Towards a Quantum Geometry: Groupoids, Clifford algebras and Shadow Manifolds.

B. J. Hiley.

b.hiley@bbk.ac.uk

Unified Structure.

1. Quantum theory through PROCESS.

   Not process in space-time but process from which space-time is abstracted.

2. Process $\nearrow$ Whitehead's notion of FUNCTION.
   Process $\searrow$ Category notion of MORPHISM.

3. Mathematically start with groupoids $\Rightarrow$ Clifford algebras.

$$\Downarrow$$

Schrödinger, Pauli, Dirac

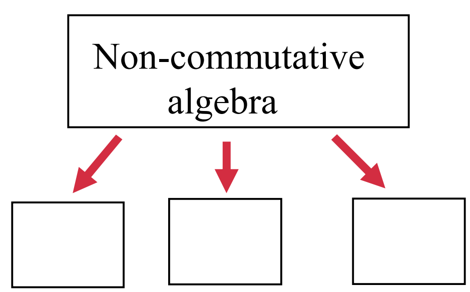
$$\mathcal{C}(0,1) \subset \mathcal{C}(3,0) \subset \mathcal{C}(1,3)$$

4. Clifford bundle $\rightarrow$ Induces light cone structure on abstracted vector space.

5. Add in symplectic structure $\rightarrow$ Non-commutative phase space. $\Rightarrow$ Moyal $\Downarrow$ Deformed Poisson

6. Construct a 'shadow' phase spaces.

   Bohm lives here $\rightarrow$



Shadow manifolds

Process and Categories.

Lawvere's "Continuously Variable Sets" (1973).

"…. the concept of motion as the presence of a body one place at one time and another place at a later time, **describes only the result of motion and does not contain an explanation of motion itself**."

He puts the emphasis on mappings or morphisms.

From these he abstracts the notion of a set.

This was just what I was looking for.

Categories or Algebra?

Hamilton:- Algebra relates successive states of some changing thing or thought.

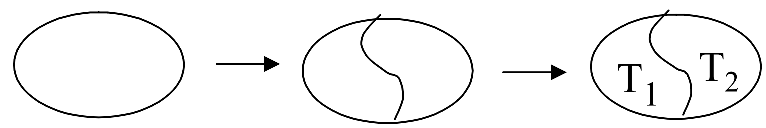
Grassmann:-

Mathematics is about THOUGHT, not MATERIAL REALITY.

It's about RELATIONSHIPS of FORM, not relationships of CONTENT.

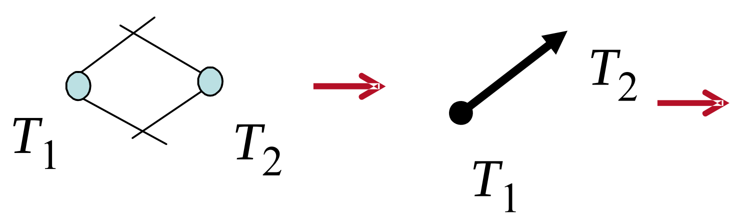
Mathematics is to do with ORDERING FORMS created in THOUGHT.

Thought $\Rightarrow$ becoming $\Rightarrow$process.



New thought contains a trace of the old thought.

Old thought contains the potential of the new thought.

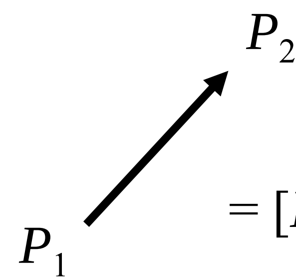


$[T_1, T_2]$

Put together via $[T_1, T_2]\bullet[T_2, T_3] = [T_1, T_3]$ $\Rightarrow$ Groupoid

Closer look at Groupoids.

We have a set X of arrows, sources and targets, s and t

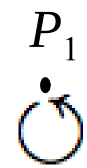


$= [P_1, P_2]$

P_1 is the source s $\qquad$ P_2 is the target t

Combine arrows via $\qquad [P_1P_2]\bullet[P_2P_3] = [P_1P_3]$

Note

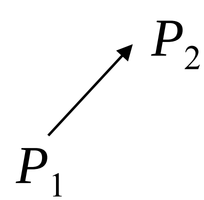


1. $[P_1, P_1]$ is a left unity.
2. $[P_2, P_2]$ is a right unity.
3. Inverse $\quad [P_1, P_2]^{-1} = [P_2, P_1]$

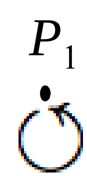
$$[P_1P_2]\bullet[P_1P_2]^{-1} = [P_1P_2]\bullet[P_2P_1] = [P_1P_1] = 1$$

Our interpretation

$[P_1, P_2]$ is P_1 **BECOMING** P_2



$[P_1, P_1]$ is our **BEING**.



