Categorical Quantum Mechanics: The "Monoidal" Approach

Samson Abramsky

Introduction

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We aim to give an introduction to the "monoidal" approach, which could as well be called "linear".

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monoidal	*vs.*	cartesian
linear	*vs.*	intuitionistic
processes	*vs.*	propositions
geometry of proofs	*vs.*	geometric logic

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- operational aspects
- interplay of quantum and classical
- compositionality
- open *vs.* closed systems.

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- operational aspects
- interplay of quantum and classical
- compositionality
- open *vs.* closed systems.

These are important for applications to quantum informatics, but also of foundational significance.

Quantum Operations/Quantum Processes

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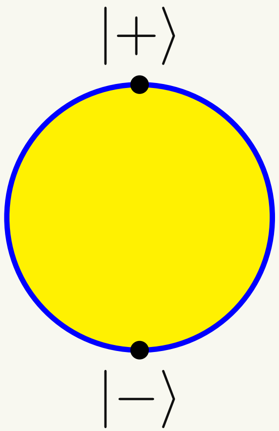

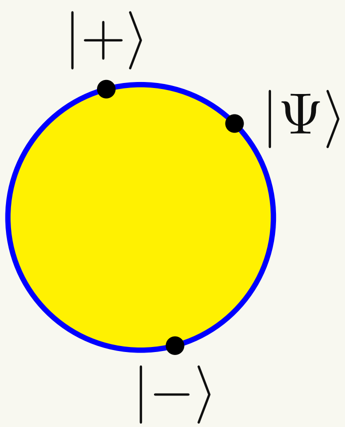

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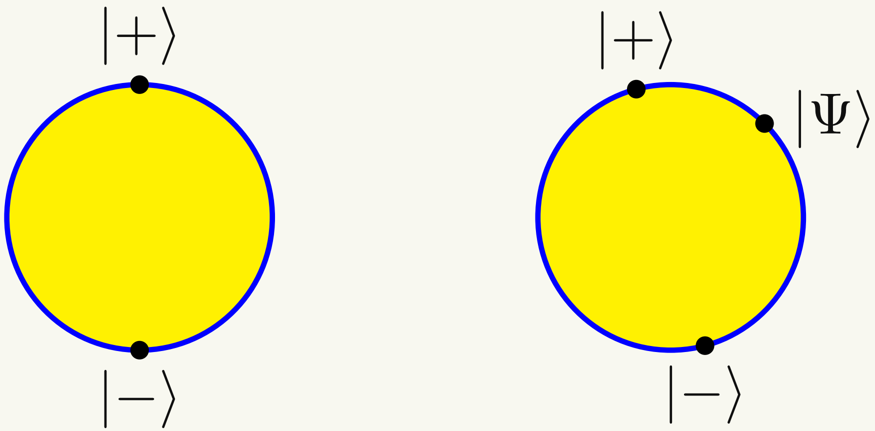

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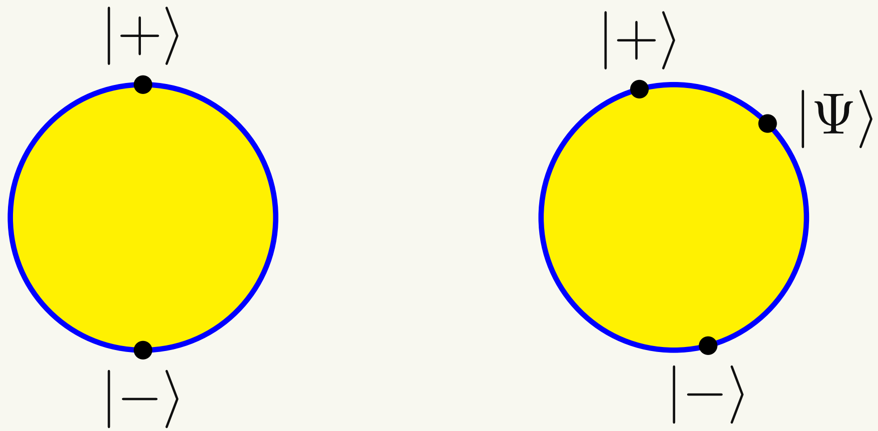

- measurements of qubits
  - have two outcomes $|-\rangle$, $|+\rangle$
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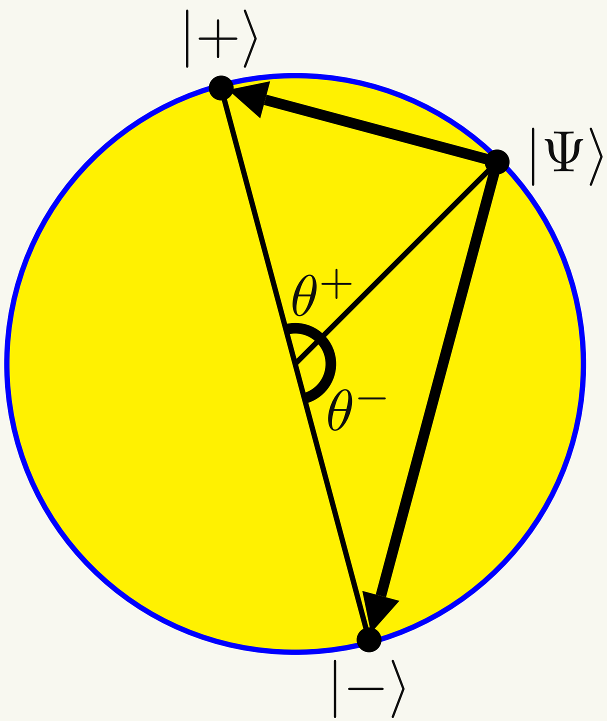
- admit **unitary transformations**, *i.e.* antipodes and angles are preserved.

‘Truth makes an angle with reality’



We have **partial constant maps** $Q \longrightarrow Q$ on the sphere Q

$$\mathcal{P}_+ : |\psi\rangle \mapsto |+\rangle \qquad \mathcal{P}_- : |\psi\rangle \mapsto |-\rangle$$

which have chance $\mathrm{prob}(\theta^\sharp)$ for $\sharp \in \{+,-\}$.
We know the value **after** the measurement, but not what the qubit was **before** the measurement!

So measurements change the state, and apparently lose information. This seems like bad news! However . . .

Bell state:

$$|00\rangle + |11\rangle$$

EPR state:

$$|01\rangle + |10\rangle$$

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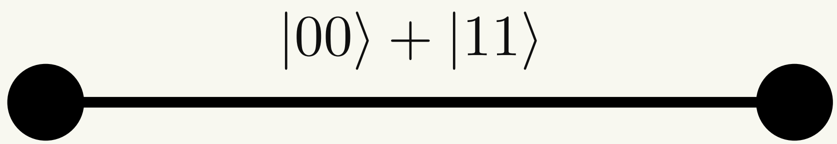

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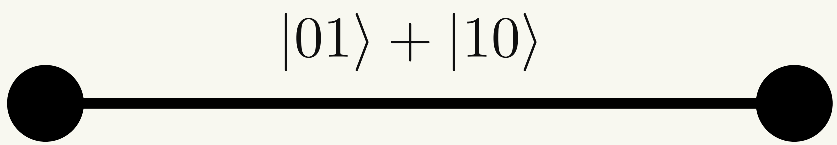

Compound systems are represented by **tensor product**: $\mathcal{H}_1 \otimes \mathcal{H}_2$.
Typical element:

$$\sum_i \lambda_i \cdot \phi_i \otimes \psi_i$$

Superposition encodes **correlation**. Einstein's 'spooky action at a distance'. Even if the particles are spatially separated, measuring one has an effect on the state of the other.
Bell's theorem: QM is **essentially non-local**.

Towards a general formalism for describing physical processes

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So we want a general setting in which we can describe processes (of whatever kind) closed under sequential and parallel composition.

Basic Setting: Symmetric Monoidal Categories

Categories

A category $\mathcal{C}$ has **objects** (types) A, B, C, ..., and for each pair of objects A, B a set of **morphisms** $\mathcal{C}(A, B)$. (Notation: $f : A \rightarrow B$). It also has identities $\mathrm{id}_A : A \rightarrow A$, and composition $g \circ f$ when types match:

$$A \xrightarrow{f} B \xrightarrow{g} C$$

Categories allow us to deal explictly with **typed processes**, e.g.

Logic	**Programming**	**Computation**
Propositions	Data Types	States
Proofs	Programs	Transitions

For QM:

Types of system (e.g. qubits Q)	$\rightsquigarrow$	objects
Processes/Operations (e.g. measurements)	$\rightsquigarrow$	morphisms

Symmetric Monoidal Categories

A **symmetric monoidal category** comes equipped with an associative operation which acts on **both** objects **and** morphisms — a **bifunctor**:

$$A \otimes B \qquad f_1 \otimes f_2 : A_1 \otimes A_2 \longrightarrow B_1 \otimes B_2$$

There is also a symmetry operation

$$\sigma_{A,B} : A \otimes B \longrightarrow B \otimes A$$

which satisfies some 'obvious' rules, e.g. naturality:

$$\begin{tikzcd}
A_1 \otimes A_2 \arrow[r, "f_1 \otimes f_2"] \arrow[d, "\sigma_{A_1,A_2}"'] & B_1 \otimes B_2 \arrow[d, "\sigma_{B_1,B_2}"] \\
A_2 \otimes A_1 \arrow[r, "f_1 \otimes f_2"] & B_2 \otimes B_1
\end{tikzcd}$$

The Logic of Tensor Product

Tensor can express **independent** or **concurrent** actions (mathematically: bifunctoriality):

$$\begin{tikzcd}
A_1 \otimes A_2 \arrow[r, "f_1 \otimes \mathsf{id}"] \arrow[d, "\mathsf{id} \otimes f_2"'] & B_1 \otimes A_2 \arrow[d, "\mathsf{id} \otimes f_2"] \\
A_1 \otimes B_2 \arrow[r, "f_1 \otimes \mathsf{id}"'] & B_1 \otimes B_2
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But tensor is **not** a cartesian product, in the sense that **we cannot reconstruct an 'element' of the tensor from its components**.

This turns out to comprise the **absence** of

$$A \xrightarrow{\Delta} A \otimes A \qquad A_1 \otimes A_2 \xrightarrow{\pi_i} A_i$$

$$\text{Cf.} \qquad A \vdash A \wedge A \qquad\qquad A_1 \wedge A_2 \vdash A_i$$

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No-Cloning and No-Deleting are built in!

