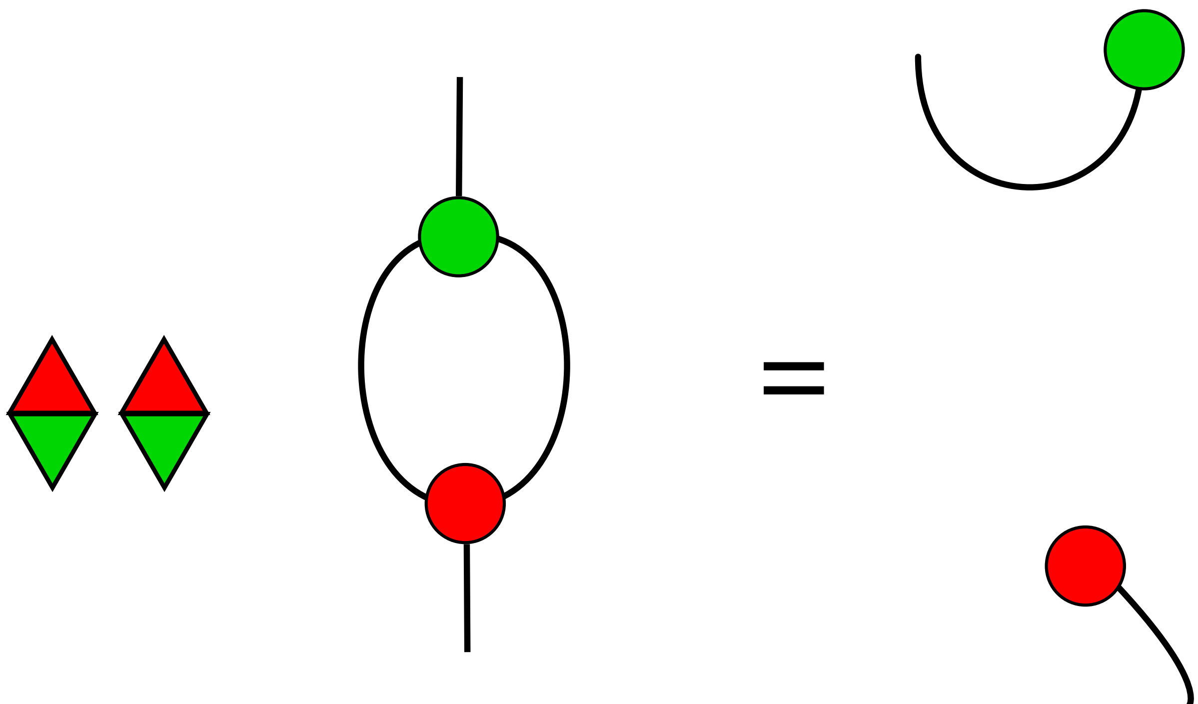

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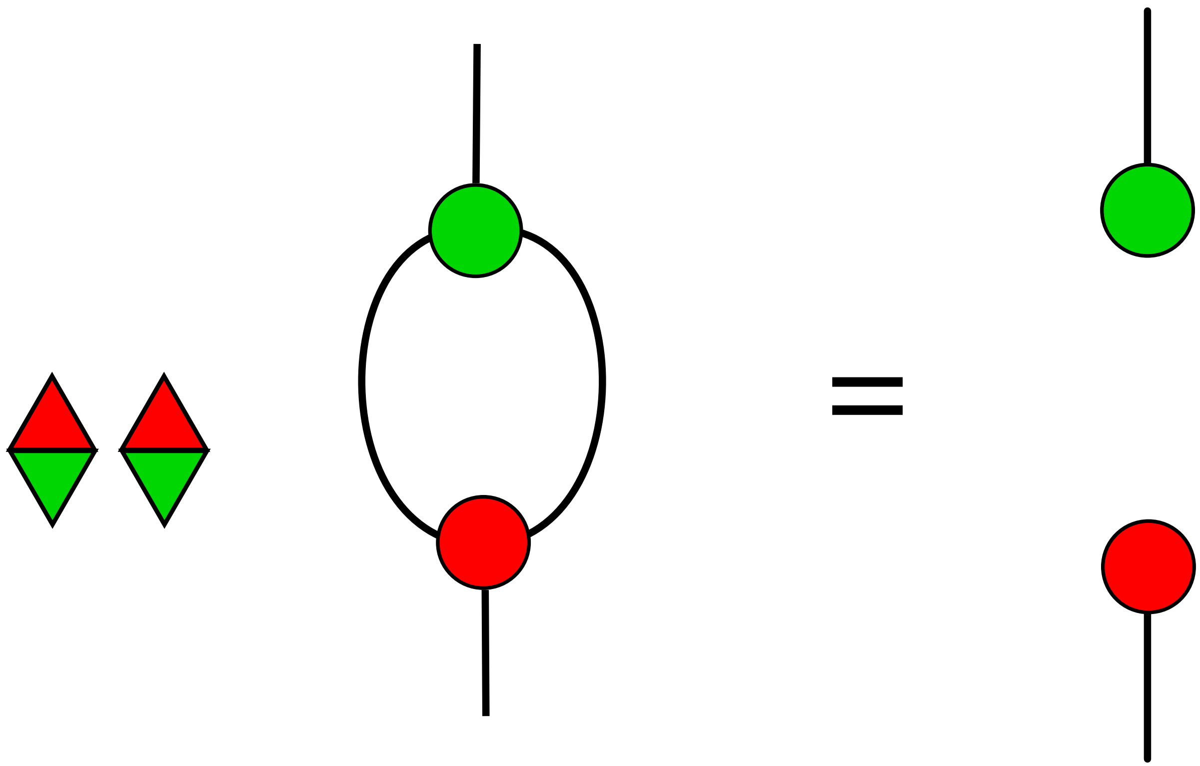

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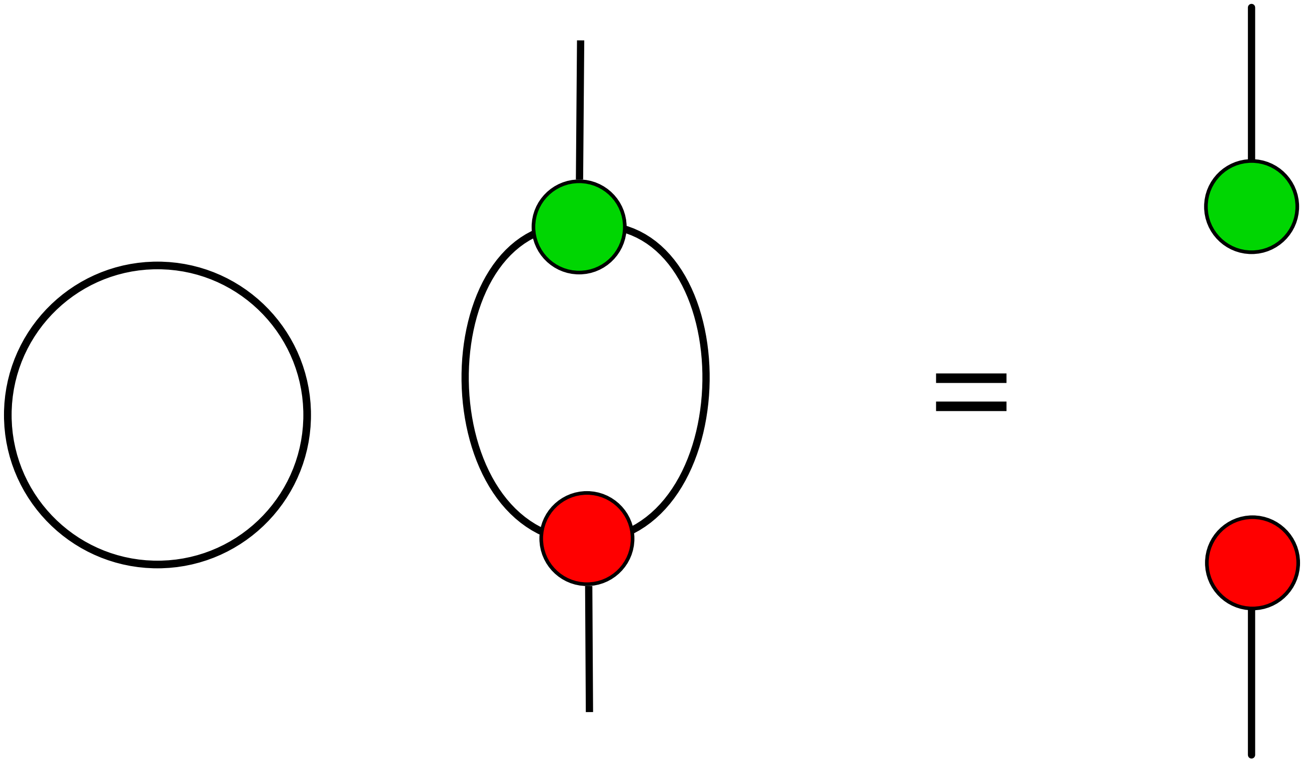
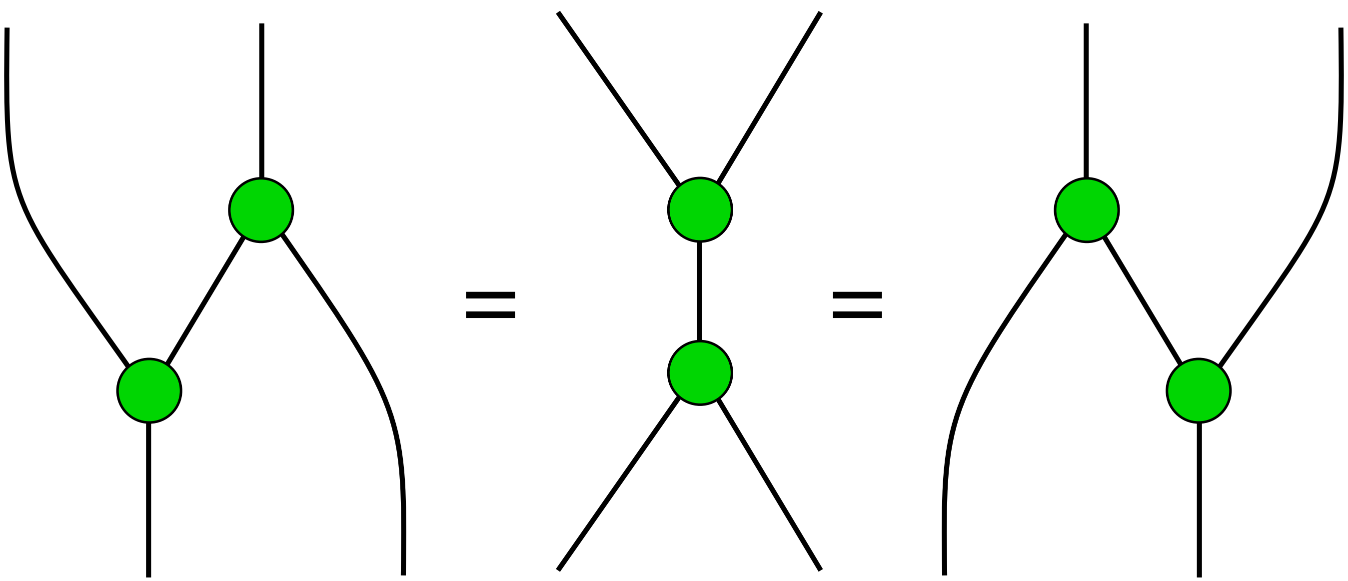


Therefore, the scaled bialgebra is in fact a *scaled Hopf algebra*, whose antipode is the identity times the dimension of the underlying space.