Since $[P_1, P_1] \bullet [P_1, P_1] = [P_1, P_1]$, being is **IDEMPOTENT.**

The Algebra of Process.

Rules of composition.

(i) $[kA, kB] = k[A, B]$ — Strength of process.

(ii) $[A, B] = -[B, A]$ — Process directed.

(iii) $\boxed{[A, B] \bullet [B, C] = \pm [A, C]}$ — Order of succession.

(iv) $[A, B] + [C, D] = [A+C, B+D]$ — Order of coexistence.

(v) $[A, [B, C]] = [A, B, C] = [[A, B], C]$

Notice $[A, B] \bullet [C, D]$ is NOT defined (yet!) [Multiplication gives a Brandt groupoid]

[Hiley, Ann. de la Fond. Louis de Broglie, **5**, 75-103 (1980). Proc. ANPA 23, 104-133 (2001)]

Lou Kauffman's iterant algebra

$$[A, B]^*[C, D] = [AC, BD]$$

[Kauffman, Physics of Knots (1993)]

Raptis and Zaptrin's causal sets.

$$\boxed{|A\rangle\langle B| * |C\rangle\langle D| \to \delta_{BC} |A\rangle\langle D|}$$

[Raptis & Zaptrin, gr-qc/9904079]

Bob Coecke's approach through categories.

$$\boxed{\text{If } f : A \to B \text{ and } g : B \to C, \quad f \circ g : A \to C}$$

[Abramsky& Coecke q-ph/0402130]

Space-time an Abstraction?

Hamilton: [Motion & Time, Space & Matter, Machambers&Turnbull, 1976]

"In algebra relationships are between successive states of some changing thing or thought". [Metaphysics of Maths--Algebra of Pure Time]

Einstein: [Physics and Reality, J, Franklin Inst. 221 (1936) 378]

"perhaps the success of the *Heisenberg method points to a purely algebraic description of nature*, that is, to the elimination of the continuous functions from physics. Then, however, we must give up, in principle, the space-time continuum…."

Wheeler: [Quantum Theory and Gravitation, 1980]

Not Day 1 Geometry Day 2 Physics.

But Day 1 The quantum principle Day 2 Geometry

Gel’fand construction.

Commutative algebras.

Traditional way

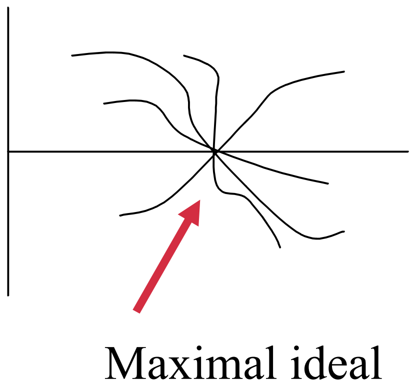
Start with a topological space or a metric space and construct the commutative algebra of functions on that space.

Alternative way.

Take a given commutative algebra and abstract the topological and metric properties from the algebra.

The points of that space are the two-sided maximal ideals.

$$F_x = \{ f \mid f(x) = 0 : \forall f \in C^\infty \}$$



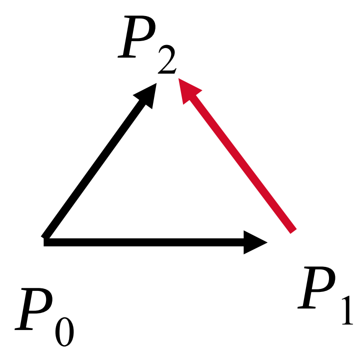
Can we do the same thing for a non-commutative structure?

[Demaret et al Fond. Sci. **2**, (1997) 137-176]

The Directional Calculus.

Consider an arrow, $[P_0P_1]$, in *some* direction with $[P_0P_1] = -[P_1P_0]$

Consider an arrow, $[P_0P_2]$, in *another* direction



How do we get from $[P_0P_1]$ to $[P_0P_2]$?

