A topos for algebraic quantum theory

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Plan

Given C*-algebra A, make topos $\mathcal{T}(A)$.

Inside $\mathcal{T}(A)$, define *commutative* C*-algebra $\underline{A}$.
Inside $\mathcal{T}(A)$, consider its spectrum $\underline{\Sigma}$.

Internalize observable a and state ρ of A,
to get truth value for "$a \in (p, q)$ in state ρ".

Have to take care when reasoning inside $\mathcal{T}(A)$ ...

Categorical logic: monoidal

Can formulate notions definable by *monoidal logic*

finite conjunction

in any *monoidal category*

monoidal structure

e.g. monoid objects ($I \xrightarrow{e} M \xleftarrow{m} M \otimes M$), semiring objects

Such notions are preserved by *monoidal functors*

Categorical logic: cartesian

Can formulate notions definable by *cartesian logic*
- finite conjunction, and
- unique existential quantification

in any *cartesian category*
- finite products, and
- equalizers

e.g. group objects, ring objects

Such notions are preserved by *cartesian functors*

Categorical logic: regular

Can formulate notions definable by *regular logic*

finite conjunction, and
existential quantification

in any *regular category*

finite products,
equalizers, and
images

e.g. divisible group objects, division ring objects

Such notions are preserved by *regular functors*

Categorical logic: geometric

Can formulate notions definable by *geometric logic*

finite conjunction,
existential quantification, and
infinite disjunction

in any *geometric category*

finite products,
equalizers,
images
well-powered, with unions of subobjects

Such notions are preserved by *geometric functors*

Categorical logic: full higher-order

Can formulate notions definable by *full higher-order logic*

finite conjunction,
finite disjunction,
existential quantification,
universal quantification,
negation, and
implication

in any *topos*

finite products,
equalizers,
exponents, and
subobject classifier

Topos logic is the summum of categorical logic.

Categorical logic: intuitionistic

Fix interpretation of types, function symbols, relation symbols.
Then interpretation $[\![\varphi]\!] \in \mathrm{Sub}([\![\mathrm{FV}(\varphi)]\!])$ of formula φ fixed.
For closed φ:

$$[\![\varphi]\!] \in \mathrm{Sub}(1) \cong \Omega$$

So Ω is 'truth value object'
It is a Heyting algebra, more general than $\{\mathtt{false}, \mathtt{true}\}$.

T topos, then $T \models \varphi$ means $[\![\varphi]\!] = 1$ ('T validates φ')

Topos logic is intuitionistic logic

C*-algebras

A *C*-algebra* is a set A with

- addition $+ : A \times A \to A$,
- complex scalar multiplication $\cdot : \mathbb{C} \times A \to A$,
- multiplication $\cdot : A \times A \to A$,
- involution $(-)^* : A \to A$, and
- norm $\|\cdot\| : A \to \mathbb{R}$

such that

- $a^{**} = a$,
- $\|ab\| \leq \|a\| \cdot \|b\|$,
- $\|a^*a\| = \|a\|^2$,
- complete, ...

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- complete, ...

Need to be careful about $\mathbb{R}$ ...

C-algebras in a topos*

A *C*-algebra* is an object A with

addition $+ : A \times A \to A$,
complex scalar multiplication $\cdot : \mathbb{C}_{\mathbb{Q}} \times A \to A$,
multiplication $\cdot : A \times A \to A$,
involution $(-)^* : A \to A$, and
norm $N \subseteq A \times \mathbb{Q}$

such that

$a^{**} = a$,
if $(a,p) \in N$ and $(b,q) \in N$, then $(ab, pq) \in N$,
$(a^*a, q^2) \in N$ iff $(a,q) \in N$,
complete, ...

Intuitionistic: relation $N \subseteq A \times \mathbb{Q}$ instead of function $\|\cdot\| : A \to \mathbb{R}$.
(idea: $(a,q) \in N$ iff $\|a\| < q$)

So can formulate this notion in any topos.

Bohr's doctrine of classical concepts

C*-algebras are time-honored way to organise quantum theory.
But Bohr says: only access to quantum physics via classical physics.
(Mathematically: via commutative C*-algebras).