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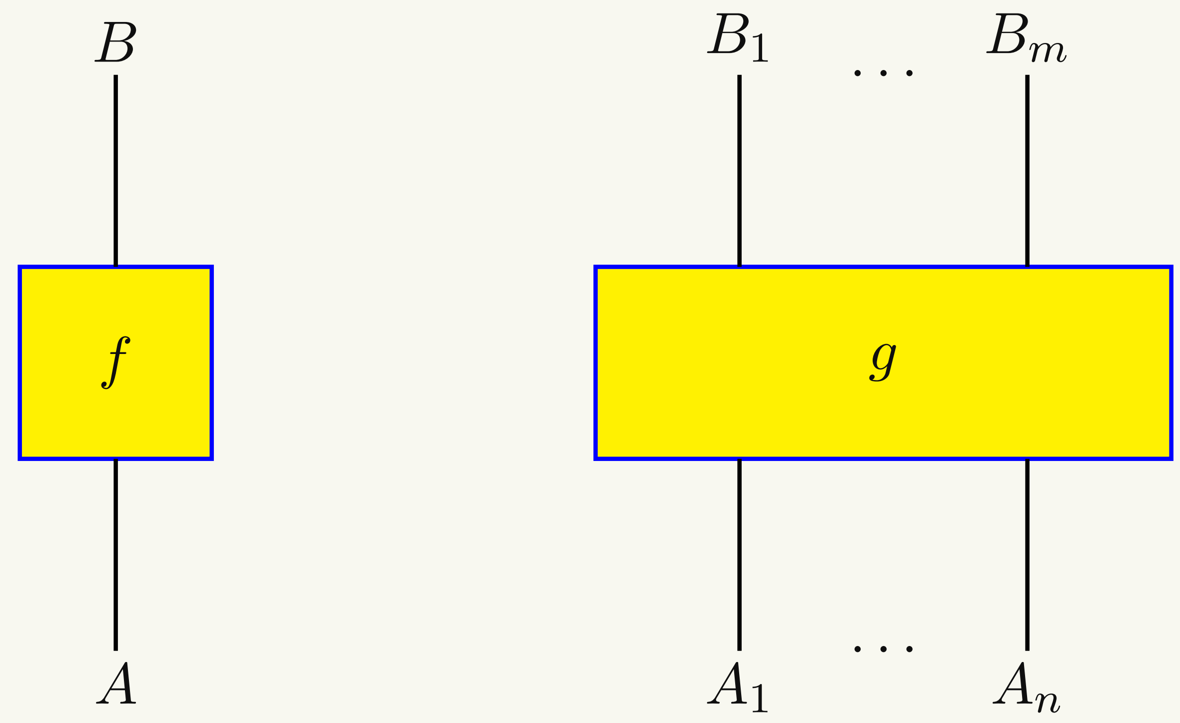
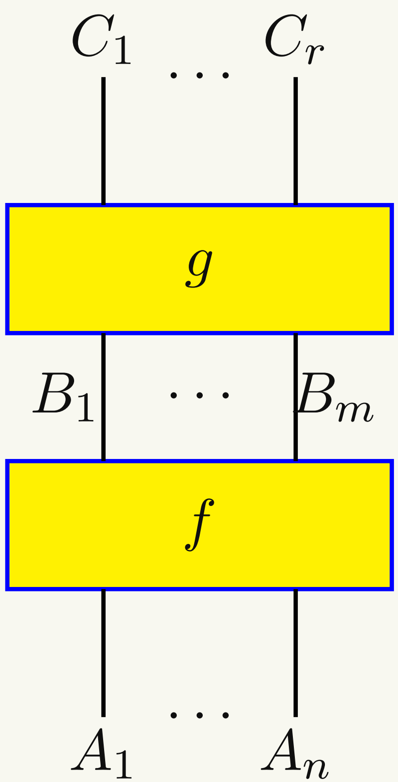
There is a natural objection to this, that this is too abstract, and we lose all grip of what is going on.

But this objection does not really hold water! Monoidal categories, quite generally, admit a beautiful graphical or diagrammatic calculus (Joyal, Street et al.) which makes equational proofs perspicuous, and is sound and complete for equational reasoning in monoidal categories. It also supports links with Logic (e.g. Proof Nets) and with Geometry (Knots, Braids, Temperley-Lieb algebra etc.)

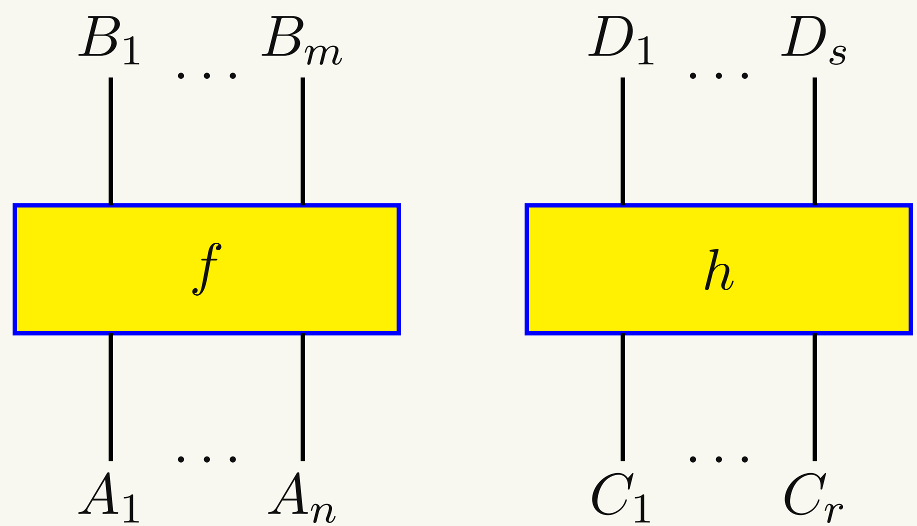


$$f : A \longrightarrow B \qquad g : A_1 \otimes \cdots \otimes A_n \longrightarrow B_1 \otimes \cdots \otimes B_m.$$

Series and Parallel Composition

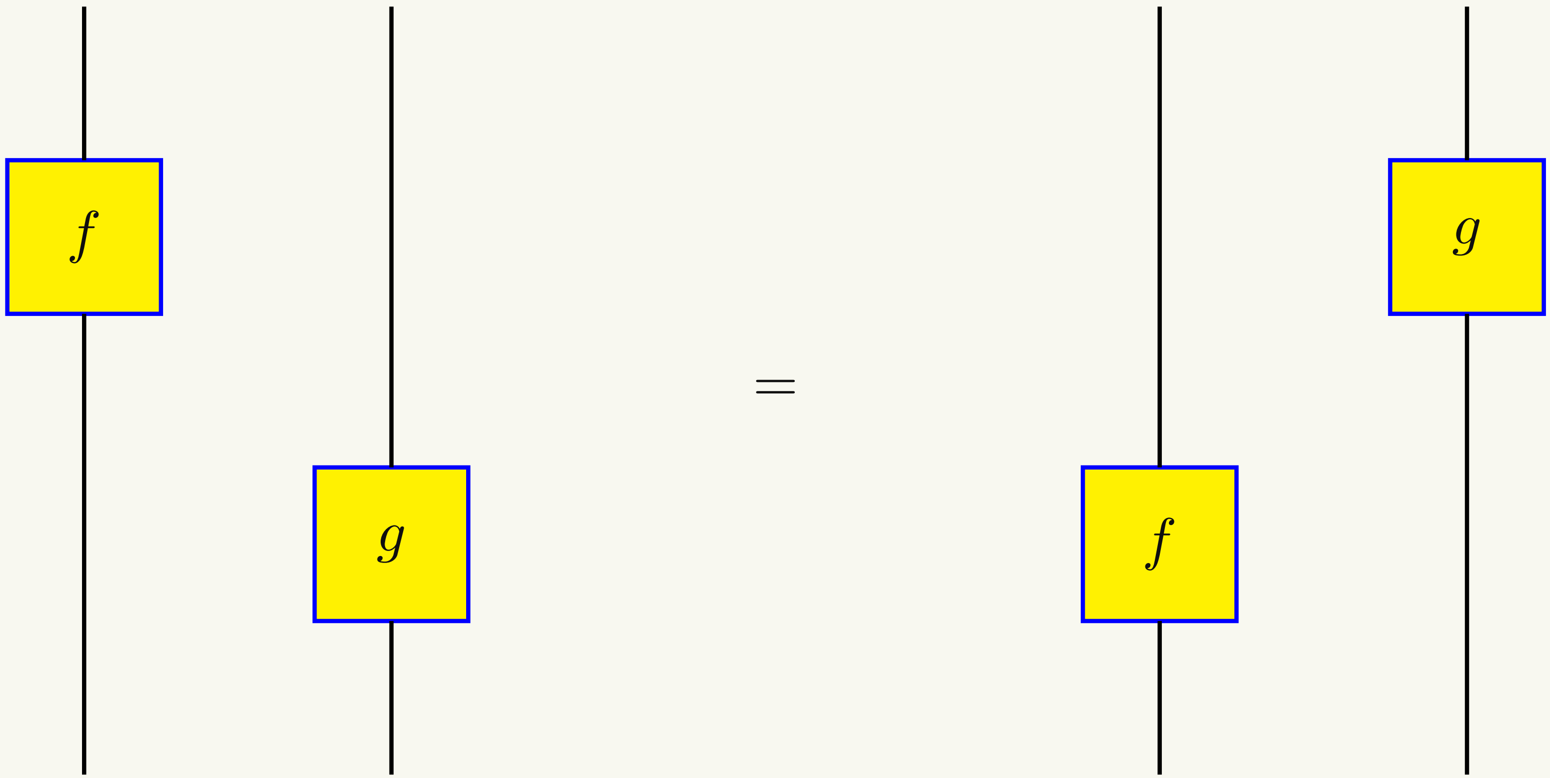


$g \circ f$



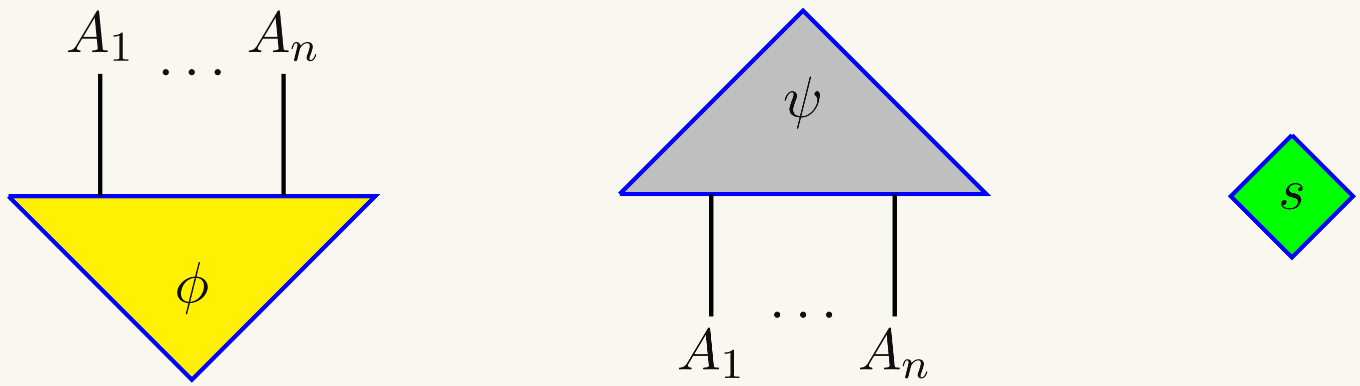
$f \otimes h$

Geometry absorbs Functoriality, Naturality



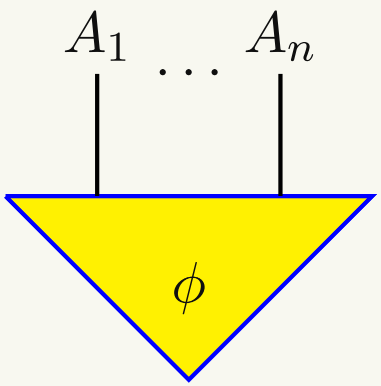
$$(f \otimes 1) \circ (1 \otimes g) \quad = \quad f \otimes g \quad = \quad (1 \otimes g) \circ (f \otimes 1)$$

