TOPOS THEORY IN THE FORMULATION OF THEORIES OF PHYSICS

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   - GR is the ultimate classical/realist theory!
   - But: 'reality of space/time points' is *very* dubious.
     - $\mathrm{Diff}(M)$ invariance/covariance.
     - Use of $\mathbb{R}$: a highly abstract concept.

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How can this be applied to space and time themselves?

The Unholy Trinity

The triangle of

real-world data $\leftrightarrow$ mathematics $\leftrightarrow$ conceptual framework

underpins all theoretical physics.

But: in QG, the first is largely missing. Consequently:

- Would we recognise the 'correct' theory if we saw it?
- What makes any particular idea a 'good' one?
  - Appointments and promotions
  - Getting research grants

In practice:

- Work by analogy with other theories (e.g., QFT)
- Indulge in one's personal mathematical interests and philosophical prejudices:-)

The Planck Length

Presumably something dramatic happens to the nature of space and time at $L_P := \sqrt{\frac{G\hbar}{c^3}} \simeq 10^{-35}m \simeq 10^{-42}secs$.

- Main QG programmes are string theory and loop quantum gravity. Both negate idea of 'points' in space & time.
- Suggests we need a non-manifold model of space-time. Consider models of space-time without points: a *locale?*

But: It is often asserted that classical space and time will 'emerge' from the correct QG formalism in some limit.

> Thus a fundamental theory may (i) have no intrinsic reference to spatio-temporal concepts; and (ii) the spatio-temporal concepts that emerge from it may be non-standard.

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2. Interpretational: *instrumentalism* versus *realism*.

   We want to talk about 'the way things are' in regard to space and time.

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(ii) as the values of *probabilities*;

(iii) as a fundamental ingredient in mathematical models of *space and time* via the ideas of differential geometry.

The use of $\mathbb{R}$ (and $\mathbb{C}$) in standard quantum theory is a reflection of (i) and (ii); and, indirectly, of (iii) too.

2. Why are Physical Quantities Assumed Real-Valued?

Traditionally, quantities are measured with rulers and pointers.

- Thus there is a direct link between the 'quantity-value space' and the assumed structure of *physical* space.
- Thus we have a potential 'category error' at L_P: if physical space is not based on $\mathbb{R}$, we should not assume *a priori* that physical quantities are real-valued.

If the quantity-value space is *not* $\mathbb{R}$, then what is the status of the Hilbert-space formalism?

3. Why Are Probabilities Assumed Real Numbers?

Relative-frequency interpretation: $\frac{N_i}{N}$ tends to $r \in [0, 1]$ as $N \to \infty$.

- This statement is instrumentalist. It does not work if there is no classical spatio-temporal background in which measurements could be made.

- In 'realist' interpretations, probability is often interpreted as *propensity* (*latency*, *potentiality*).

  – But why should a propensity be a real number in $[0, 1]$?

  – Minimal requirement is, presumably, an ordered set, but this need not even be *totally* ordered.

The Big Problem

Standard QT is grounded in Newtonian space and time; or SR.

How can the formalism be modified, or generalised, so as

(i) to be‘realist’; and

(ii) not to be dependent *a priori* on real and complex numbers?

- For example, if we have a given ‘non-standard’ background $\mathcal{C}$, what is the quantum formalism that is *adapted* to $\mathcal{C}$?
- Very difficult: usual Hilbert-space formalism is very rigid

What *are* the basic principles of ‘quantum theory’; or beyond?

III. Theories of a Physical System

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"A thing is always something that has such and such properties, always something that is constituted in such and such a way. This something is the bearer of the properties; the something, as it were, that underlies the qualities." **Heidegger**

In constructing a theory of some branch of physics, the key ingredients are the *mathematical models* for

1. Space, time, or space-time;
2. Physical quantities;
3. 'States' of a system—"The way things are" (or *history* equivalent)

Some key questions:

- What is meant by a 'value' of a 'physical quantity'?
- How is this related to the idea of a 'property'?
- How does a state determine the properties of a system?

The Realism of Classical Physics:

1. A physical quantity A is represented by $\tilde{A} : \mathcal{S} \to \mathbb{R}$.

2. A state $s \in \mathcal{S}$ specifies 'how things are': i.e., the value of any physical quantity A in that state is $\tilde{A}(s) \in \mathbb{R}$.