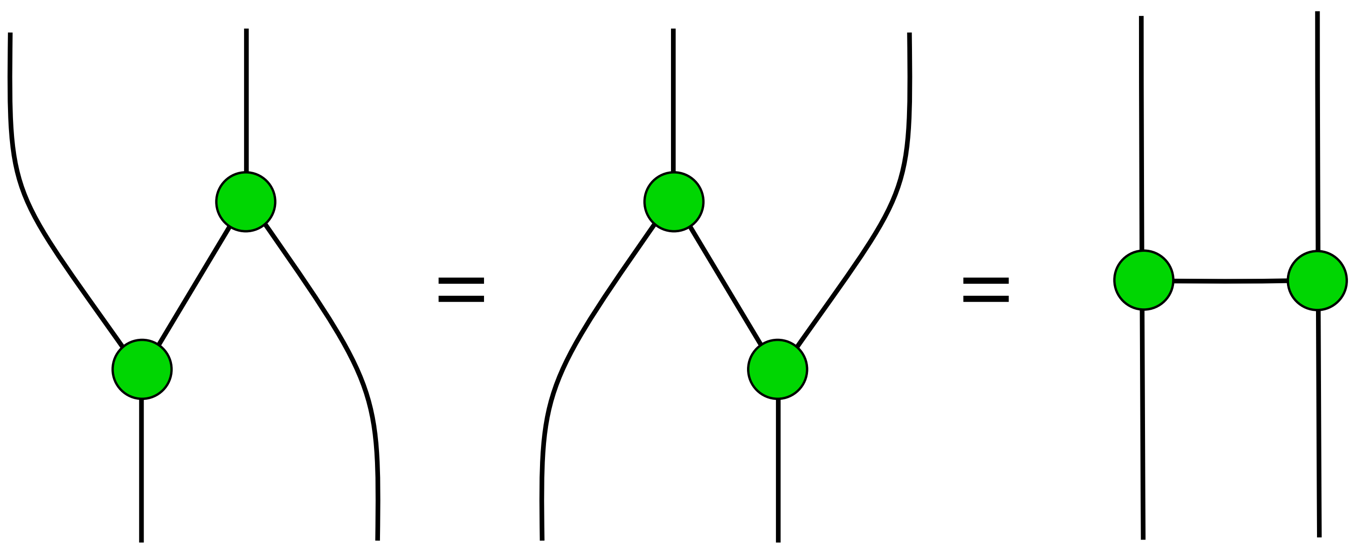
Temporality?

We have the following equation:



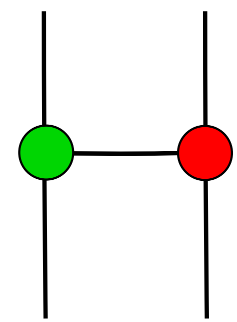
Temporality?

Hence the following is well defined:

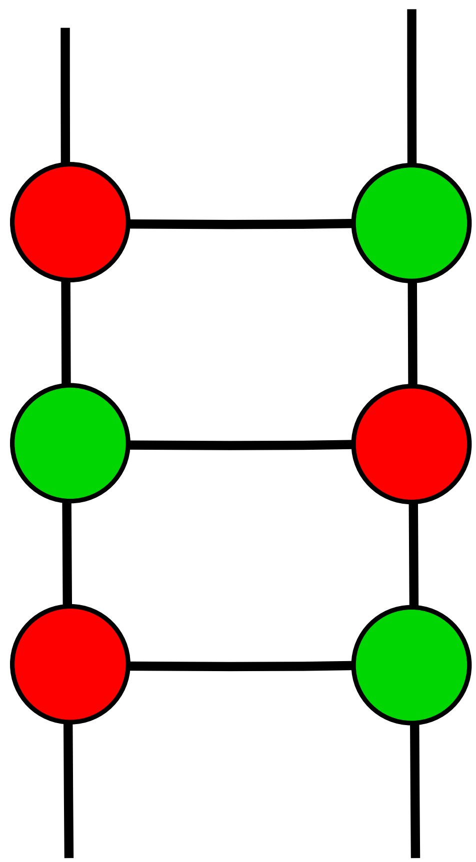


Unlike usual logic gate notation, both vertical and horizontal lines have the same meaning.

Representing Quantum Logic Gates (1)

$$\wedge X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} =$$ 

Example: $3 \times \wedge X =$ swap



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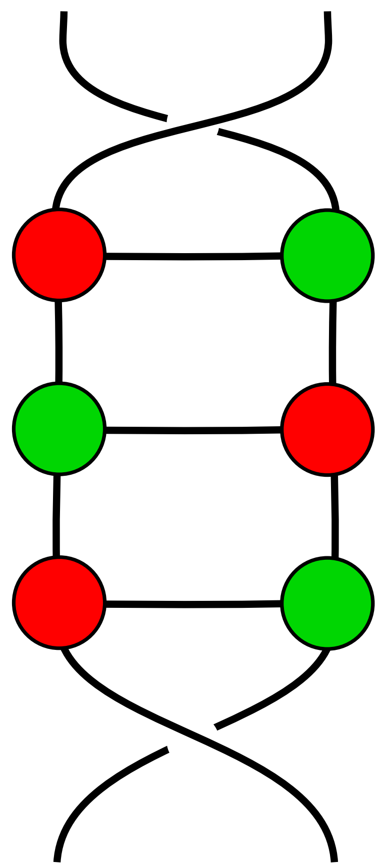

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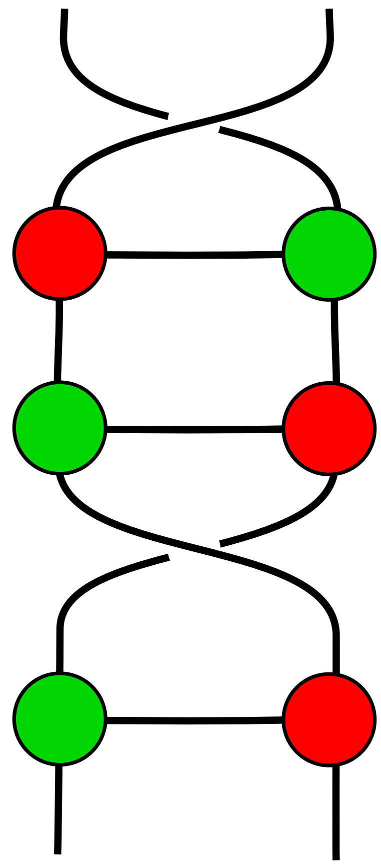

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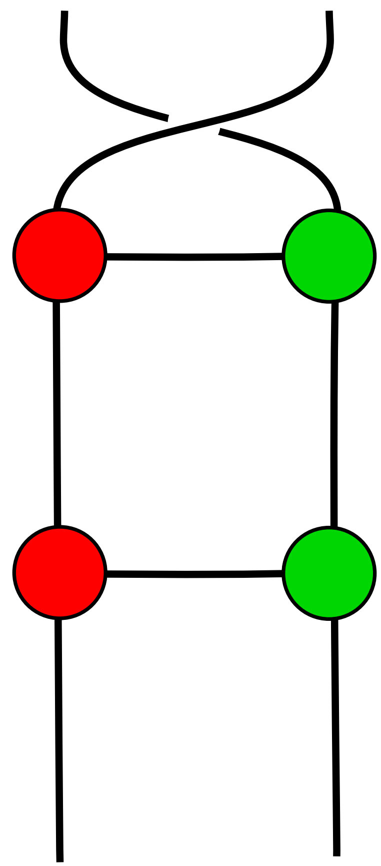

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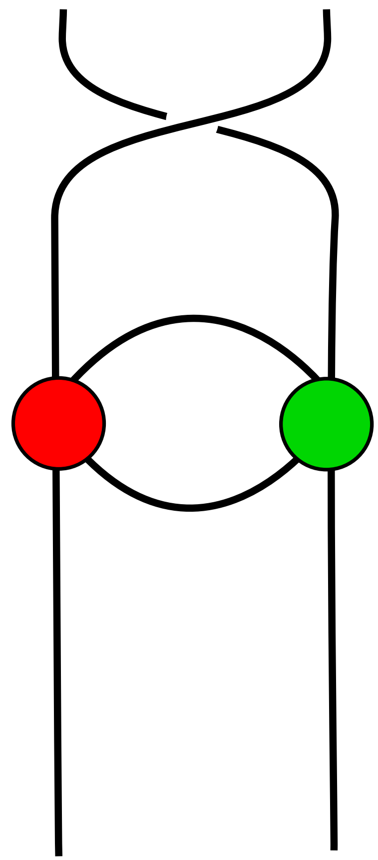

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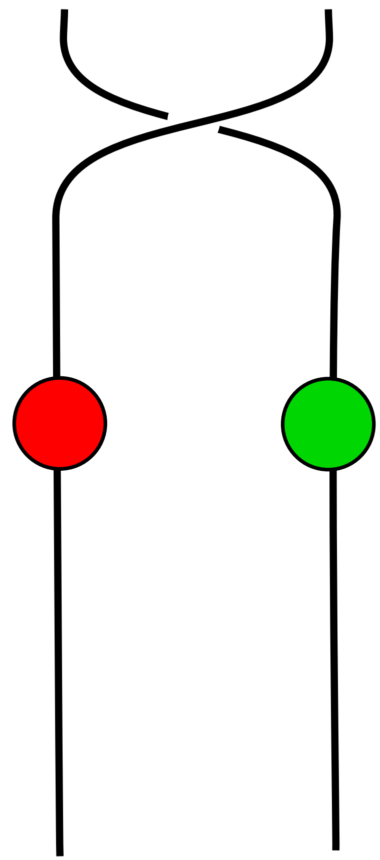