Introduce $[P_1P_2]$ so that $[P_0P_1]\bullet[P_1P_2] = [P_0P_2]$

Do it again so that $[P_0P_2]\bullet[P_1P_2] = -[P_0P_2]\bullet[P_2P_1] = -[P_0P_1]$

$$[P_0P_1]\cdot\big([P_1P_2]\cdot[P_1P_2]\big) = -[P_0P_1]$$

$$\Downarrow$$

$$[P_1P_2]\bullet[P_1P_2], = -1 \qquad \Rightarrow \qquad \boxed{i^2 = -1}$$

[Hiley, Quantum Interactions, 1-10 OXFORD 2008]

The Quaternions.

Identify $[P_0P_1] = e_1; \quad [P_0P_2] = e_2; \quad [P_1P_2] = e_{12}.$

C **(0,2)⇔SO(2)**

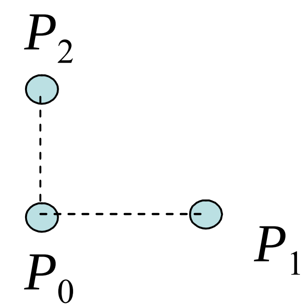
1	e_1	e_2	e_1e_2
e_1	-1	e_1e_2	$-e_2$
e_2	$-e_1e_2$	-1	e_1
e_1e_2	e_2	$-e_1$	-1

Isomorphic to the QUATERNIONS, {*i, j, k*}

Elements anti-commute $\left[[P_iP_j],[P_nP_m]\right]_+ = 2\delta_{jn}\delta_{im}$

$\boxed{cf : \left[\gamma_\mu, \gamma_\nu\right]_+ = 2g_{\mu\nu}}$

Introduce mapping: η : Cliff $\to$ Vect.

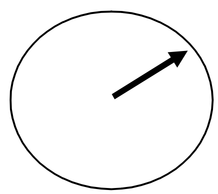


The three idempotents become points of *V*.

$\eta : [P_0P_0] \to P_0, \quad [P_1P_1] \to P_1, \quad \text{and} \quad [P_2P_2] \to P_2$

Allow addition and exploit the Clifford group $A' = gAg^{-1}$ with $g = a + be_{12}$

If $\eta(a) = \cos(\theta/2)$ and $\eta(b) = \sin(\theta/2)$



Directional calculus.

2-dim Lorentz Group. C(1,1) ⇔ SO(1,1)

Introduce **polar** processes $[P_0\, P\,]$ and **temporal** process $[P_0\, T\,]$

with $[P_0 T] \bullet [TP] = -[P_0 P]$

This gives

$$[P_0\, P] \bullet [P_0\, P] = -\, [P_0\, P] \bullet [P\, P_0] = -\, [P_0\, P_0] = -\, 1$$

$$[P_0\, T] \bullet [P_0\, T] = -\, [P_0\, T] \bullet [T\, P_0] = +\, [P_0\, P_0] = +\, 1$$

Write $[P_0\, T] = e_0$, $[P_0 P] = e_1$ and $[PT] = e_{01}$.

We then get the multiplication table

1	e_0	e_1	e_{01}
e_0	1	e_{01}	e_0
e_1	$-e_{01}$	-1	e_0
e_{01}	$-e_1$	$-e_0$	1

Clifford group $R_{1,1}$.

SO(1,1)

The Lorentz Group

Allow addition $\rightarrow$ light cone coordinates $\eta(e_0 \pm e_1) = t \pm x$

Velocity?

Consider $[P_0P] \bullet [P_0T]^{-1} = -[PT]$ or $[P_0T]^{-1} \bullet [P_0P] = [PT] = e_{01} = \alpha$

$$\eta(e_{01}) \rightarrow \mathbf{v}$$

Dirac's α

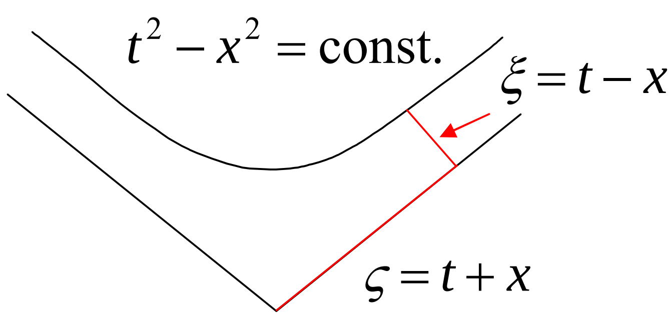
Again exploit the Clifford group $A' = gAg^{-1}$ with $g = a + be_{01}$