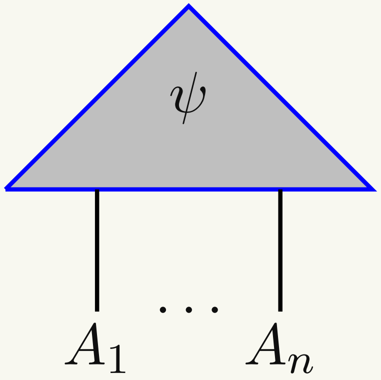
Bras, Kets and Scalars

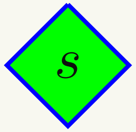


$$\phi : A_1 \otimes \cdots \otimes A_n \longrightarrow I \qquad \psi : I \longrightarrow A_1 \otimes \cdots \otimes A_n \qquad s : I \longrightarrow I.$$

Bras:	no outputs
Kets:	no inputs
Scalars:	no inputs or outputs.







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Two-dimensional generalization of **Dirac notation**!

$$\langle \phi \mid \qquad\qquad \mid \psi \rangle \qquad\qquad \langle \phi \mid \psi \rangle$$

Interlude: The Miracle of Scalars

Scalars in monoidal categories

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Examples: $(\mathbf{FdVec}_{\mathbb{K}}, \otimes)$, $(\mathbf{Rel}, \times)$.

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(1) $\mathcal{C}(I, I)$ is a commutative monoid

$$\begin{tikzcd}
I \arrow[r, "\rho_I^{-1}"] & I \otimes I \arrow[r, equal] & I \otimes I \arrow[r, "\lambda_I"] \arrow[d, "1\otimes t"] & I \arrow[d, "t"] \\
I \arrow[u, "s"] \arrow[r, "\rho_I^{-1}"] \arrow[d, "t"'] & I \otimes I \arrow[u, "s\otimes 1"'] \arrow[r, "s\otimes t"] \arrow[d, "1\otimes t"'] & I \otimes I \arrow[r, "\lambda_I"] & I \\
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\end{tikzcd}$$

using the coherence equation $\lambda_I = \rho_I$.

(2) Each scalar $s : I \to I$ induces a natural transformation

$$s_A : A \xrightarrow{\simeq} I \otimes A \xrightarrow{s \otimes 1_A} I \otimes A \xrightarrow{\simeq} A\,.$$

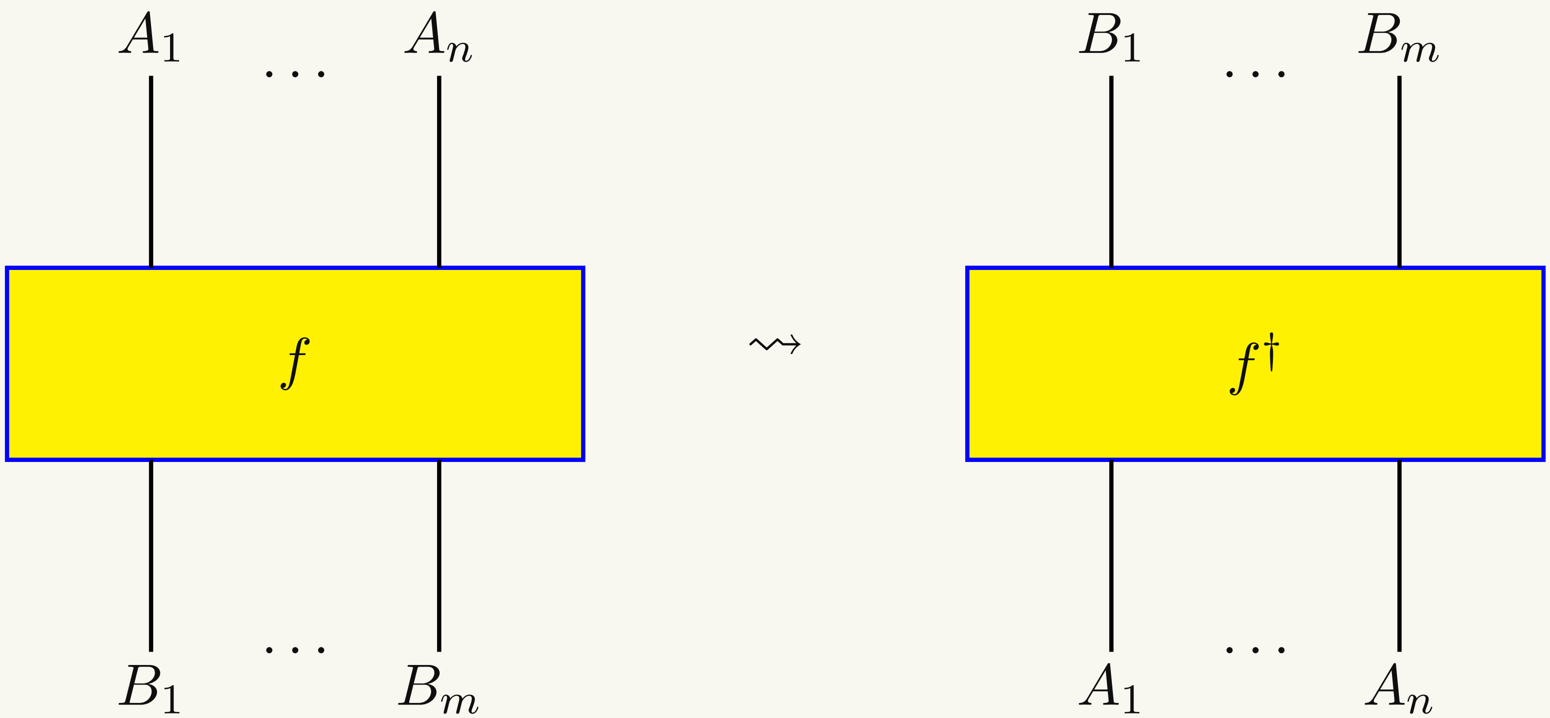
$$\begin{tikzcd} A \arrow[r, "s_A"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ B \arrow[r, "s_B"'] & B \end{tikzcd}$$

We write $s \bullet f$ for $f \circ s_A = s_B \circ f$. Note that

$$\begin{array}{rcl} s \bullet (t \bullet f) & = & (s \circ t) \bullet f \\ (s \bullet g) \circ (r \bullet f) & = & (s \circ r) \bullet (g \circ f) \\ (s \bullet f) \otimes (t \bullet g) & = & (s \circ t) \bullet (f \otimes g) \end{array}$$

Dagger Monoidal Categories

Adjoint Arrows — reflection in the x-axis



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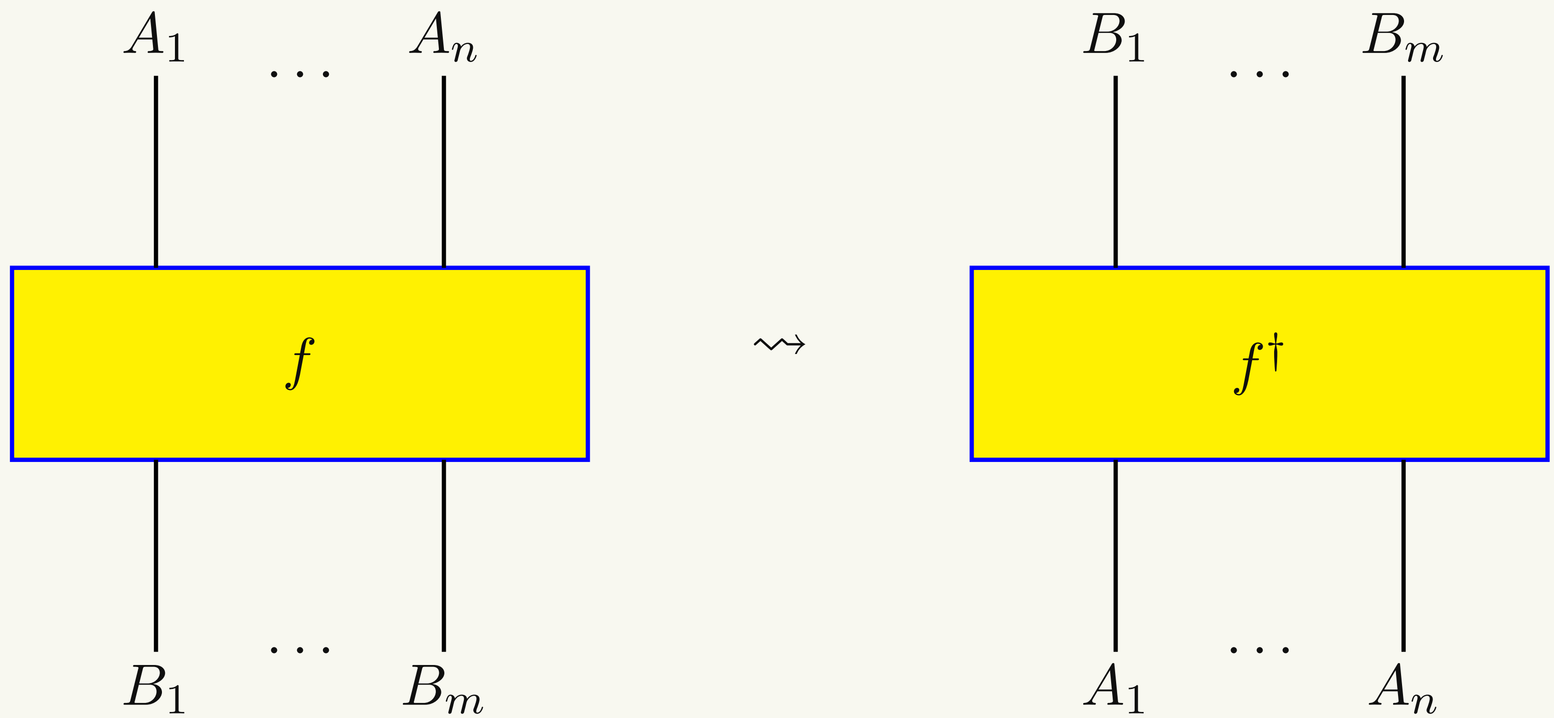

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We can turn kets into bras and vice versa — full scale Dirac notation! Given $\phi, \psi : I \longrightarrow A$,

$$\langle \phi \mid \psi \rangle = \phi^\dagger \circ \psi : I \longrightarrow I$$

which is indeed a scalar!