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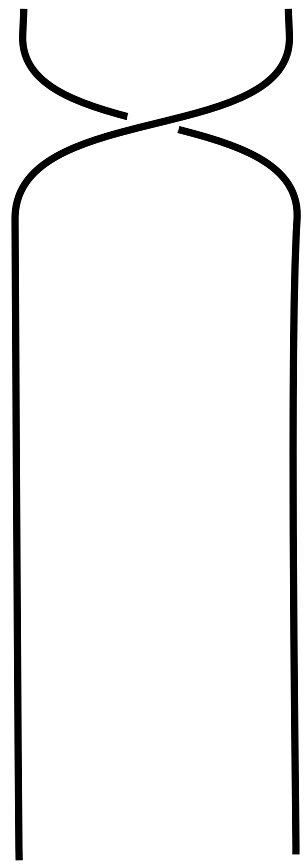
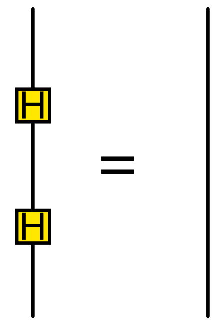


The Hadamard Map

The *Hadamard map* $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ enjoys a number of useful properties:

- Self adjointness: $H = H^\dagger$; and unitarity: $HH = \mathrm{id}$;



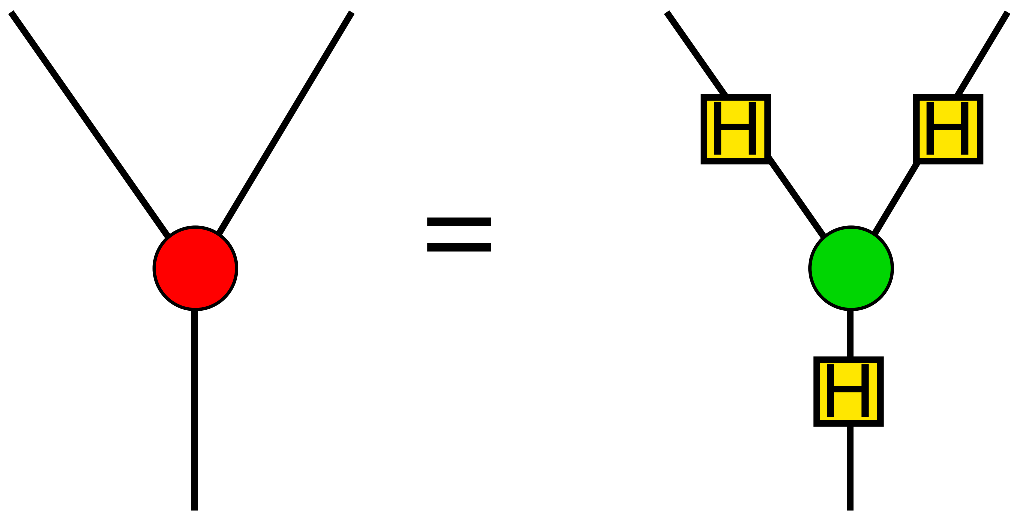
- The Hadamard exchanges the X and Z bases.

Hence:

$$\delta_X = (H \otimes H)\delta_Z H \qquad\qquad \epsilon_X = \epsilon_Z H$$

Hadamard as a Mediating Map

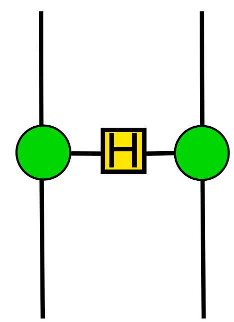
We can define the red classical structure in terms of H and the green structure:



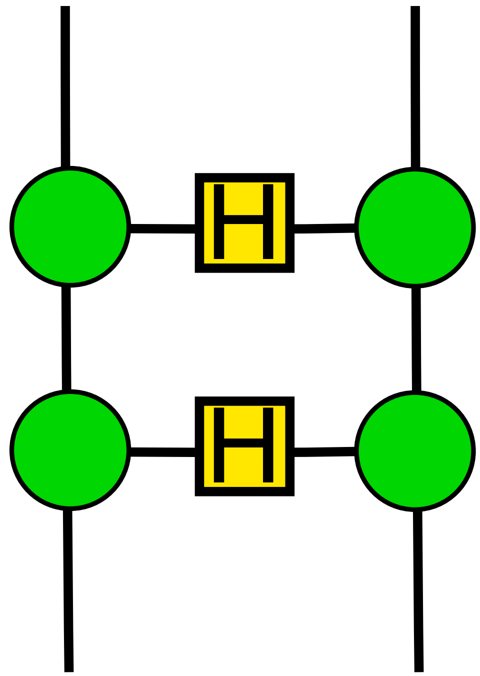
We can immediately derive a law for changing the colour of dots by introducing H boxes – in fact this gives a general "colour duality".

Representing Quantum Logic Gates (2)

$$\wedge Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} =$$



Example: $\wedge Z \circ \wedge Z = \mathrm{id}$



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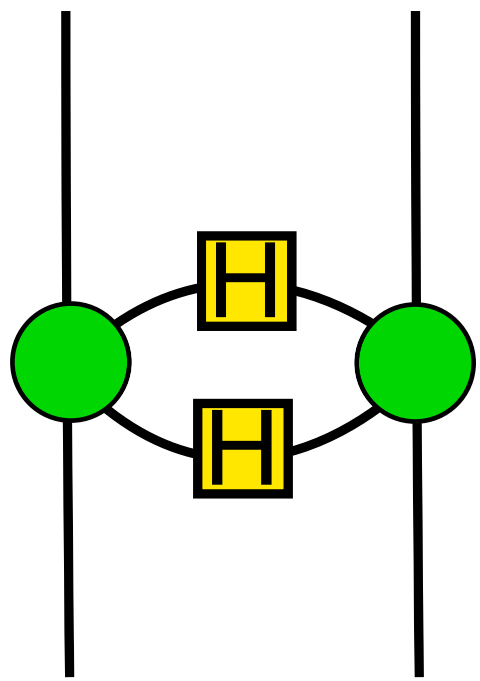

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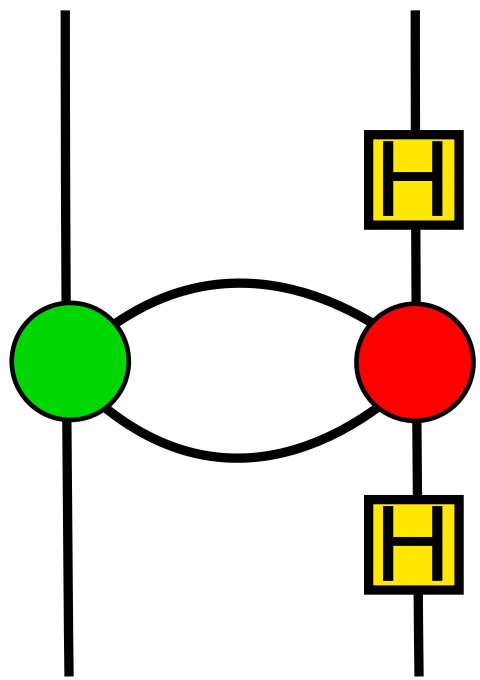

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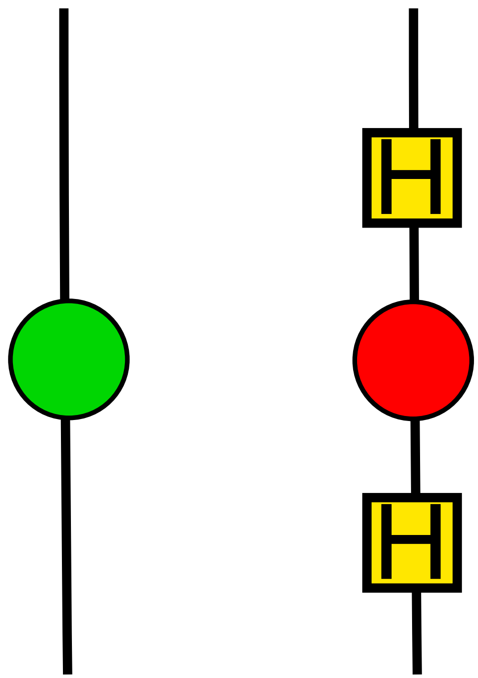

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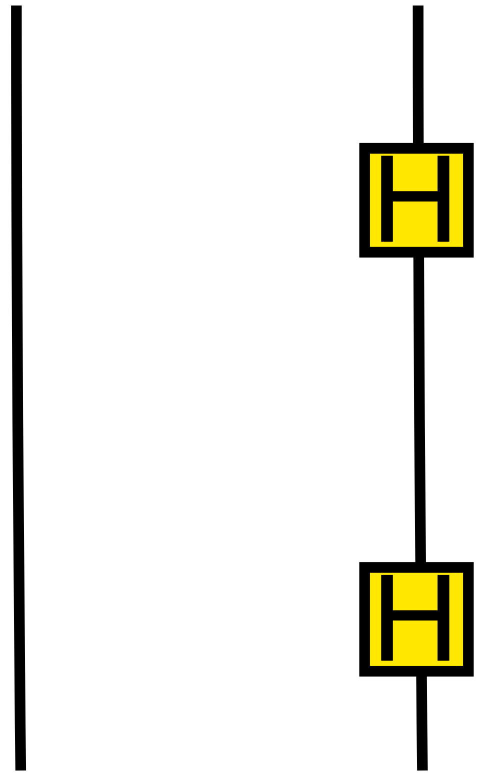