Here $\eta(a) = \cosh(\lambda/2)$, $\eta(b) = \sinh(\lambda/2)$ $\tanh(\lambda/2) = \mathrm{v}$

$$\eta\ :\quad e_0' \pm e_1' = \kappa^{\pm}(e_0 \pm e_1) \quad \rightarrow \quad t' \pm x' = k^{\pm}(t \pm x) \qquad k = \sqrt{\frac{1+\mathrm{v}}{1-\mathrm{v}}}$$

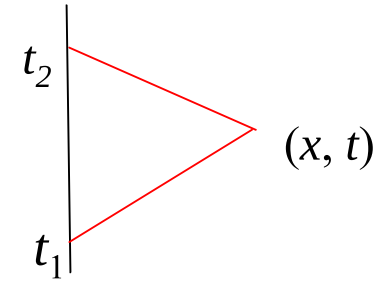
This is the *k*-calculus ⇒ Lorentz transformations of SR.

[Kauffman, Physics of Knots (1993)]



Kaufman and the k-calculus.

[Kauffman, Physics of Knots (1993)]



Primary connections.

$$t_2 - t_1 = 2x,$$

$$t_2 + t_1 = 2t. \quad \Rightarrow \quad t_2 = t + x, \qquad t_1 = t - x.$$

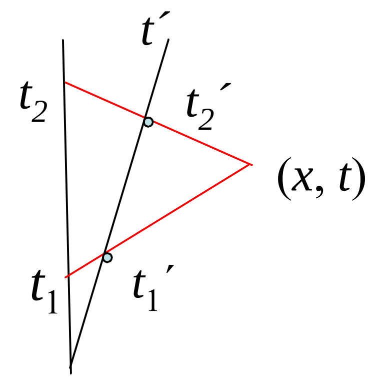
$$[t_2,\ t_1] = [t + x,\ t - x] = t*[1, 1] + x*[1, -1] = t\,1 + x\,\sigma \quad \text{where } \sigma * \sigma = \sigma$$

The *k*-calculus gives

$$t = k\,t_1 \text{ and } t_2 = kt \text{ so that } \quad \frac{t_2}{t_1} = \frac{t+x}{t-x} = k^2 \quad \Rightarrow \quad k = \sqrt{\frac{1+\text{v}}{1-\text{v}}}$$

Lorentz transform T_L is $\quad t_1' = kt_1 \quad$ and $\quad t_2 = kt_2'$

$$[t_2',\ t_1'] = [k^{-1}t_2,\ kt_1] = T_L*[t_2,\ t_1] = [k^{-1}, k]*[t_2,\ t_1]$$



Write

$$T_L = \left[k^{-1}, k\right] = k^{-1}\left[1, k^2\right] = \frac{1}{\sqrt{1-\text{v}^2}}\left[1-\text{v}, 1+\text{v}\right] = \gamma\left[1, -\text{v}\right] = \gamma\left(1 - \text{v}\sigma\right)$$

Then

$$T_L*(t + x\sigma) = \gamma(1 - v\sigma)*(t + x\sigma) = [\gamma(1-\text{v}),\ \gamma(1+\text{v})]*[t + x, t - x]$$

$$= \gamma(t - \text{v}x) + \gamma(x - \text{v}t)\sigma = \gamma(t' + x'\sigma)$$

Lorentz boost: $\quad t' = \gamma(t - \text{v}x); \qquad x' = \gamma(x - \text{v}t)$

Higher Dimensional Clifford Algebras.