This is how ('naive') realism enters into classical physics.

3. The subset $\tilde{A}^{-1}(\Delta) \subseteq \mathcal{S}$ represents the proposition "$A \in \Delta$": asserts that the system has a certain *property*.

Thus, the *mathematical* structure of set theory, implies that, of necessity, the propositions in classical physics form a *Boolean logic*.

The collection of such propositions forms a *deductive system*: i.e., there is a sequent calculus for constructing proofs.

Failure of Naive Realism in Quantum Physics

Kochen-Specker theorem: it is impossible to assign consistent true-false values to all the propositions in quantum theory.

Equivalently: it is not possible to assign consistent values to all the physical quantities in a quantum theory.

Conclusion:

- From Heidegger's perspective, there is 'no way things are'.
- Instead an *instrumentalist* interpretation is used.

Representing Physical Quantities

Let A be a physical quantity in a system S, with associated propositions "$A \in \Delta$", where $\Delta \subseteq \mathbb{R}$.

Classical theory of S:

- State space is a symplectic manifold, $\mathcal{S}$
- $A \rightsquigarrow \tilde{A} : \mathcal{S} \to \mathbb{R}$
- For $\Delta \subseteq \mathbb{R}$, "$A \in \Delta$" $\rightsquigarrow \tilde{A}^{-1}(\Delta) \subseteq \mathcal{S}$; Boolean lattice.

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- $A \rightsquigarrow \hat{A}$
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Category generalisation: objects Σ and $\mathcal{R}$ in a category τ:

- $A \rightsquigarrow \breve{A} : \Sigma \to \mathcal{R}$.
- For Δ a sub-object of $\mathcal{R}$, "$A \in \Delta$" $\rightsquigarrow$ a sub-object of Σ?

IV. Introducing Topos Theory

Does such ‘categorification’ work?

1. Not in general: usually, sub-objects of an object do not have a logical structure. However, they *do* in a *topos*!

2. A topos is a category that ‘behaves much like **Sets**’. In particular there are:
   - $0, 1$; pull-backs & push-outs (hence, products & co-products)
   - Exponentiation:

     $$\mathrm{Hom}(C, A^B) \simeq \mathrm{Hom}(C \times B, A)$$

   - A ‘sub-object classifier’, Ω: to any sub-object A of B, $\exists \chi_A : B \to \Omega$ such that $A = \chi_A^{-1}(1)$.

The Logical Structure of Sub-objects

In a topos:

1. The collection, $\mathrm{Sub}(A)$, of sub-objects of an object A forms a *Heyting algebra*.
2. The same applies to $\Gamma\Omega := \mathrm{Hom}(1, \Omega)$, 'global elements'

A Heyting algebra is a distributive lattice, $\mathfrak{H}$, with 0 and 1, and such that to each $\alpha, \beta \in \mathfrak{H}$ there exists $\alpha \Rightarrow \beta \in \mathfrak{H}$ such that

$$\gamma \preceq (\alpha \Rightarrow \beta) \text{ iff } \gamma \wedge \alpha \preceq \beta.$$

- Negation is defined as $\neg\alpha := (\alpha \Rightarrow 0)$.
- *Excluded middle* may not hold: there may exist $\alpha \in \mathfrak{H}$ such that $\alpha \vee \neg\alpha \prec 1$.

  Equivalently there may be β such that $\beta \prec \neg\neg\beta$.

The Mathematics of 'Neo-Realism'

- **In set theory:** let $K \subseteq X$ and $x \in X$. Consider the proposition "$x \in K$". The truth value is

$$\nu(x \in K) = \begin{cases} 1 & \text{if } x \text{ belongs to } K; \\ 0 & \text{otherwise.} \end{cases}$$

Thus "$x \in K$" is true if, and only if, x belongs to K.

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- **In a topos:** a proposition can be only 'partly true':

  Let $K \in \mathrm{Sub}(X)$ with $\chi_K : X \to \Omega$ and let $x \in X$, i.e., $\ulcorner x \urcorner : 1 \to X$ is a global element of X. Then

$$\nu(x \in K) := \chi_K \circ \ulcorner x \urcorner$$

  where $\chi_K \circ \ulcorner x \urcorner : 1 \to \Omega$. Thus the 'generalised truth value' of "$x \in K$" belongs to the Heyting algebra $\Gamma\Omega$.