Given C*-algebra A, define

$$\mathcal{C}(A) = \{C \subseteq A \mid C \in \mathbf{cCStar}\}$$

It is a posetal category

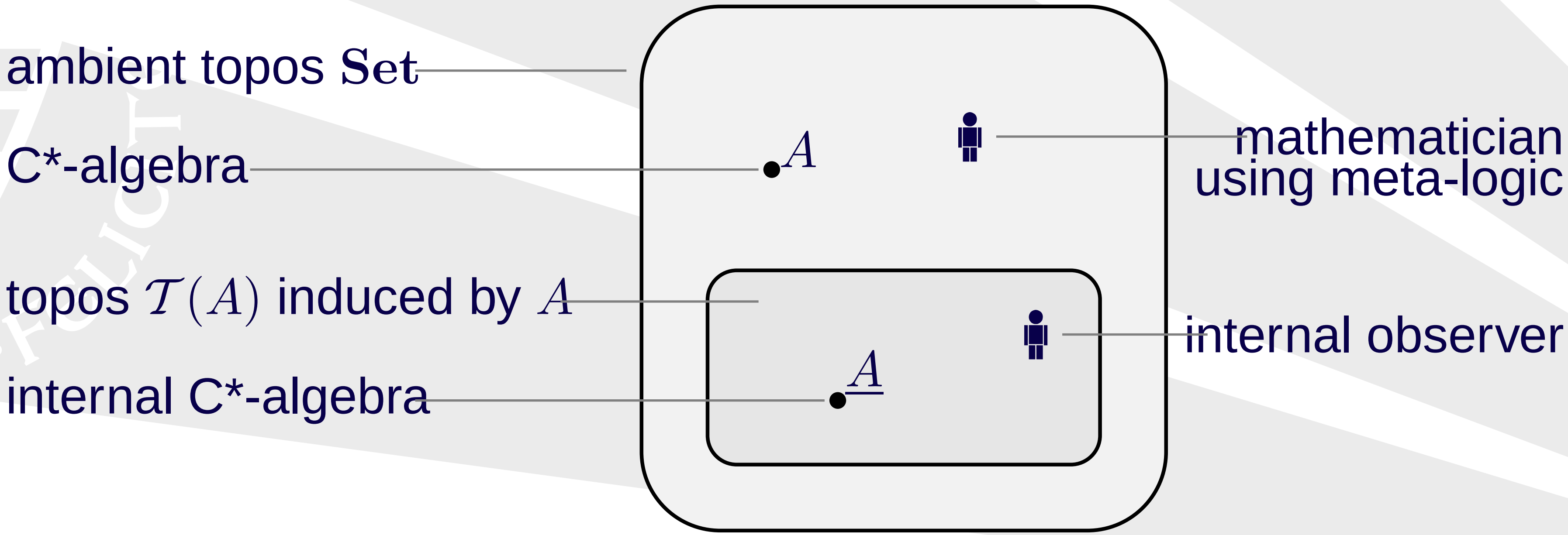
Then $\mathcal{T}(A) = \mathbf{Set}^{\mathcal{C}(A)}$ is a topos

Internal C-algebra*

Define object $\underline{A}$ in topos $\mathcal{T}(A)$ by

$$\underline{A}(C) = C$$
$$\underline{A}(C \hookrightarrow D) = \text{inclusion}$$

Then $\mathcal{T}(A) \models$ "$\underline{A}$ is a *commutative* C*-algebra"!



Gelfand duality

Gelfand duality characterizes commutative algebras.

$$\mathbf{cCStar} \underset{\mathbf{Top}(-,\mathbb{C})}{\overset{\sigma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathbf{KHaus}^{\mathrm{op}}$$

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Need to be careful about $\mathbf{KHaus}$ ...

Gelfand duality in a topos

Gelfand duality characterizes commutative algebras.

$$\mathbf{cCStar} \underset{\mathbf{Frm}(\mathcal{O}\mathbb{C}_{\mathbb{Q}}, -)}{\overset{\sigma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathbf{KRegFrm}$$

Use frame $\mathcal{O}X$ (lattice of open sets) instead of topological space X.
("pointless topology")

Intuitionistic formulation and proof
(Banaschewski-Mulvey / Coquand-Spitters)
so valid in any topos.

Internal spectrum

There is an object $\underline{\Sigma}$ in topos $\mathcal{T}(A)$ such that

$\mathcal{T}(A) \models$ "$\underline{\Sigma}$ is a frame".