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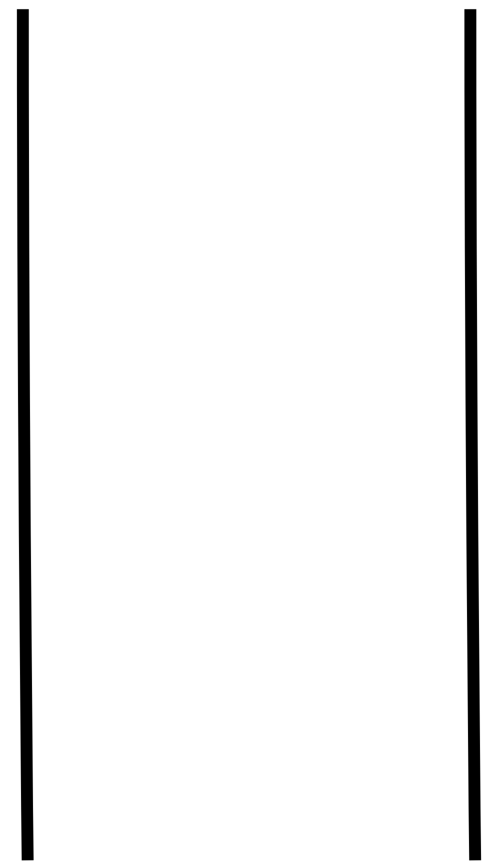
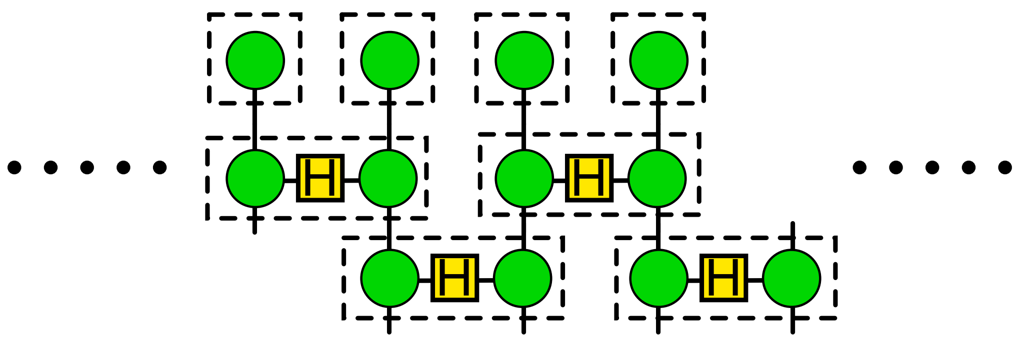


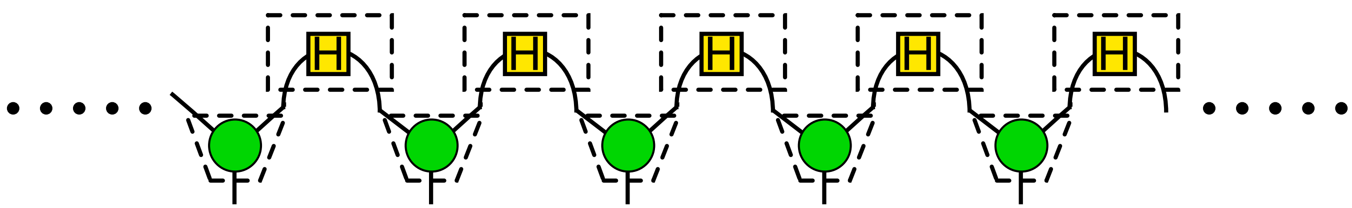
Preparing a 1D-Cluster State

The cluster state can be prepared by applying a $\wedge Z$ operation between pairs of qubits in the $|+\rangle$ state:



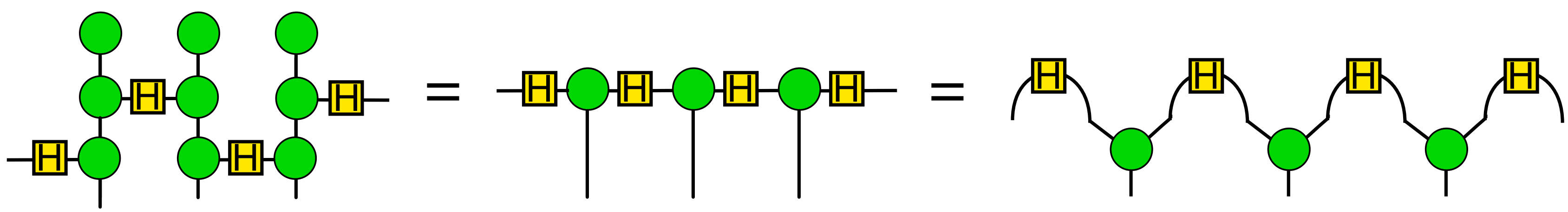
Preparing a 1D-Cluster State

Alternatively, the cluster state can be prepared by fusion of states of the form $|0+\rangle + |1-\rangle$. Recalling that $\delta_Z^\dagger$ is the fusion operation, this method of preparation can be represented as:



Preparing a 1D-Cluster State

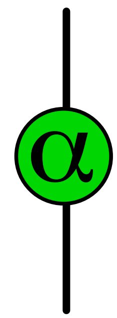
By the spider law, these are equivalent:



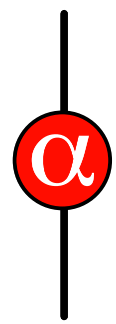
Incorporating Phases

Let $\alpha \in (0, 2\pi)$; consider the maps:

$$Z_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} =$$

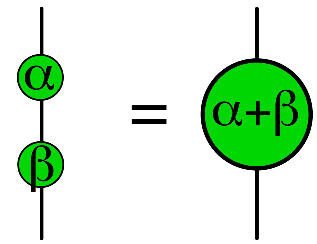


$$X_\alpha = HZ_\alpha H =$$

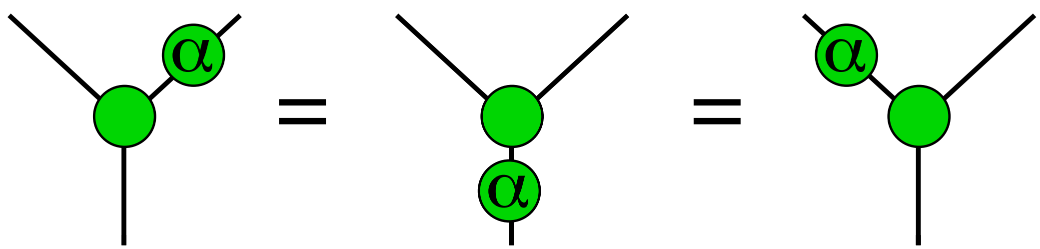


Incorporating Phases

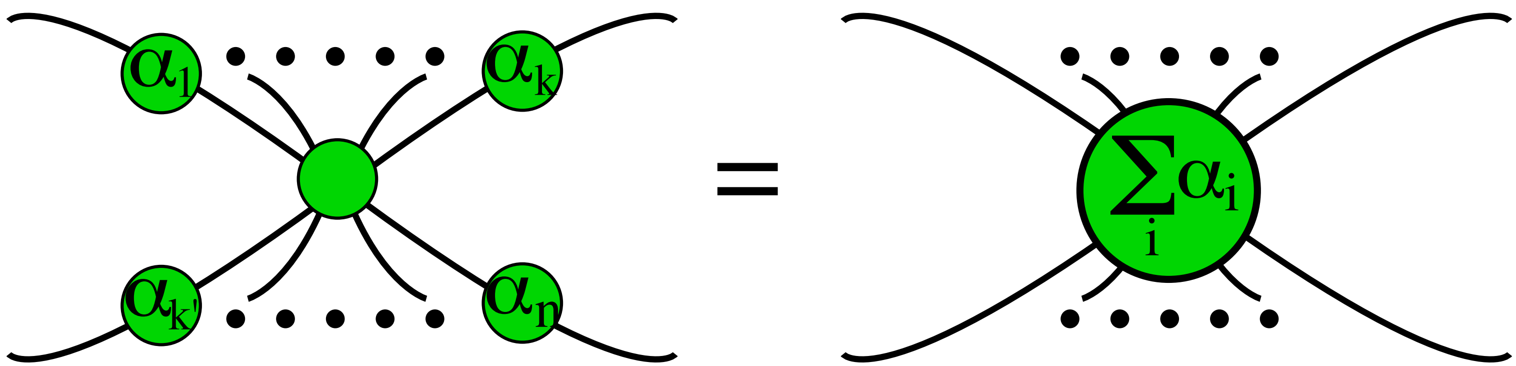
$$Z_\alpha \circ Z_\beta = Z_{\alpha+\beta}$$



$$\begin{tikzcd}
Q \arrow[r, "\delta"] \arrow[d, "Z_\alpha"'] & Q \otimes Q \arrow[d, "Z_\alpha \otimes \mathrm{id}"] \\
Q \arrow[r, "\delta"'] & Q \otimes Q
\end{tikzcd}$$