This can be used to represent a type of 'neo-realism'

Our Main Contention

For a given theory-type, each system S to which the theory is applicable can be formulated and interpreted neo-realistically within the framework of a particular topos $\tau_\phi(S)$.

Different theory-types have *different* topoi representations ϕ.

Conceptually, this structure is 'neo-realist' in the sense:

1. A physical quantity, A, is represented by an arrow $A_{\phi,S} : \Sigma_{\phi,S} \to \mathcal{R}_{\phi,S}$ where $\Sigma_{\phi,S}$ and $\mathcal{R}_{\phi,S}$ are two special objects in the topos $\tau_\phi(S)$.
2. Propositions about S are represented by sub-objects of $\Sigma_{\phi,S}$. These form a Heyting algebra.
3. The topos analogue of a state is a 'truth object' or 'pseudo-state'.

   Propositions are assigned truth values in $\Gamma\Omega_{\tau_\phi(S)}$.

- Thus a theory expressed in this way *looks* like classical physics except that classical physics always employs the topos **Sets**, whereas other theories—including quantum theory—use a different topos.

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- A topos can be used as a *foundation* for mathematics itself, just as set theory is used in the foundations of 'normal' (or 'classical') mathematics.

- In fact, any topos has an 'internal language' that is similar to the formal language on which set theory is based.

$$\boxed{\text{This is used to } \textit{interpret} \text{ the theory in a 'neo-realist' way.}}$$

The Idea of a Truth Object

In set theory, basic *mathematical* propositions are of the form

- "$x \in K$"; or, equivalently,
- "$\{x\} \subseteq K$"

In classical physics, a truth value is assigned to propositions by specifying a micro-state, $s \in \mathcal{S}$. Then, truth value of "$A \in \Delta$" is

$$\nu(A \in \Delta; s) = \begin{cases} 1 & \text{if } \tilde{A}(s) \in \Delta; \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

- But in a topos, the state object $\Sigma_{\phi,S}$ may have *no* global elements.

  For example, this *is* the case' in quantum theory.

- So, what is the analogue of a state in a general topos?

In Classical Physics

For each $s \in \mathcal{S}$, the *truth object*, T^s is the collection of sub-sets:

$$T^s := \{K \subseteq \mathcal{S} \mid s \in K\}$$

i.e., $T^s \subseteq P\mathcal{S}$, or, equivalently, $T^s \in PP\mathcal{S}$. Then

$$\begin{aligned}
\nu(A \in \Delta; s) &= \begin{cases} 1 & \text{if } s \in \tilde{A}^{-1}(\Delta); \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \tilde{A}^{-1}(\Delta) \in T^s; \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \{s\} \subseteq \tilde{A}^{-1}(\Delta); \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

which exploits the fact that $\{s\} = \bigcap_{K \in T^s} K$

In a General Topos

Version 1: A truth object is $\ulcorner T \urcorner : 1_\tau \to PP\Sigma_{\phi,s}$, a sub-object of $P\Sigma_{\phi,s}$. Then, if $K \in \mathrm{Sub}(\Sigma_{\phi,s})$, the mathematical proposition "$K \in T$" has truth value $\nu(K \in T) \in \Gamma\Omega_{\phi,s}$.

Version 2: A truth object is $\ulcorner \mathfrak{w} \urcorner : 1_\tau \to P\Sigma_{\phi,s}$, a sub-object of $\Sigma_{\phi,s}$. Then if $K \in \mathrm{Sub}(\Sigma_{\phi,s})$, the mathematical proposition "$\mathfrak{w} \subseteq K$" has truth value $\nu(\mathfrak{w} \subseteq K) \in \Gamma\Omega_{\phi,s}$.

To relate them we define

$$\mathfrak{w}^T := \bigwedge_{K \in T} K$$

which is the topos analogue of $\{s\} = \bigcap_{K \in T^s} K$.

$\boxed{\text{We can think of } \mathfrak{w}^T \text{ as being a `pseudo-state' of the system.}}$

V. Formal Languages

There is a very elegant way of describing what we are doing. Namely, to construct a theory of a system S is equivalent to finding a *representation*, ϕ, in a topos of a certain formal language, $\mathcal{L}(S)$, that is attached to S.

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  However, the representation *does* depend on theory type.

- The language includes axioms for an *intuitionistic* logic.

Equivalently, we construct a *translation* of $\mathcal{L}(S)$ into the internal language of the topos.