By adding more degrees of freedom, ie more generators we obtain

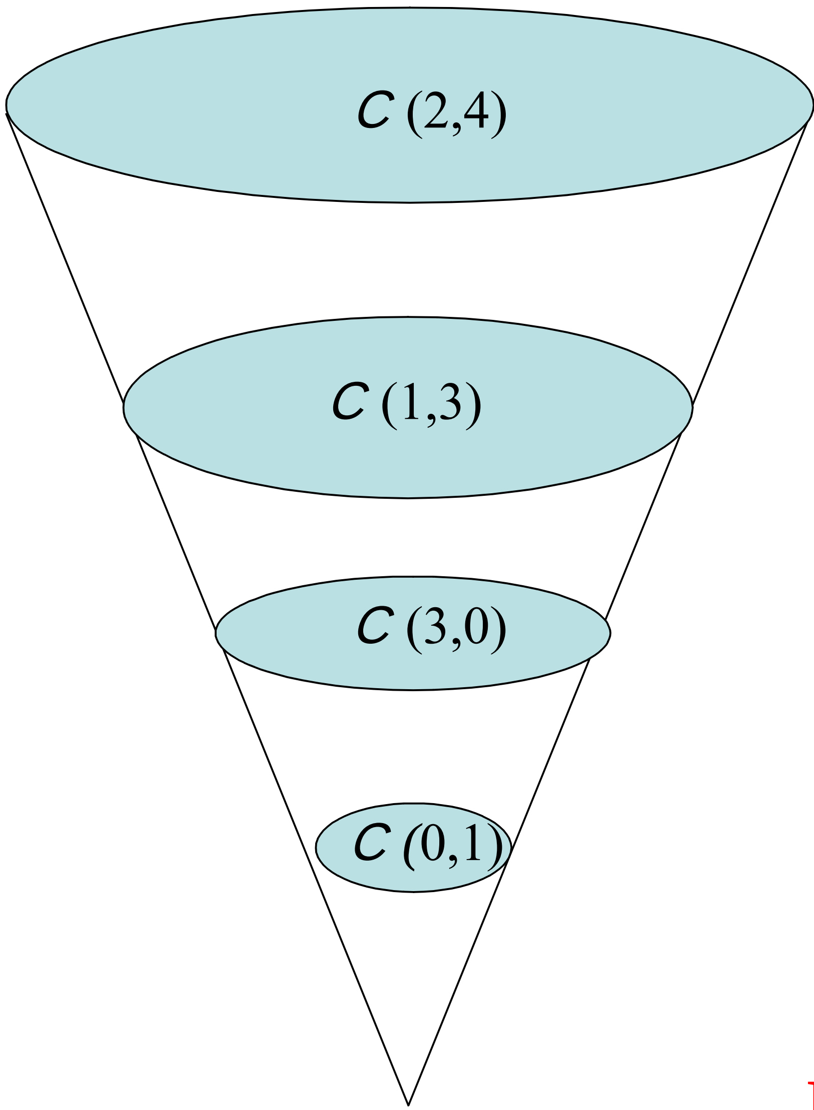
The Pauli Clifford	Non-relativistic with spin
The Dirac Clifford	Relativistic with spin
The conformal Clifford.	Twistor

The one generator Clifford I will call

The Schrödinger Clifford.

So far NOTHING is QUANTUM.