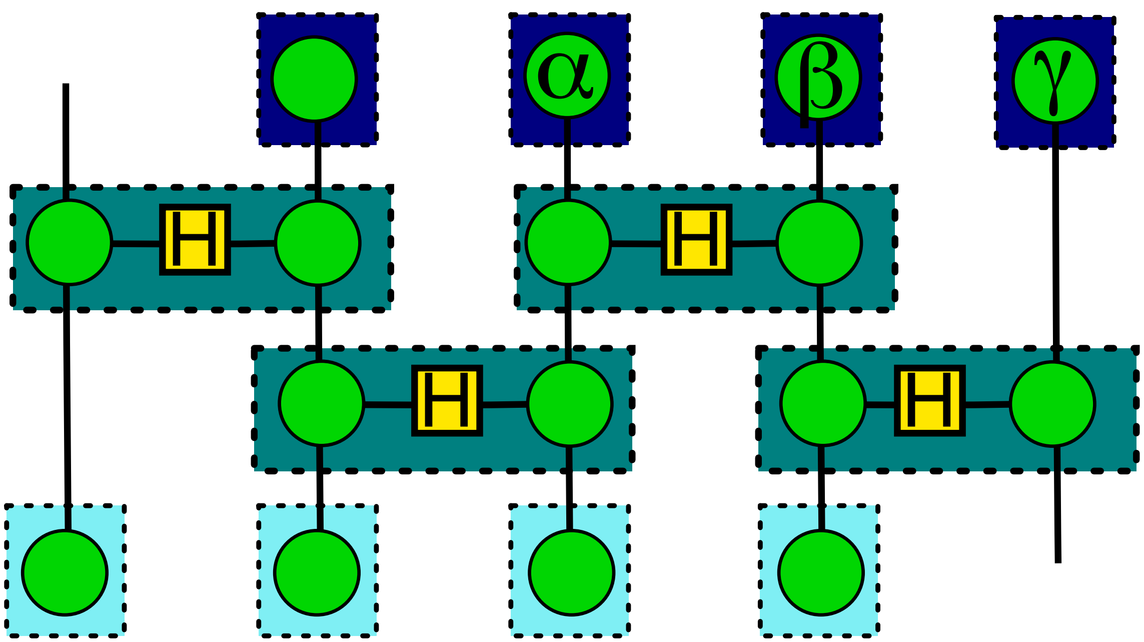
Generalised Spider Law



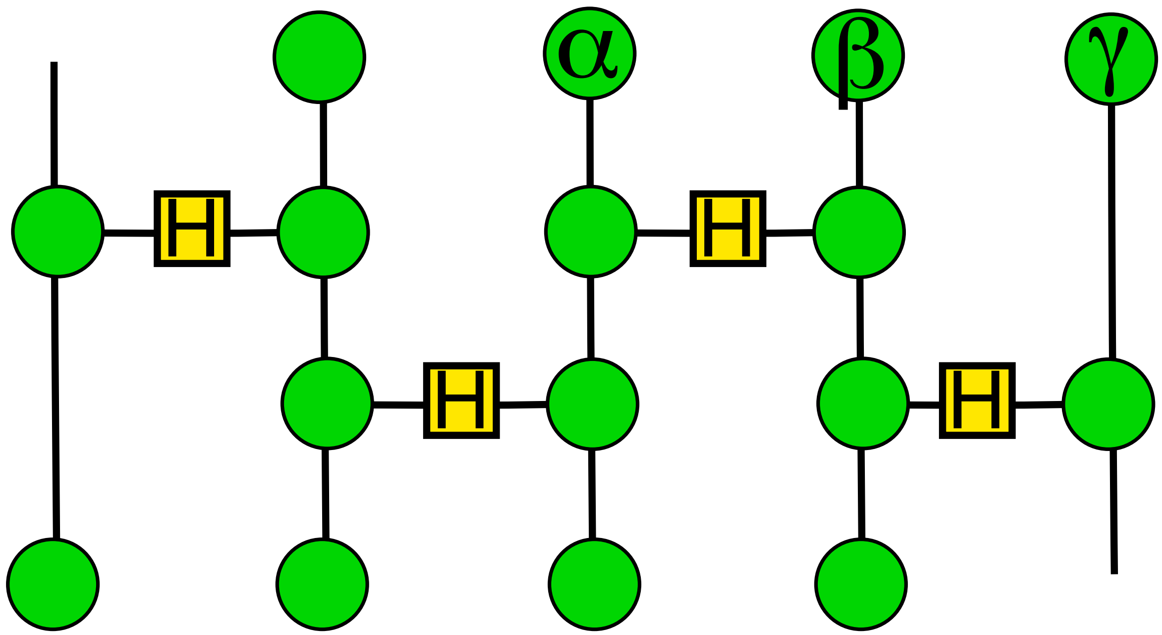
General unitary U

Proposition 2. *If U is a unitary on $\mathbb{C}^2$ there exist α, β, γ such that $U = Z_\alpha X_\beta Z_\gamma$.*

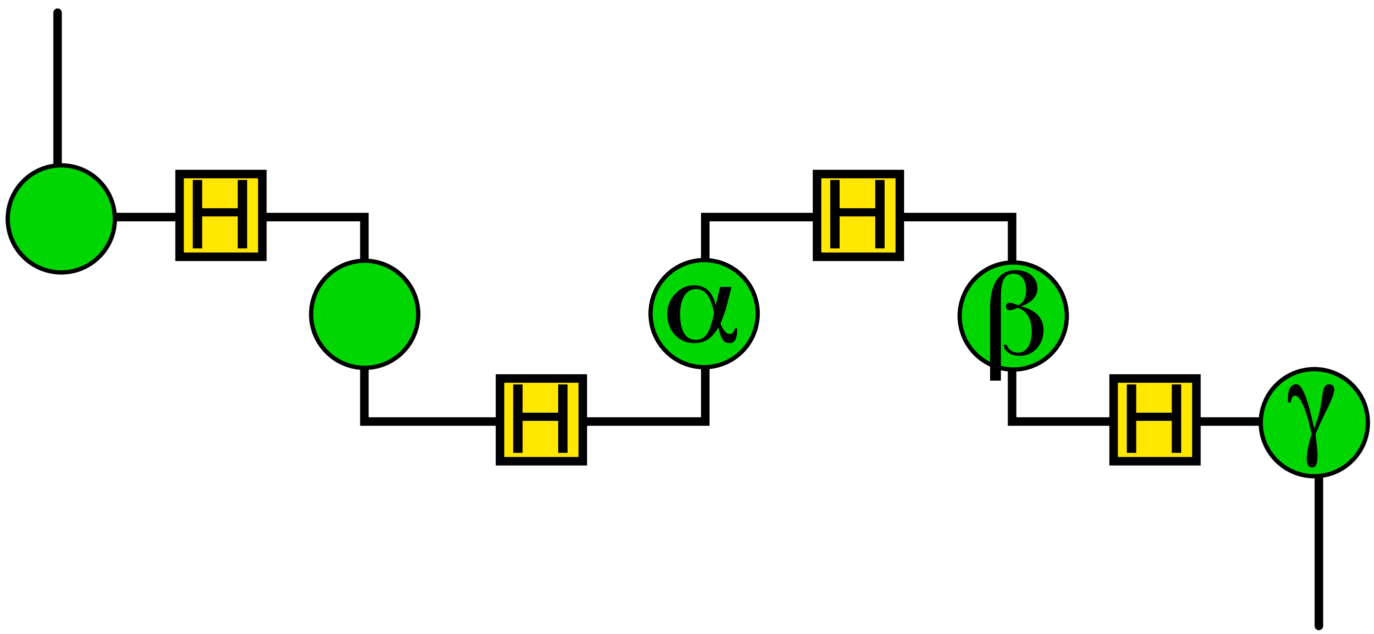
Here is (part of) a measurement based program to compute this:



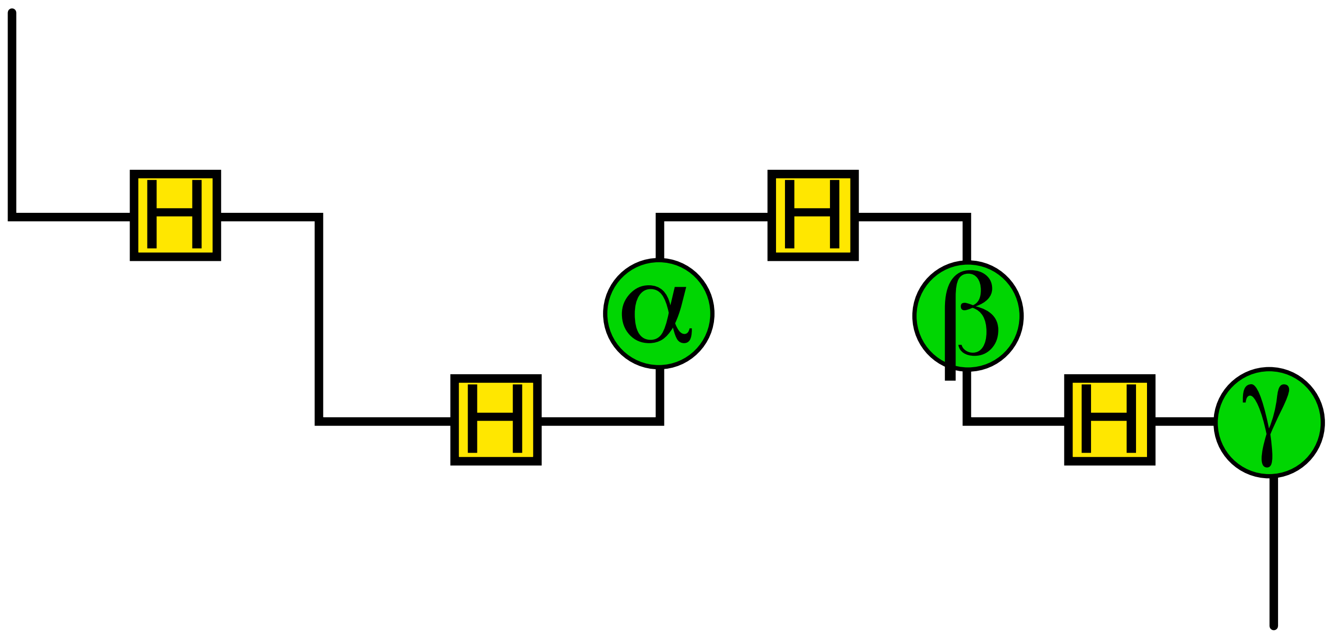
General unitary U



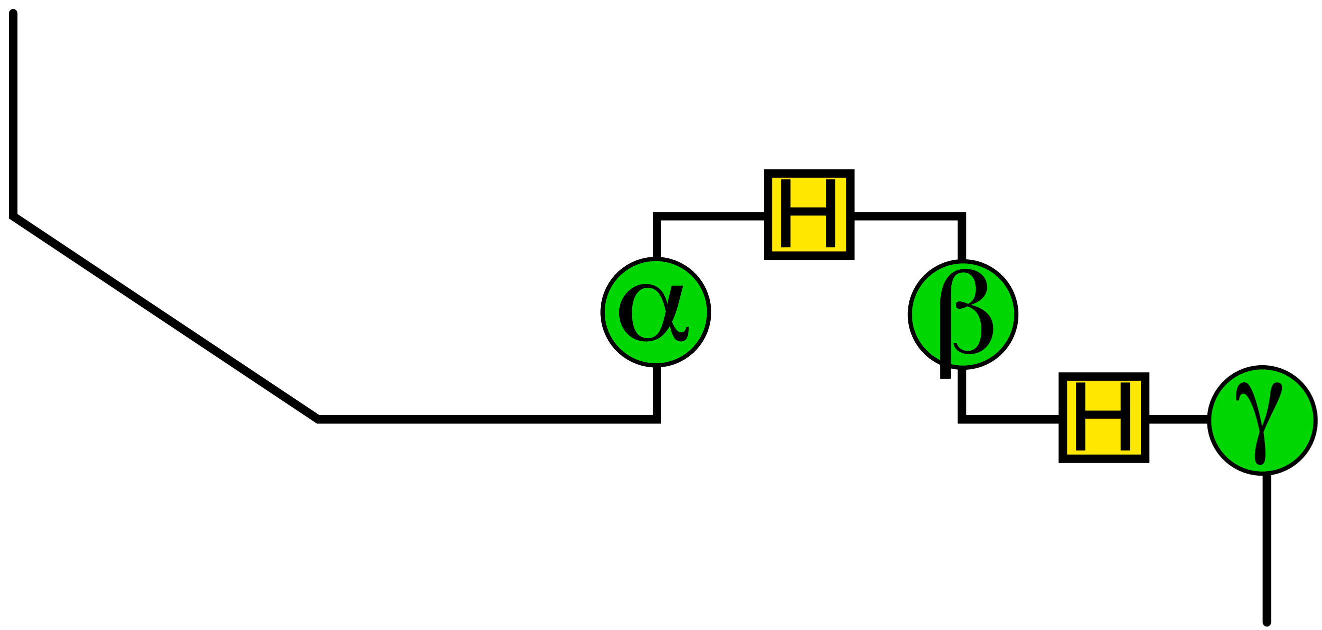
General unitary U



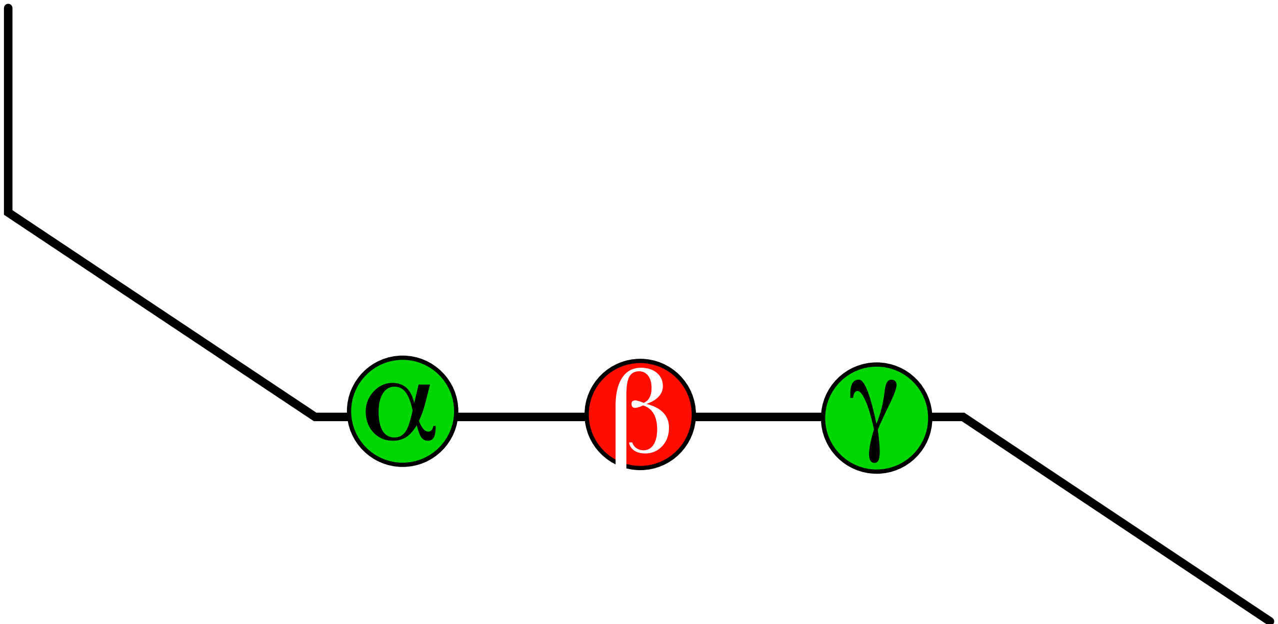
General unitary U



General unitary U



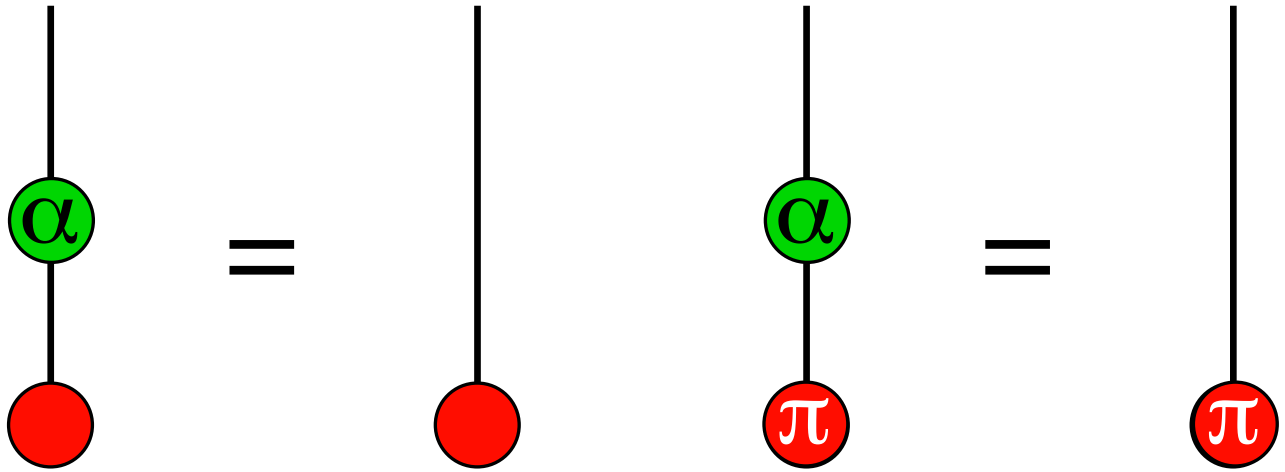
General unitary U



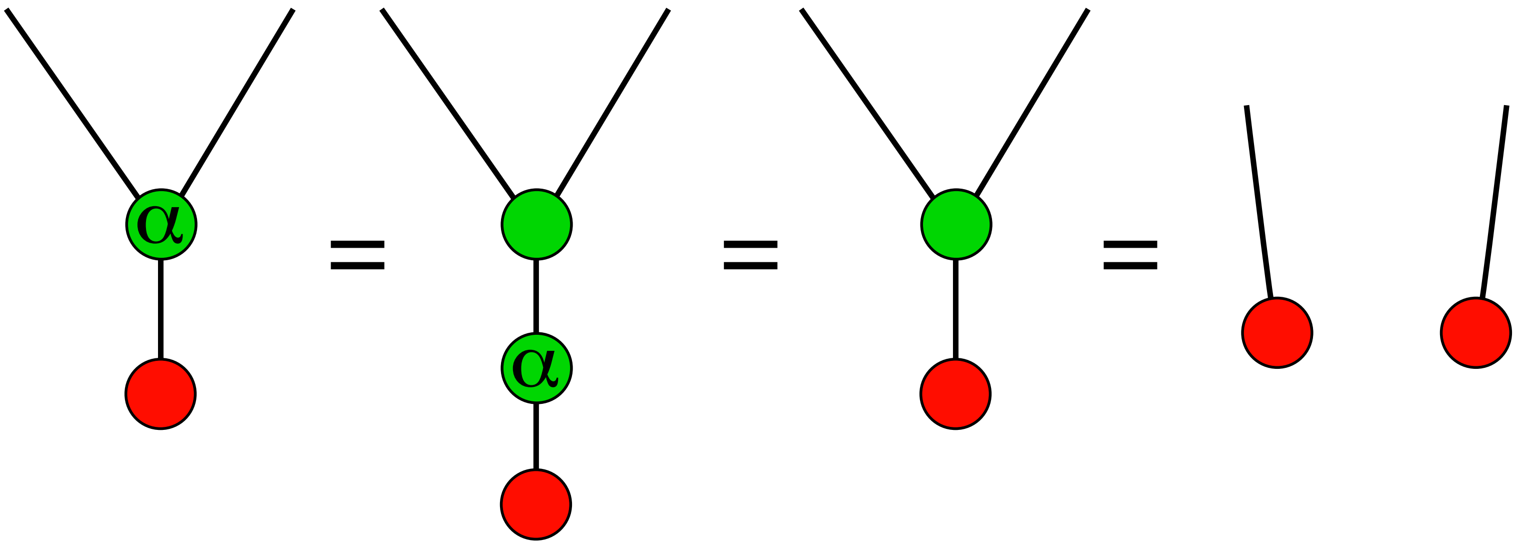
$$= Z_{\alpha} X_{\beta} Z_{\gamma}$$

How do phases interact?

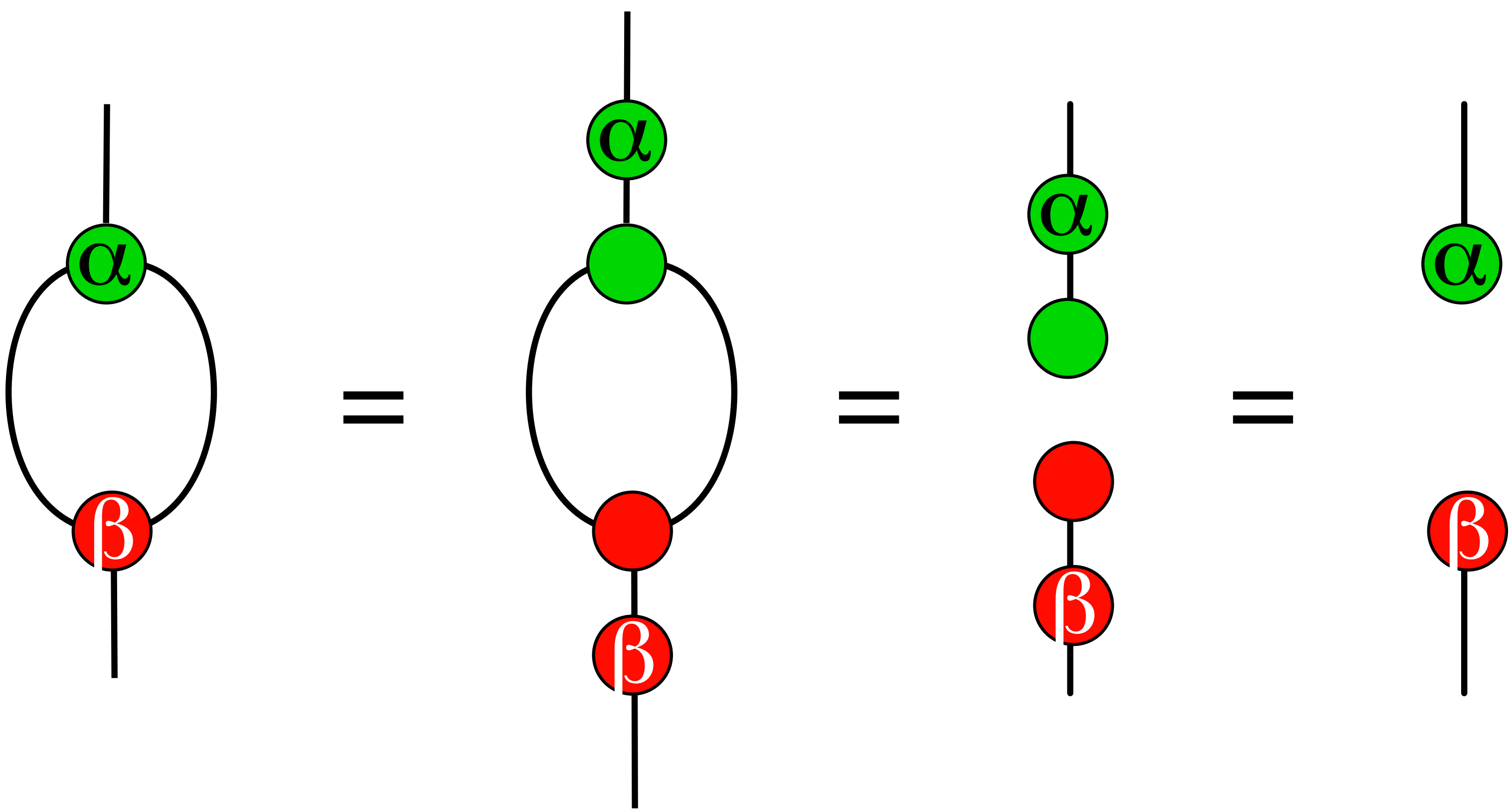
$$Z_\alpha |0\rangle = |0\rangle \qquad Z_\alpha |1\rangle = e^{i\alpha} |1\rangle = |1\rangle$$



How do phases interact?