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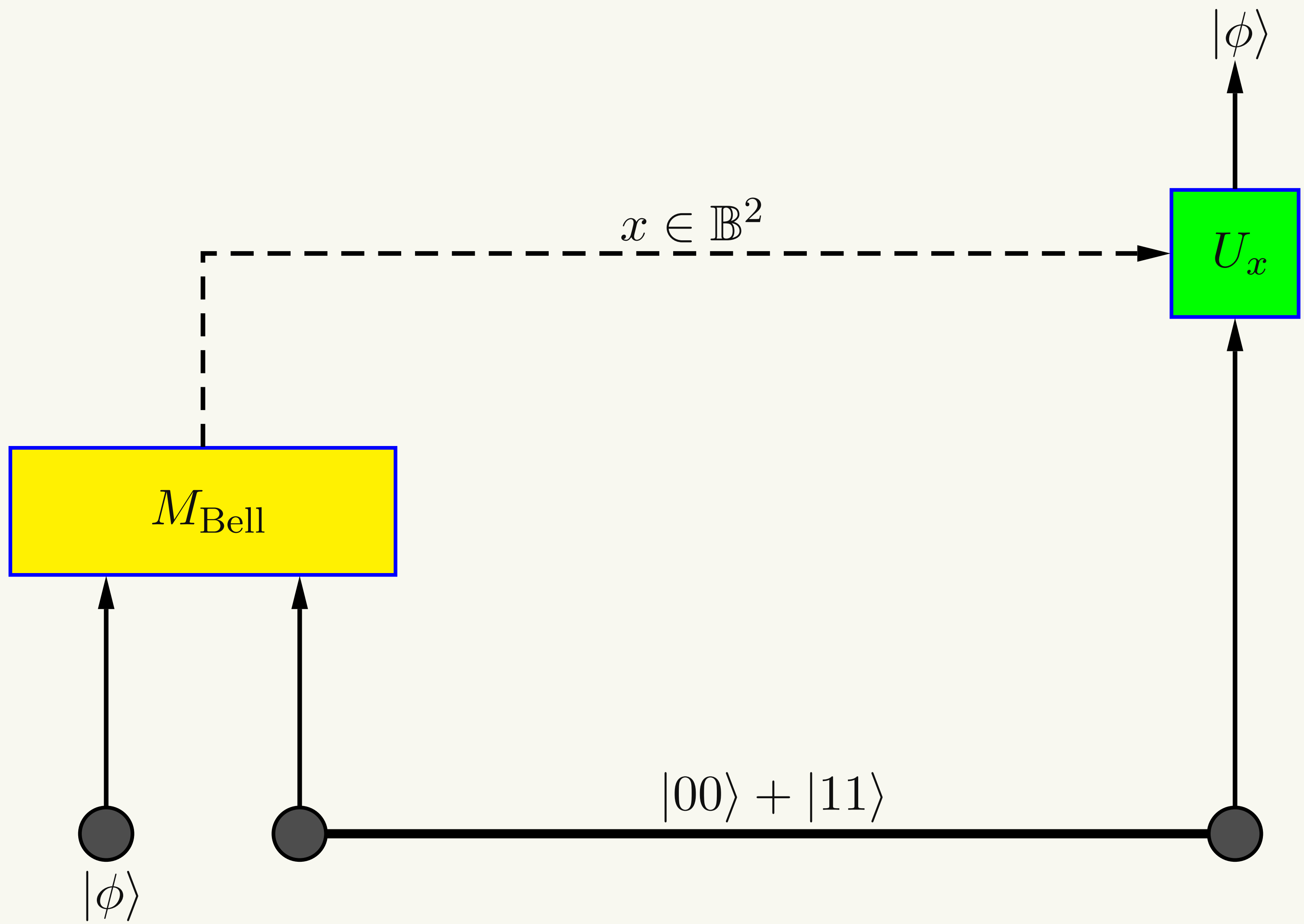
Examples: **Hilb**, **Rel**.

Entanglement, Bell States and Compact Categories

From 'paradox' to 'feature': Teleportation



Entangled states as linear maps

Indeed, $\mathcal{H}_1 \otimes \mathcal{H}_2$ is spanned by

$$\begin{matrix} |11\rangle & \cdots & |1m\rangle \\ \vdots & \ddots & \vdots \\ |n1\rangle & \cdots & |nm\rangle \end{matrix}$$

hence

$$\sum_{i,j} \alpha_{ij} |ij\rangle \quad \longleftrightarrow \quad \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nm} \end{pmatrix} \quad \longleftrightarrow \quad |i\rangle \mapsto \sum_j \alpha_{ij} |j\rangle$$

Pairs $|\psi_1, \psi_2\rangle$ are a special case — $|ij\rangle$ in a well-chosen basis.

This is **Map-State Duality**.

A projector onto the 1-dimensional subspace spanned by a vector $|\psi\rangle$ will be written $\mathcal{P}_\psi$. It is essentially (up to scalar multiples) a “partial constant map”

$$\mathcal{P}_\psi : |\phi\rangle \mapsto |\psi\rangle.$$

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We combine this notation with Map-State Duality: we write a projector $\mathcal{P}_\psi$ on a tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$ as $\mathcal{P}_f$, where f is the linear map $\mathcal{H}_1 \longrightarrow \mathcal{H}_2$ associated to ψ under Map-State Duality.

On the trail of structure

The identity map

$$|\mathsf{id} : Q \to Q\rangle \in Q \otimes Q \qquad |11\rangle + \cdots + |nn\rangle \;\longleftrightarrow\; |i\rangle \mapsto |i\rangle$$

is the **Bell state**.
A measurement of $Q \otimes Q$ has four outcomes

$$|f_1\rangle, |f_2\rangle, |f_3\rangle, |f_4\rangle \qquad (\textit{cf.}\ |00\rangle, |01\rangle, |10\rangle, |11\rangle)$$

and corresponding projectors

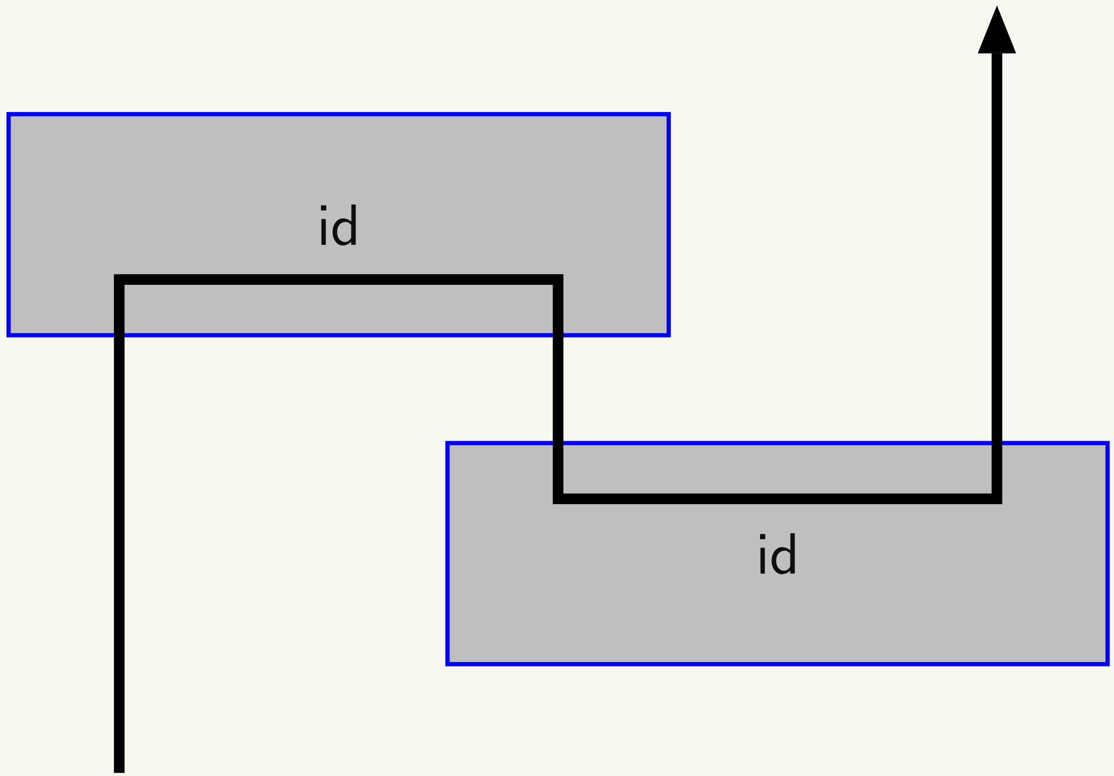
$$\mathcal{P}_f : Q \otimes Q \to Q \otimes Q :: |g\rangle \mapsto |f\rangle$$