The Language $\mathcal{L}(S)$

The language $\mathcal{L}(S)$ of a system S is *typed*. It includes:

- A symbol Σ: the linguistic precursor of the state object.
- A symbol $\mathcal{R}$: the linguistic precursor of the quantity-value object.
- A set, $F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$ of 'function symbols' $A : \Sigma \to \mathcal{R}$: the linguistic precursors of physical quantities.
- A symbol Ω: the linguistic precursor of the sub-object classifier.
- A 'set builder' $\{\tilde{x} \mid \omega\}$. This is a term of type PT, where $\tilde{x}$ is a variable of type T, and ω is a term of type Ω.
- May also include symbols for space/time

Representing the Language $\mathcal{L}(S)$

Next step: find a representation of $\mathcal{L}(S)$ in a suitable topos.

A classical theory of S: The representation σ is:

- The topos $\tau_\sigma(S)$ is **Sets**.
- Σ is represented by a symplectic manifold $\Sigma_{\sigma,S}$ (was $\mathcal{S}$).
- $\mathcal{R}$ is represented by the real numbers $\mathbb{R}$; i.e., $\mathcal{R}_{\sigma,S} := \mathbb{R}$.
- The function symbols $A : \Sigma \to \mathcal{R}$ become functions $A_{\sigma,S} : \Sigma_{\sigma,S} \to \mathbb{R}$ (was $\tilde{A}$)
- Ω is represented by the set $\{0, 1\}$ of truth values.

The Topos of Quantum Theory

- We focus on the intrinsic *contextuality* implied by the Kocken-Specher theorem.

- In standard theory, we can potentially assign 'actual values' only to members of a commuting set of operators.

  This is used in *modal* interpretations: for example, Bohm.

  > We think of such a set as a *context*, in which to view the system.
  >
  > A context is a 'classical snapshot', or 'world-view'.

- We want to consider all contexts at once! A theory of 'many world-views'.

  This motivates considering the topos of presheaves over the category of abelian subalgebras of $\mathcal{B}(\mathcal{H})$: a poset under the operation of sub-algebra inclusion.

The state object that represents symbol Σ is the '*spectral presheaf*' $\underline{\Sigma}$. For each abelian subalgebra V, $\underline{\Sigma}(V)$ is the spectrum of V.

> The K-S theorem is equivalent to the statement that $\underline{\Sigma}$ has no global elements.

- $\underline{\Sigma}$ replaces the (non-existent) state space
- A proposition represented by a projector $\hat{P}$ in QT is mapped to a sub-object $\delta(\hat{P})$ of $\underline{\Sigma}$: '*daseinisation*'.
- The quantity-value symbol $\mathcal{R}$ is represented by a presheaf $\underline{\mathcal{R}}$. This is *not* the real-number object in the topos.
- Physical quantities: arrows $\breve{A} : \underline{\Sigma} \to \underline{\mathcal{R}}$. Constructed from the Gel'fand transforms of the spectra in $\underline{\Sigma}$.
- Each state $|\psi\rangle$ gives a truth object $\underline{T}^{|\psi\rangle}$, and associated pseudo-state $\underline{\mathfrak{w}}^{|\psi\rangle} := \delta(\,|\psi\rangle\langle\psi|\,)$.

VI. Conclusions

1. General considerations of quantum gravity suggest the need to go 'beyond' standard quantum theory:
   - Must escape from *a priori* use of $\mathbb{R}$ and $\mathbb{C}$.
   - Need a 'realist' interpretation (K-S not withstanding)
2. Main idea: construct theories in a topos other than **Sets**.
   - A physical quantity, A, is represented by an arrow $A_{\phi,S} : \Sigma_{\phi,S} \to \mathcal{R}_{\phi,S}$; $\Sigma_{\phi,S}$ and $\mathcal{R}_{\phi,S}$ are special objects in $\tau_\phi(S)$.
   - The interpretation is 'neo-realist' with truth values that lie in the Heyting algebra $\Gamma\Omega_{\tau_\phi(S)}$.

     Propositions are represented by elements of Heyting algebra $\mathrm{Sub}(\Sigma_{\phi,S})$
3. Our scheme involves representing language $\mathcal{L}(S)$ in $\tau_\phi(S)$.
4. The idea of a 'state' is replaced with 'pseudo-state'/ truth-object.