How do phases interact?

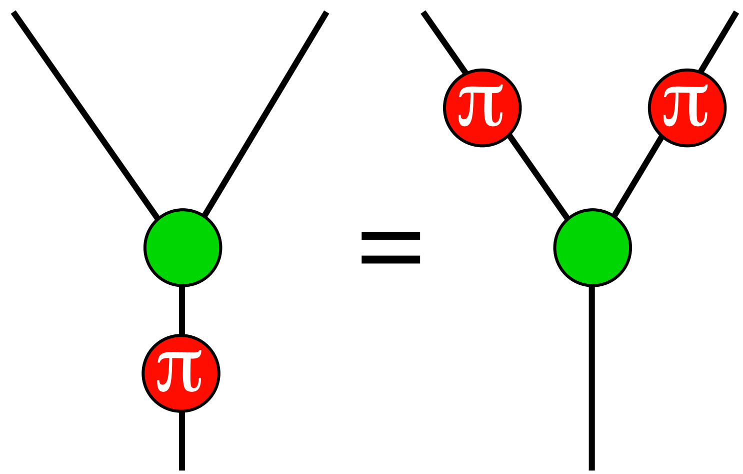


“Negation”

$$X_\pi = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad :: \left\{ \begin{array}{l} |0\rangle \mapsto |1\rangle \\ |1\rangle \mapsto |0\rangle \end{array} \right.$$

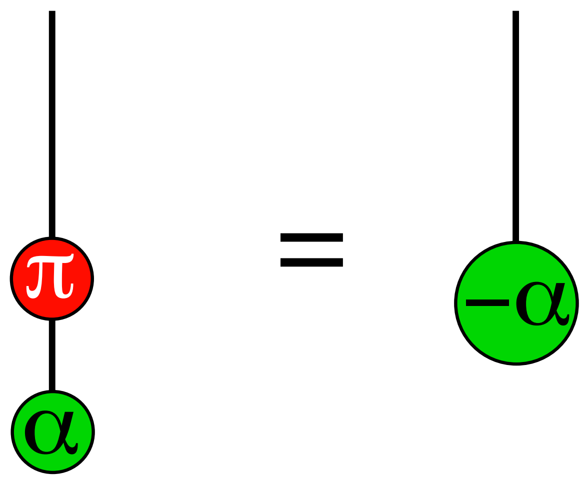
$$\begin{tikzcd} Q \arrow[r, "\delta"] \arrow[d, "X"'] & Q \otimes Q \arrow[d, "X \otimes X"] \\ Q \arrow[r, "\delta"'] & Q \otimes Q \end{tikzcd}$$

“Negation”



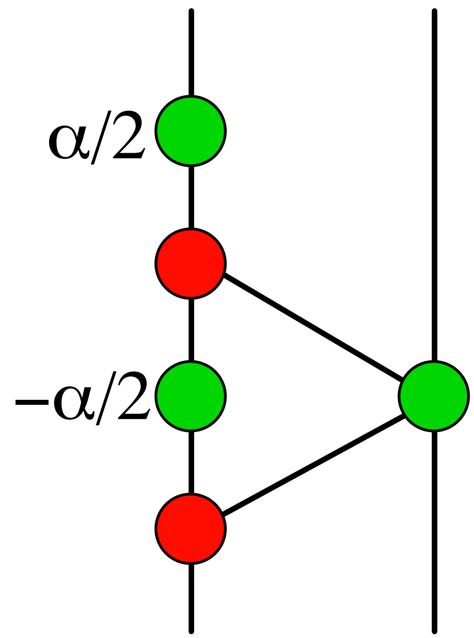
“Negation”

$$X :: |0\rangle + e^{i\alpha}|1\rangle \mapsto e^{i\alpha}|1\rangle + |0\rangle = |0\rangle + e^{-i\alpha}|1\rangle$$



Representing Controlled Phase

$$\wedge Z_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\alpha} \end{pmatrix} =$$



Example: Quantum Fourier Transform

Among the most important quantum algorithms, the quantum fourier transform is a key stage of factoring.

$$|j_0 j_1 \cdots j_n\rangle \mapsto (|0\rangle + e^{2\pi i \alpha_0}|1\rangle)(|0\rangle + e^{2\pi i \alpha_1}|1\rangle) \cdots (|0\rangle + e^{2\pi i \alpha_n}|1\rangle)$$

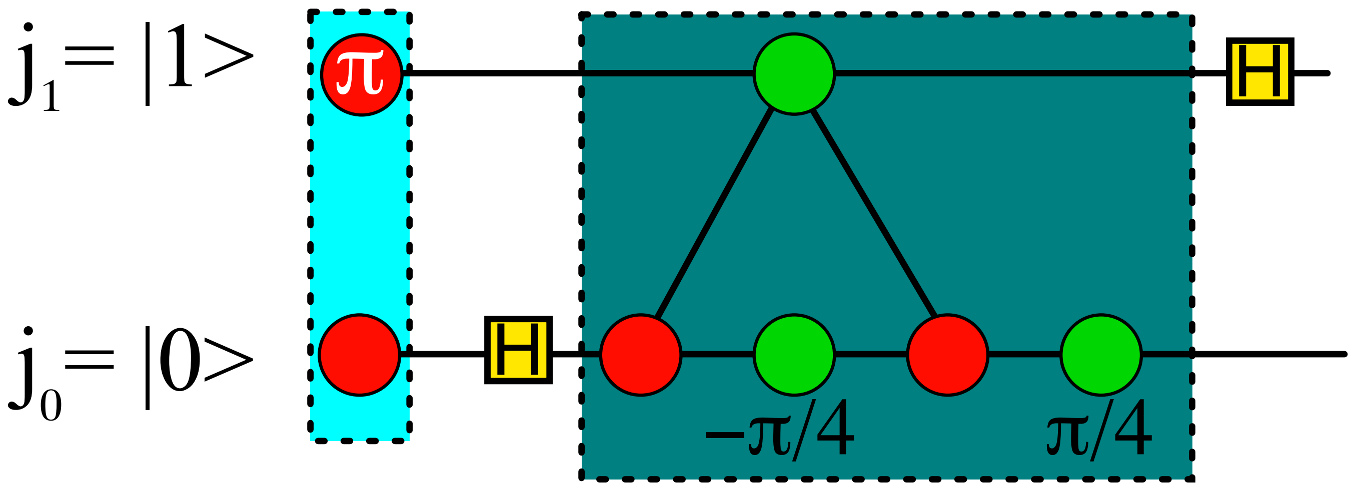
where $\alpha_k = 0.j_k \cdots j_n = \sum_{l=k}^{n} j_l / 2^k$

For 2 qubits:

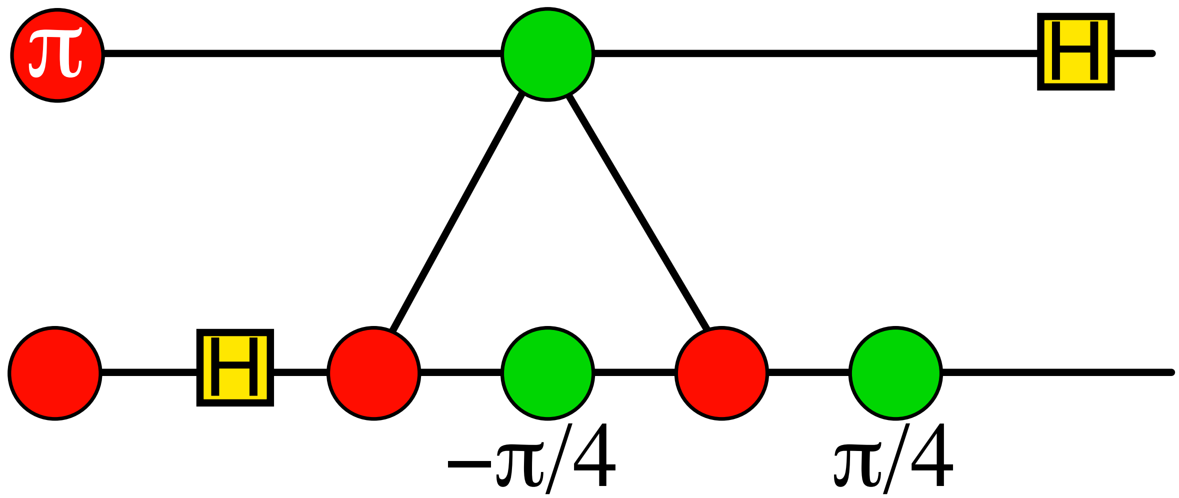
$$|00\rangle \mapsto (|0\rangle + |1\rangle)(|0\rangle + |1\rangle) \qquad |10\rangle \mapsto (|0\rangle + e^{i\pi}|1\rangle)(|0\rangle + |1\rangle)$$

$$|01\rangle \mapsto (|0\rangle + e^{i\pi/2}|1\rangle)(|0\rangle + e^{i\pi}|1\rangle) \qquad |11\rangle \mapsto (|0\rangle + e^{i3\pi/2}|1\rangle)(|0\rangle + e^{i\pi}|1\rangle)$$