E.g. the Bell state is produced by

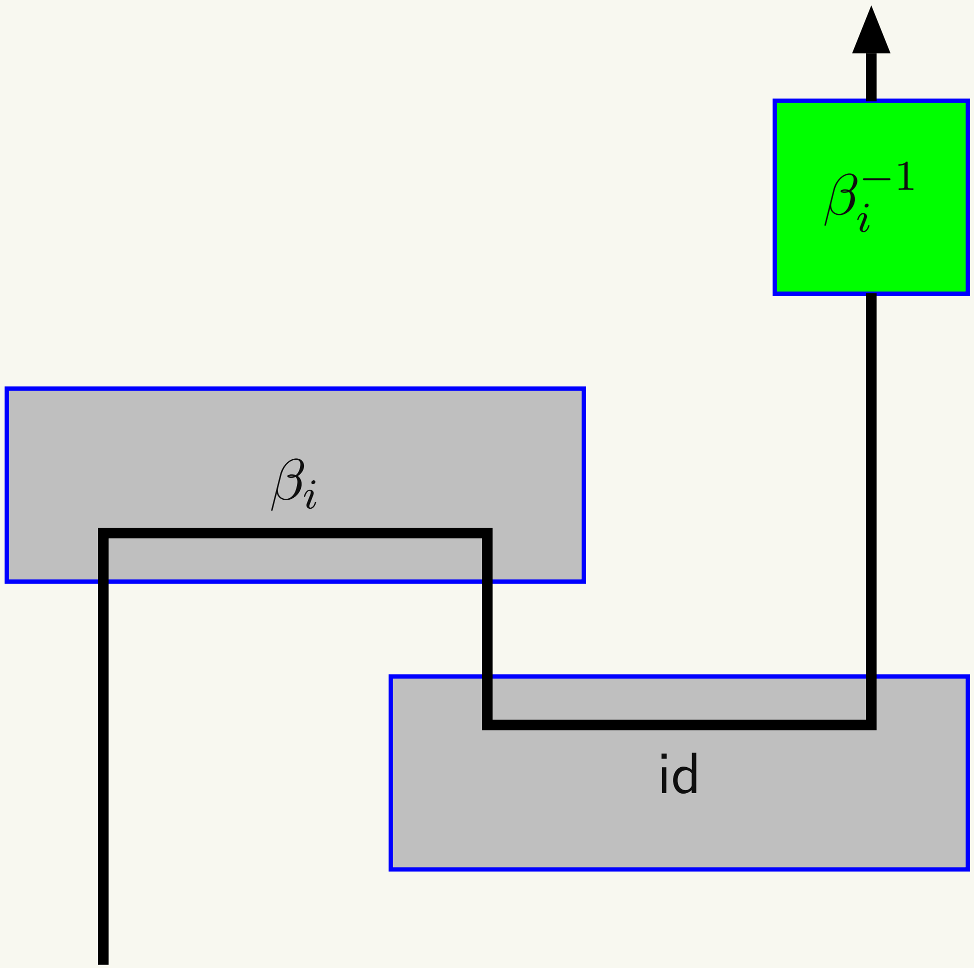
$$\mathcal{P}_{\mathsf{id}} : Q \otimes Q \to Q \otimes Q :: |g\rangle \mapsto |\mathsf{id}\rangle$$

Key Question: Do entangled states *qua* functions compose (somehow)?

Teleportation: basic case



$$\mathrm{id} \circ \mathrm{id} = \mathrm{id}$$



$$\beta_i^{-1} \circ \mathrm{id} \circ \beta_i = \mathrm{id}, \quad 1 \leq i \leq 4$$

Categorical Axiomatics

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- From this structure we can define the **trace** and **partial trace** with all the key algebraic properties.

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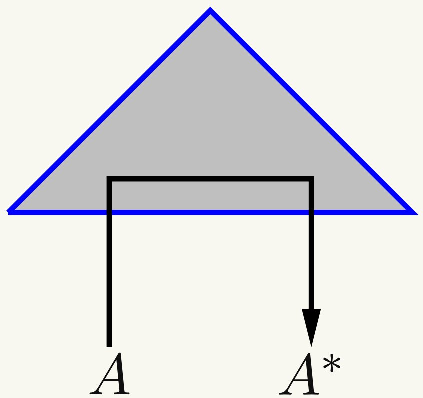
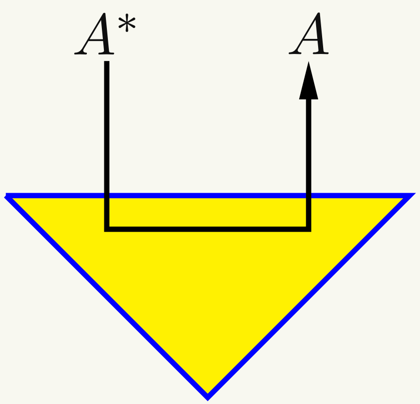
Moreover, this structure is very canonical mathematically:

- Algebraically, it gives exactly the structure of **compact categories**; the fundamental laws are the triangular identities for adjunctions.
- Logically, Bell states correspnd to **axiom links** and costates to **cut links**, and we can track entangled quantum information flows essentially by **Cut-elimination**.
- Diagrammatically, these are basic geometric simplifications: planar versions yield the **Temperley-Lieb algebra**.
- From this structure we can define the **trace** and **partial trace** with all the key algebraic properties.
- Diagrammatically, the (partial) trace closes (part of) the system; when we close the whole system we get **loops** — *i.e.* scalars!

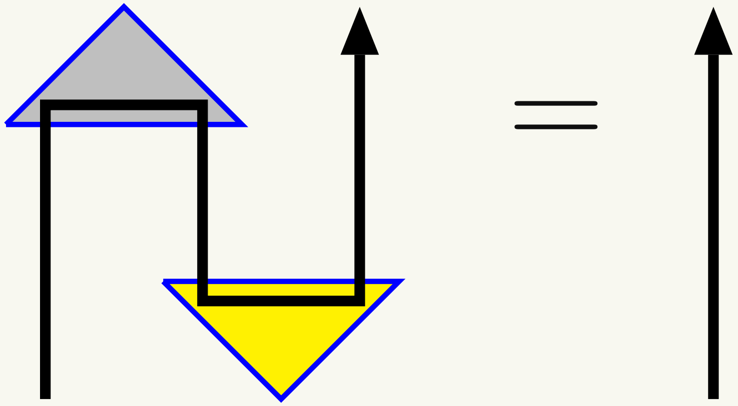


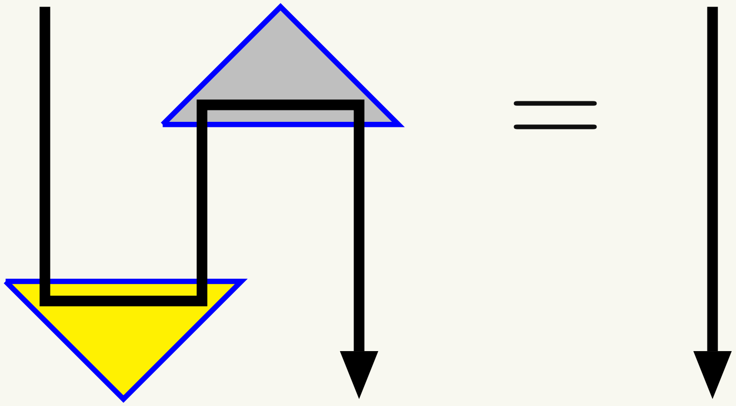


$$\epsilon_A : A \otimes A^* \longrightarrow I \qquad\qquad \eta_A : I \longrightarrow A^* \otimes A.$$

Caps = Bell States; Cups = Bell Tests.

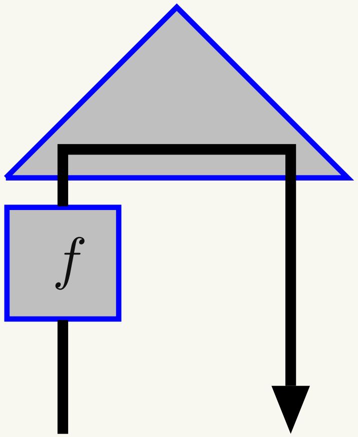
Compact Closure: The basic algebraic laws for units and counits.

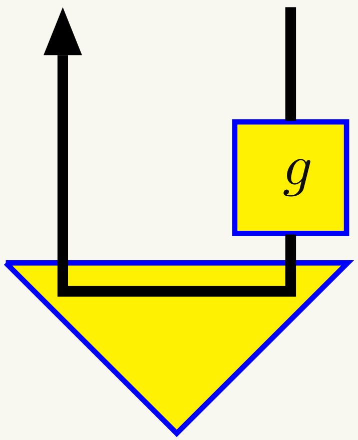




$(\epsilon_A \otimes 1_A) \circ (1_A \otimes \eta_A) = 1_A$ $\qquad$ $(1_{A^*} \otimes \epsilon_A) \circ (\eta_A \otimes 1_{A^*}) = 1_{A^*}$

For coherence with the dagger structure, we require that $\epsilon_A = \eta_A^\dagger$.

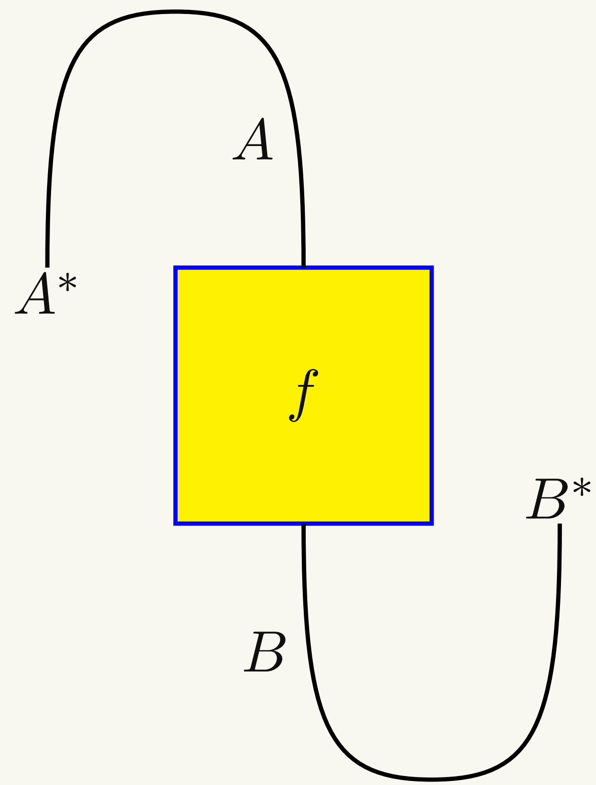




$\llcorner f \lrcorner : A \otimes B^* \longrightarrow I$ $\qquad$ $\ulcorner f \urcorner : I \longrightarrow A^* \otimes B$