$\mathcal{T}(A) \models \underline{\Sigma} = \sigma(\underline{A})$!

So have 'phase space'

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Theorem: (when A is non-commutative and has no summand M_2)
the frame $\underline{\Sigma}$ has no points.

(idea: Kochen-Specker-Isham-Butterfield)

So far

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Internalize observable a and state ρ of A,
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States

A (quasi)state of A is a positive functional
$\rho : A \to \mathbb{C}$ that is linear (on commutative parts)

An integral on $\underline{A}$ is a positive functional
$I : \underline{A} \to \mathbb{C}$ that is linear

Theorem: there is a bijective correspondence between
quasistates of A and integrals on $\underline{A}$

So can speak of states internally as integrals/measures
Hence as valuations on internal spectrum (Coquand-Spitters)

External state gives internal map $\underline{\Sigma} \to [0,1]^{\leftarrow}$

Observables

$$A_{\mathrm{sa}} \xrightarrow{\delta_1} \underline{A}_{\mathrm{sa}}^{\leftrightarrow}$$

$$\delta_1(a)(C) = (\{f \in C \mid f < a\}, \{g \in C \mid a < g\})$$

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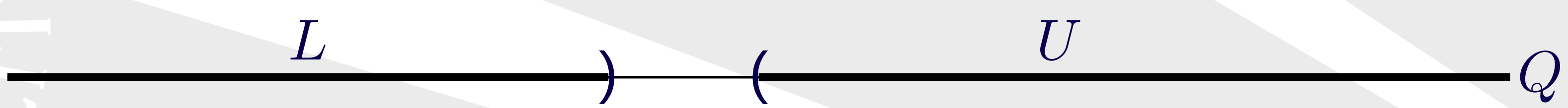
$$Q^{\leftrightarrow} = \{(L, U) \in \mathcal{P}Q \times \mathcal{P}Q \mid L \text{ lower open}, U \text{ upper open}\}$$



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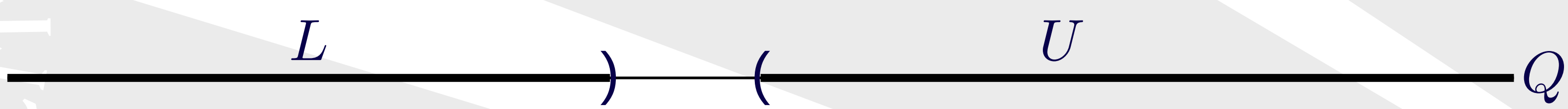

$\mathbb{IR} = \mathbb{Q}^{\leftrightarrow}$ (with Scott topology) is frame

Observables

$$A_{\mathrm{sa}} \xrightarrow{\delta_1} \underline{A}_{\mathrm{sa}}^{\leftrightarrow} \xrightarrow{\delta_2} [\![\mathbf{Frm}(\mathbb{IR}, \underline{\Sigma})\,]\!]_{\mathcal{T}(A)}$$

Observables

$$\underline{A}_{\text{sa}} \xrightarrow{\delta_1} \underline{A}_{\text{sa}}^{\leftrightarrow} \xrightarrow{\delta_2} [\![\mathbf{Frm}(\mathbb{IR}, \underline{\Sigma})\,]\!]_{\mathcal{T}(A)}$$

$$[\![\,\mathbf{Frm}(\mathbb{IR}, \underline{\Sigma})\,]\!]_{\mathcal{T}(A)} = [\![\,[\![\,\mathbb{IR}\,]\!]_{\mathrm{Sh}(\underline{\Sigma})}\,]\!]_{\mathcal{T}(A)}$$

external observable gives internal map $\underline{\mathbb{IR}} \to \underline{\Sigma}$

Propositions

Physics considers propositions $a \in (p,q)$ in state ρ

Interval (p,q) gives map $\underline{1} \to \underline{\mathbb{IR}}$.
Observable a gives map $\underline{\mathbb{IR}} \to \underline{\Sigma}$.
State ρ gives map $\underline{\Sigma} \to [0,1]^{\leftarrow}$.

Composition gives
probability $r : \underline{1} \to [0,1]^{\leftarrow}$
truth value $[\![r = 1]\!] : \underline{1} \to \underline{\Omega}$!

Conclusion

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