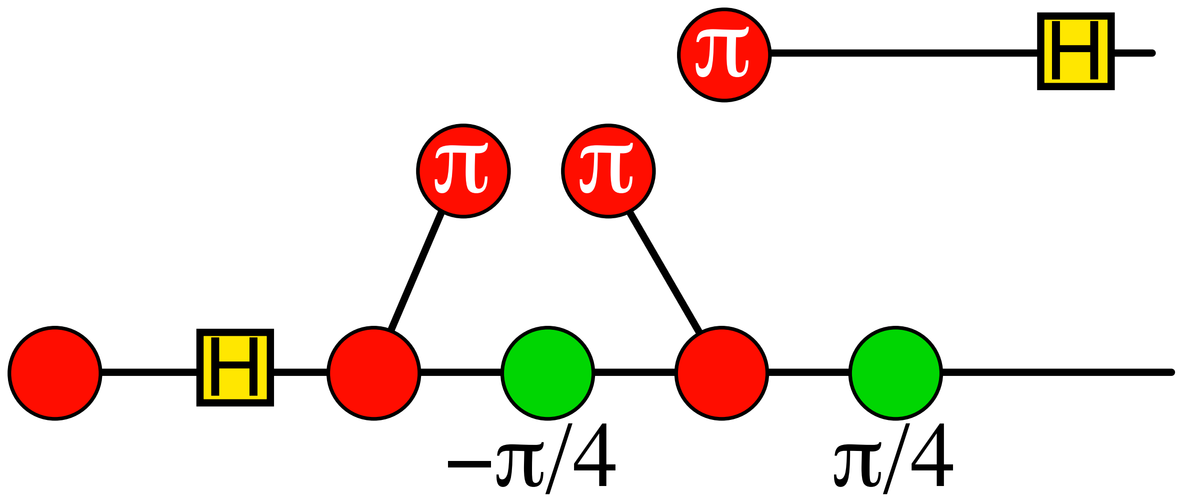
Example: Quantum Fourier Transform



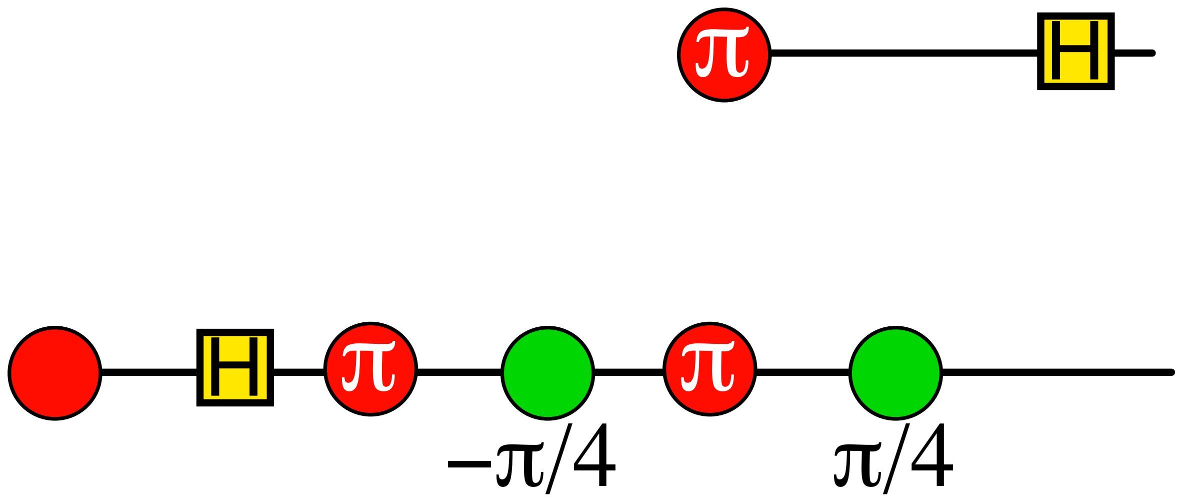
Example: Quantum Fourier Transform



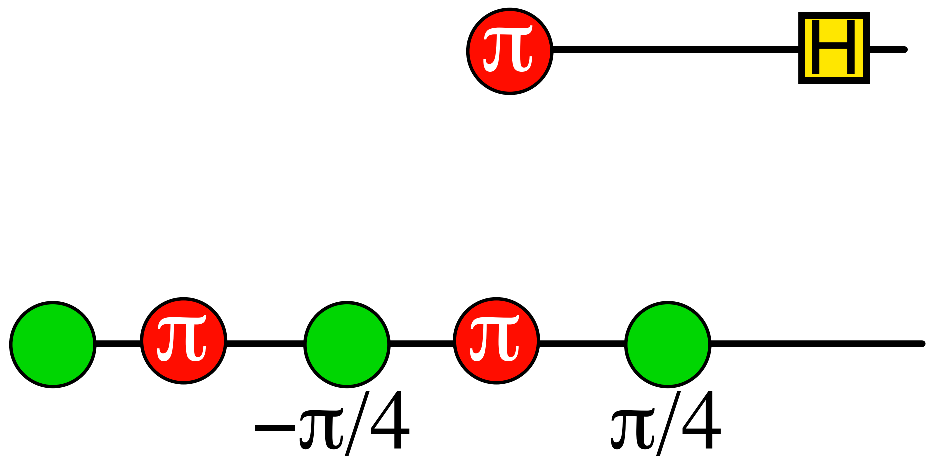
Example: Quantum Fourier Transform



Example: Quantum Fourier Transform



Example: Quantum Fourier Transform



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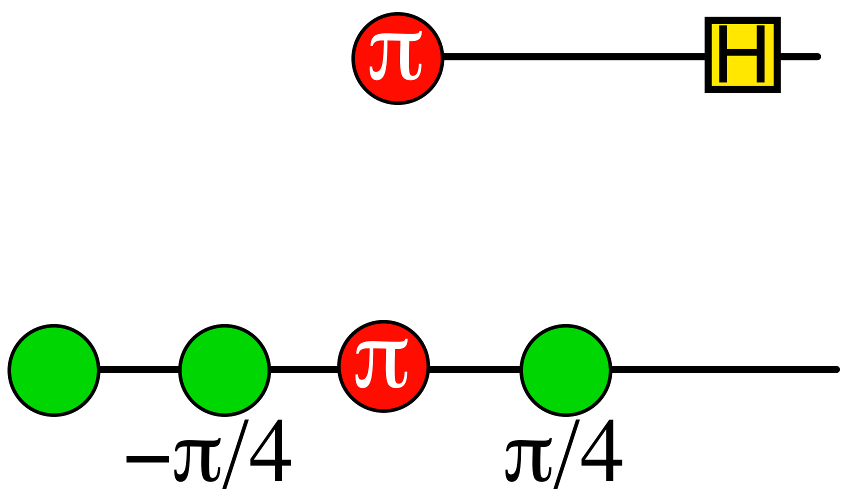
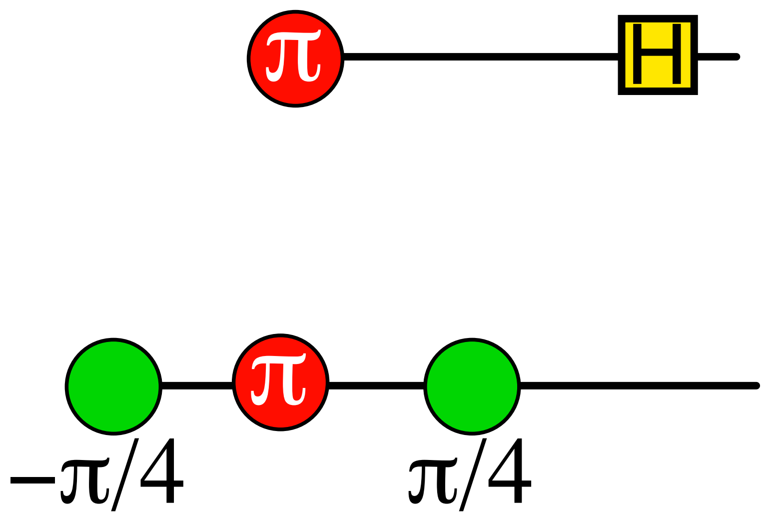
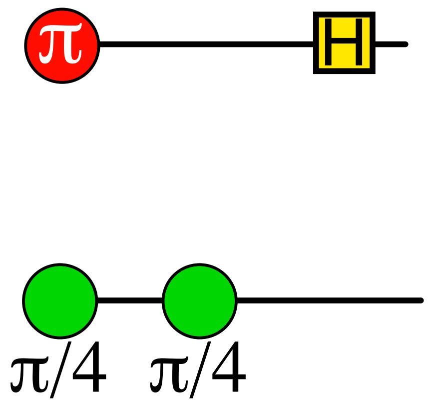


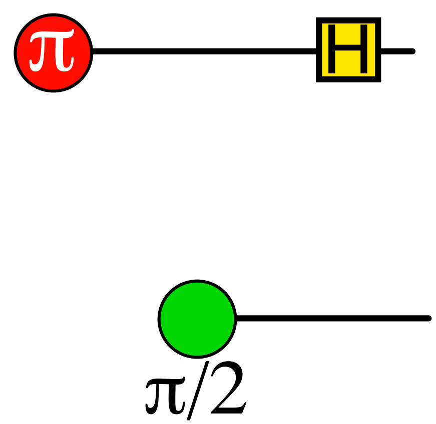
Example: Quantum Fourier Transform



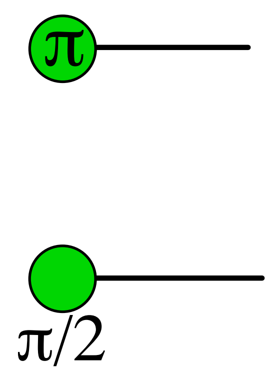
Example: Quantum Fourier Transform



Example: Quantum Fourier Transform



Example: Quantum Fourier Transform



which is the correct result! YAY!

Conclusions

- Pairs of incompatible observables form a Hopf algebra-like structure.
- This structure captures a fundamental aspect of quantum mechanics.
- The axioms are sufficiently strong to derive the properties of quantum logic gates and prove the correctness of important quantum algorithms.

Ongoing Work

- Relating the general theory of MUBs to the underlying classical operations;
- Graphical characterisations of multipartite entangled states;
- Flow and GFlow?
- Formal properties:
  - Rewriting: Confluence? Termination?
  - Mechanisation (in progress with Lucas Dixon)
  - Induction principles for reasoning about graphical rewriting?
  - Model-theoretic completeness?