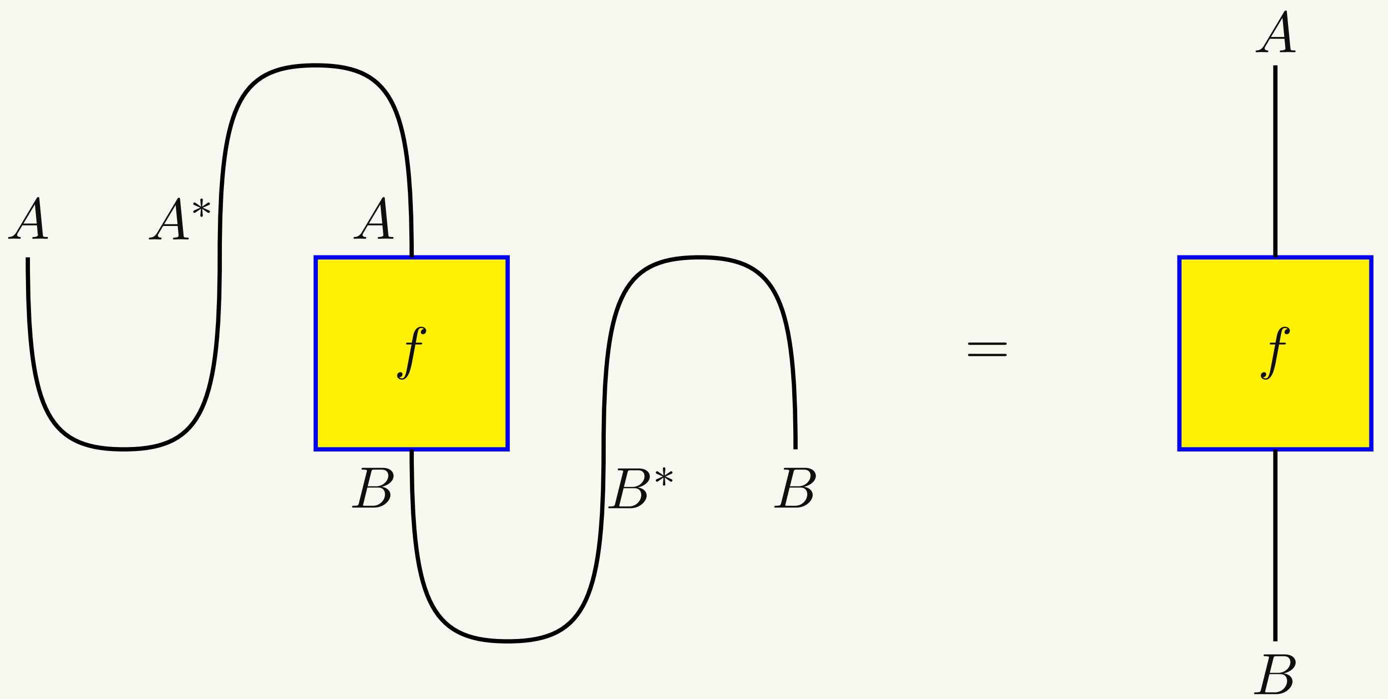
$$\mathcal{C}(A \otimes B^*, I) \simeq \mathcal{C}(A, B) \simeq \mathcal{C}(I, A^* \otimes B).$$

This is the general form of Map-State duality.

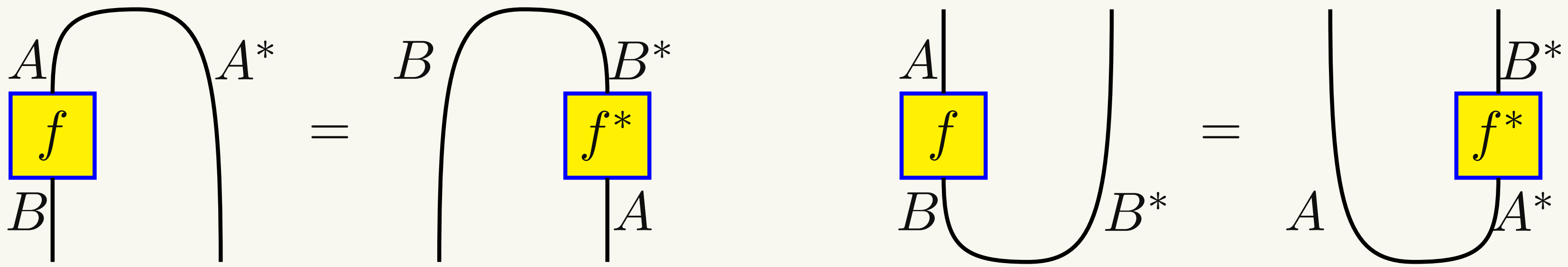


$$f^* = (1 \otimes \epsilon_B) \circ (1 \otimes f \otimes 1) \circ (\eta_A \otimes 1).$$

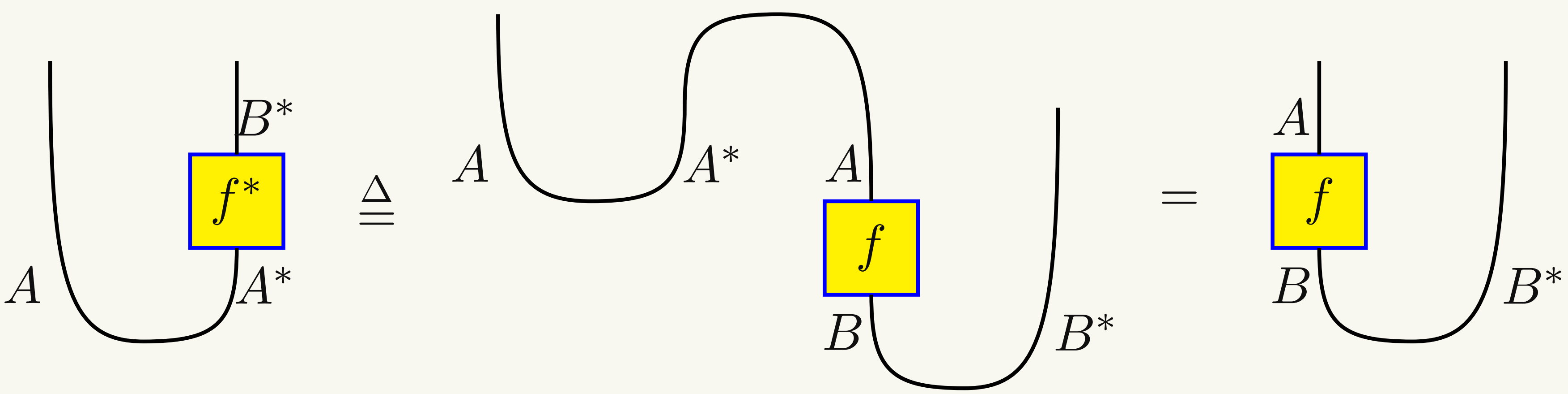
A Little Taste of Diagrammatic Reasoning

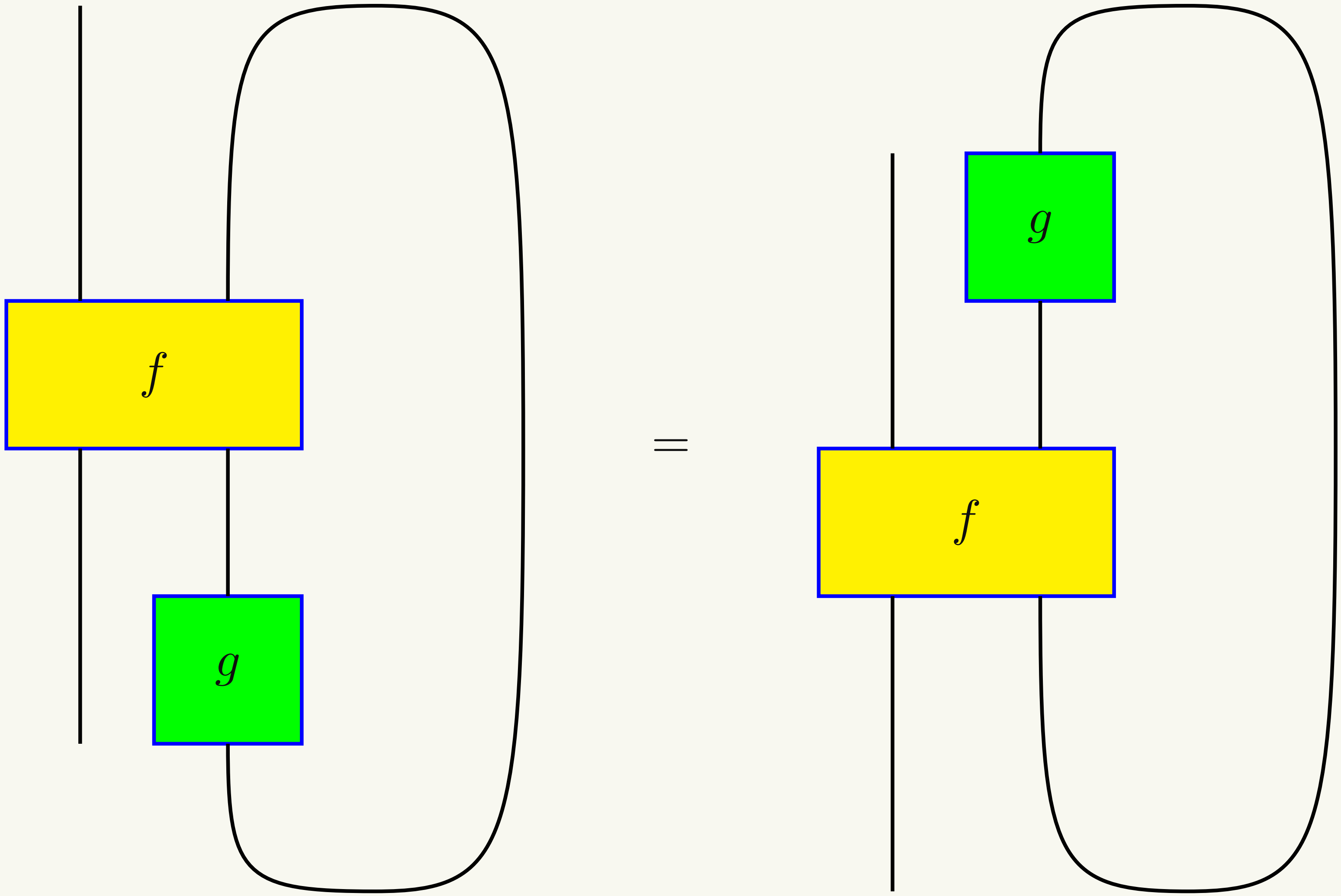


$$f^{**} = f.$$

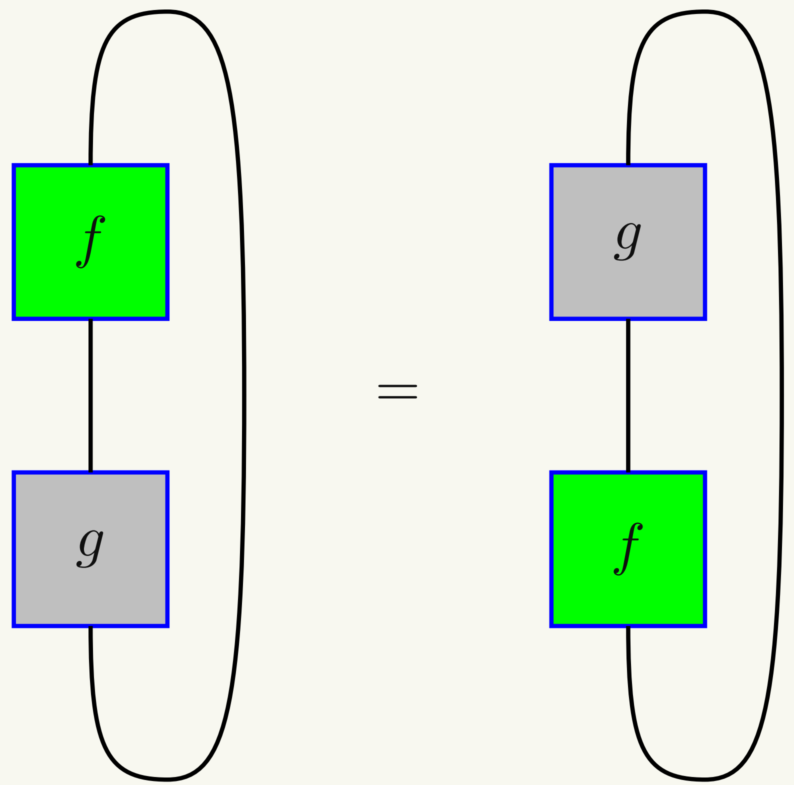


Diagrammatic Proof

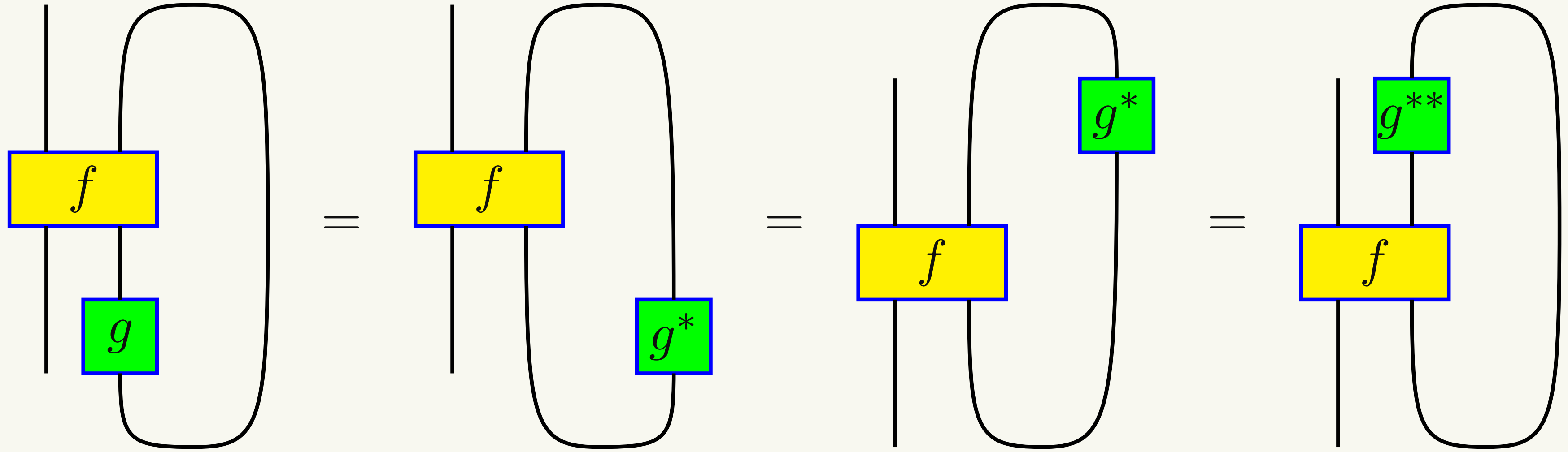



f
g
=
g
f

Application: Invariance of Trace Under Cyclic Permutations



Graphical Proof of Feedback Dinaturality



We use $g^{**} = g$ to conclude.

Illuminating Quantum Information Flow in Entangled Systems

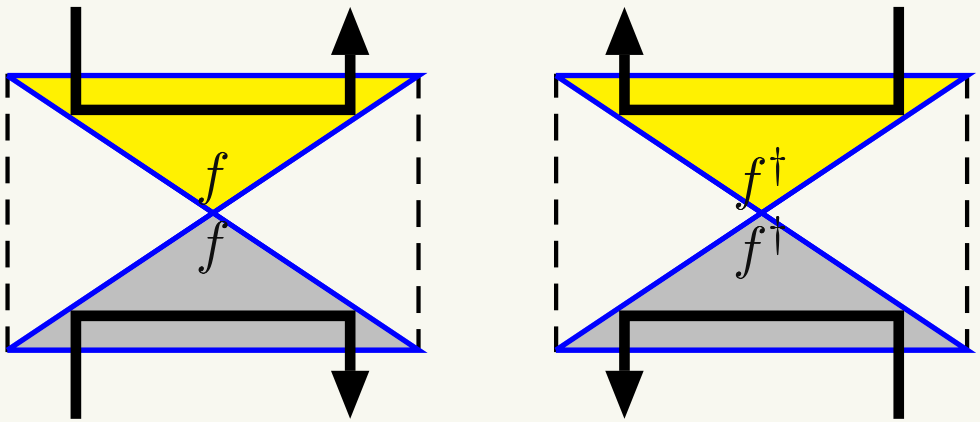
Bipartite Projectors

Information flow in entangled states can be captured mathematically by the isomorphism

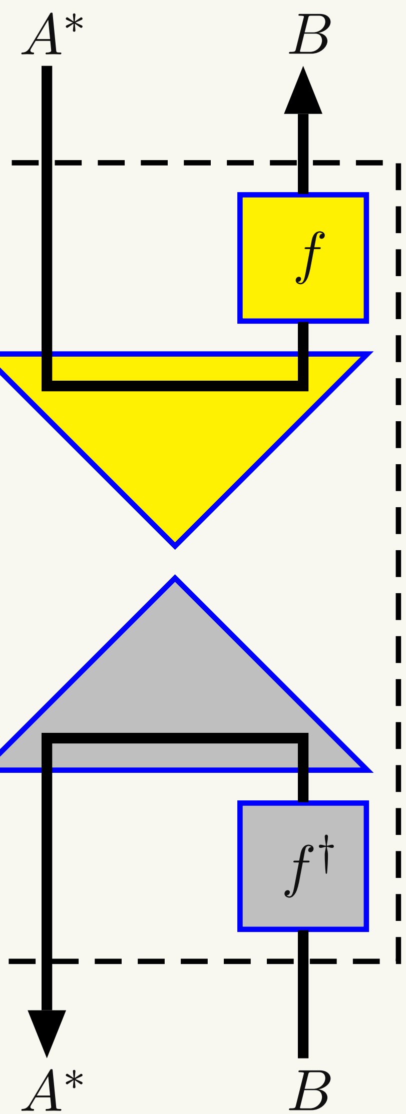
$$\mathbf{Hom}(A, B) \cong A^* \otimes B.$$

This leads to a **decomposition** of bipartite projectors into "names" (preparations) and "conames" (measurements).

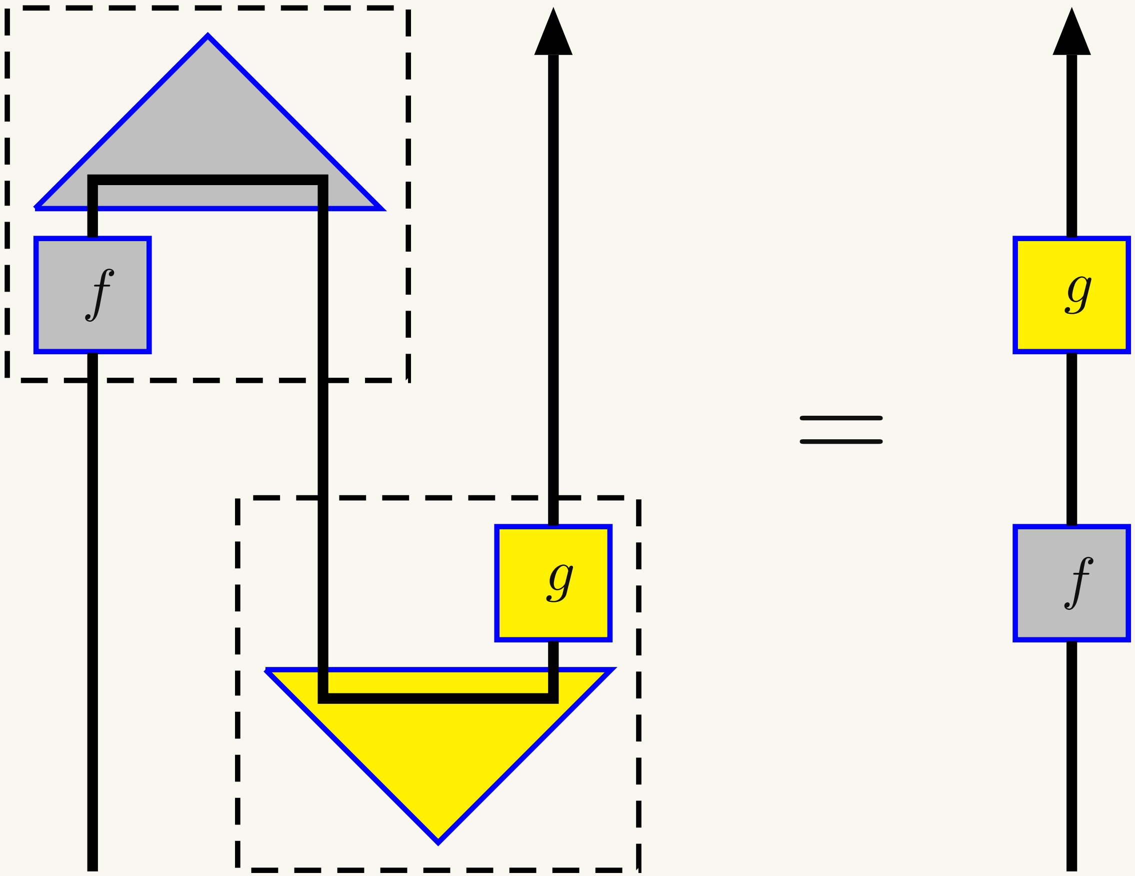
In graphical notation:

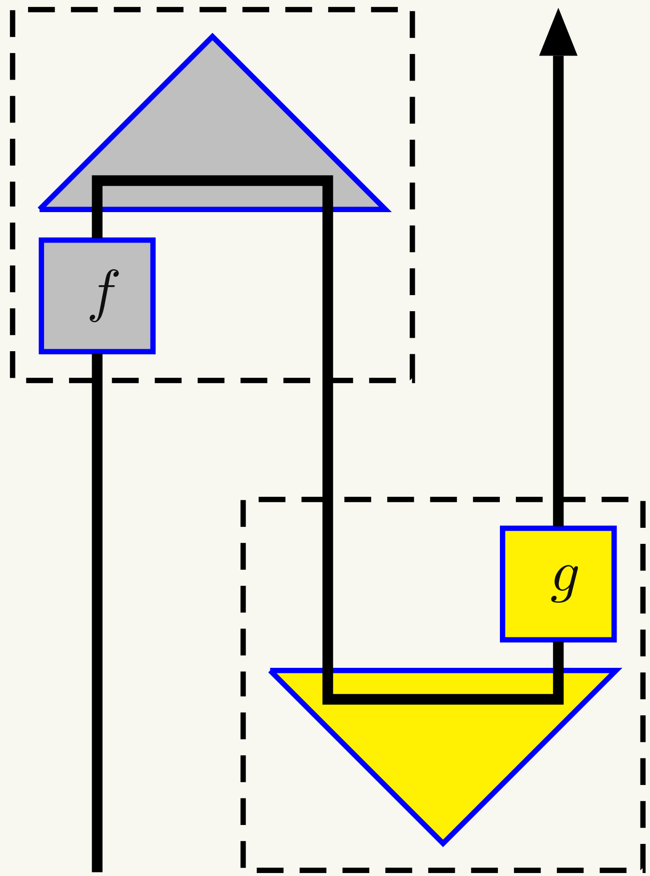


Projectors Decomposed



The key algebraic fact from which teleportation (and many other protocols) can be derived.

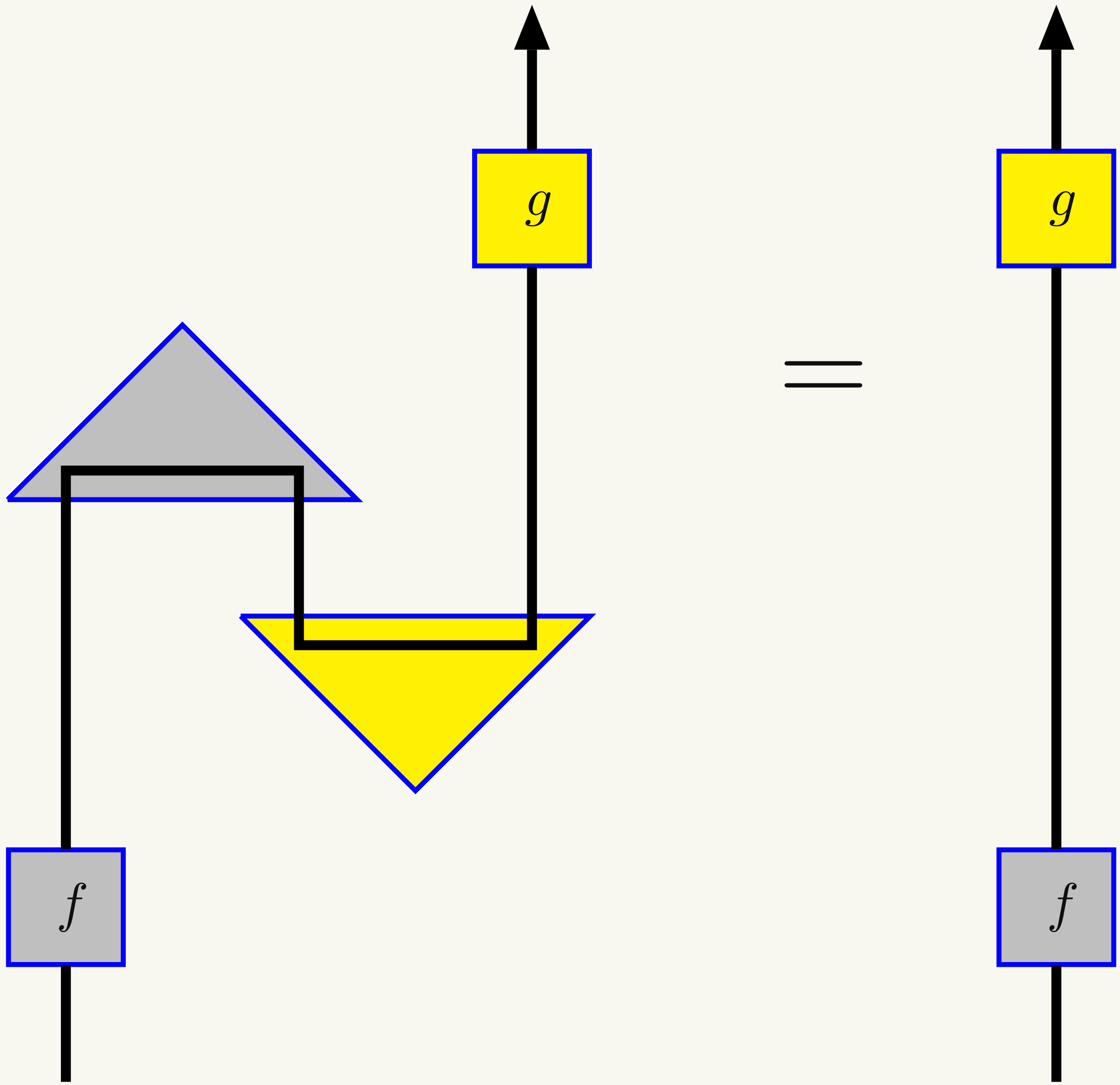




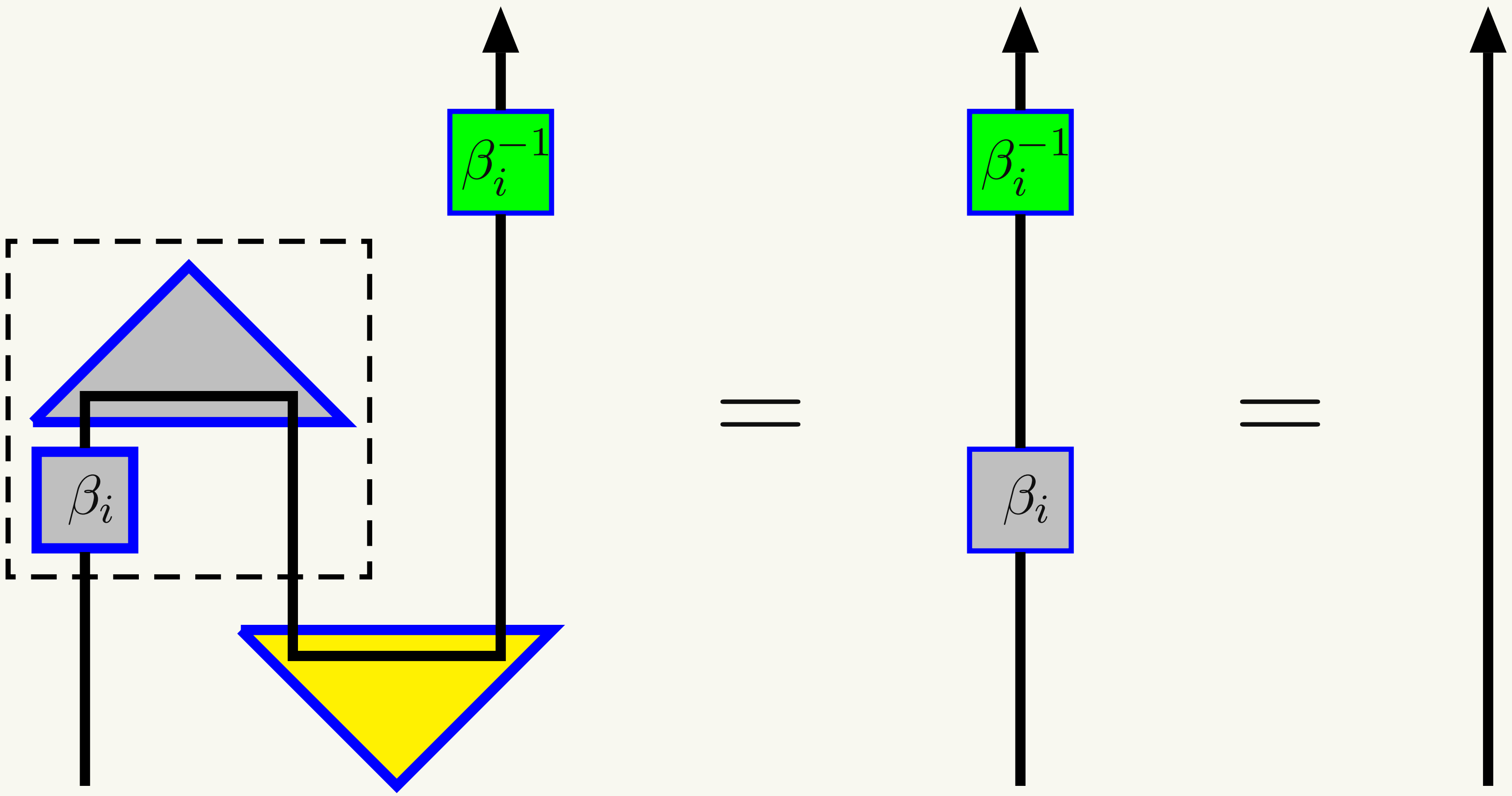
f
g

$=$

Compositionality ctd



Teleportation diagrammatically



Measurements and Classical Information

First Approach: Biproducts

Suppose we assume biproducts (need only assume products: Robin Houston) in a dagger compact category.
These can be used to represent **branching on measurement outcomes**:

$$A \xrightarrow{\quad M \quad} \bigoplus_{i=1}^{n} A_i \xrightarrow{\quad \bigoplus_{i=1}^{n} U_i \quad} \bigoplus_{i=1}^{n} B_i$$

Propagation of the outcome of a measurement performed on one part of a compound system to other parts — "classical communication" — can be expressed using **distributivity**:

$$(A_1 \oplus A_2) \otimes B \;\cong\; (A_1 \otimes B) \oplus (A_2 \otimes B).$$

Teleportation categorically

$$
\begin{tikzcd}[column sep=large]
Q \arrow[r, "(1\otimes\eta)", "(1)"'] \arrow[d, equal] & Q \otimes Q \otimes Q \arrow[r, "\langle \llcorner U_i \lrcorner \rangle_{i=1}^{4} \otimes 1", "(2)"'] & \bigoplus_{i=1}^{4} I \otimes Q \arrow[r, "\text{dist}", "(3)"'] & \bigoplus_{i=1}^{4} Q \arrow[r, "\bigoplus_{i=1}^{4} U_i^{-1}", "(4)"'] & \bigoplus_{i=1}^{4} Q \arrow[d, equal] \\
Q \arrow[rrrr, "\langle 1 \rangle_{i=1}^{i=4}"'] & & & & \bigoplus_{i=1}^{4} Q
\end{tikzcd}
$$

(1) **Produce EPR pair**

(2) **Perform measurement in Bell-basis**

(3) **Propagate classical information**

(4) **Perform unitary correction**.

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N.B. Alternative approach (which brings important new structure to light): **classical structures** (Coecke and Pavlovic), *i.e.* Frobenius dagger-algebras.

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- Distributed QM, “discrete QFT”??