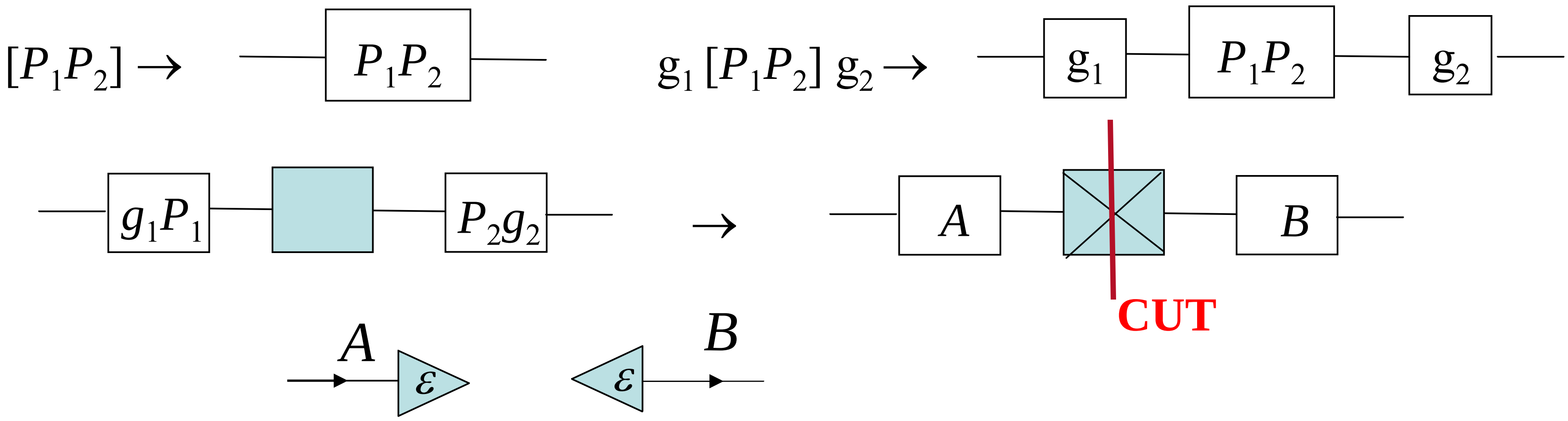
Hierarchy of Clifford Algebras



	Generating elements.
Conformal	$\{1, e_0, e_1, e_2, e_3, e_4, e_5\}$
Dirac	$\{1, e_0, e_1, e_2, e_3\}$
Pauli	$\{1, e_1, e_2, e_3\}$
Schrödinger	$\{1, e_1\}$

Nothing quantum here.

Where are the Spinors?

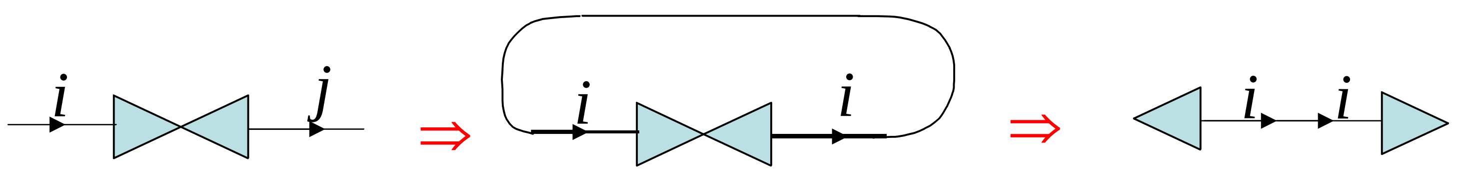


$\in$ left ideal $\Psi_L = A\varepsilon$

$\in$ right ideal $\Psi_R = \varepsilon B$

Idempotent $\varepsilon^2 = \varepsilon$

These are our SPINORS



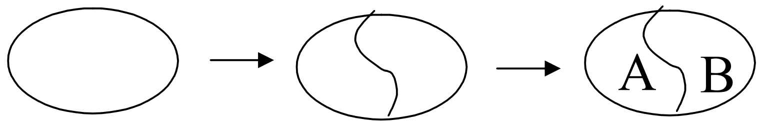
Operator **Trace** **Complex number**

$\Psi_L \Psi_R = A\varepsilon B \quad \to \quad A\rangle\langle B \Rightarrow \psi\rangle\langle\phi \Rightarrow \hat{\rho}$

Standard ket $\psi\rangle$

[Dirac, Quantum Mechanics 3rd Edition p. 79 (1947)]
[Dirac, Spinors in Hilbert Space (1974)]

Meaning of Symbolism?



$[A, B] \rightarrow A\varepsilon B \rightarrow A\varepsilon \;\; \varepsilon B \rightarrow \Psi_{\mathrm{L}} \; \Psi_{\mathrm{R}} \rightarrow \psi\rangle \; \langle\phi$ **Taking apart**

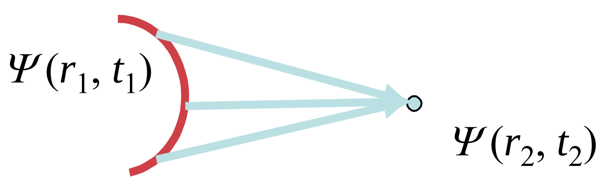
The choice of idempotent determines the distinction.

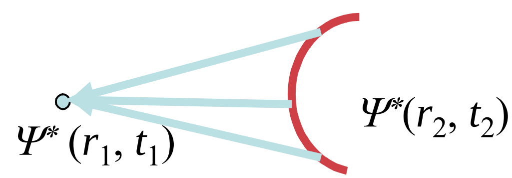
Feynman suggested we put this in time.

$\psi(\mathrm{R}_1)$ was information coming from the past.

$\psi^*(\mathrm{R}_0)$ was information coming from the 'future'.

Huygens





$$\psi(r_2,t_2) = \int G(r_1,r_2,t_1,t_2)\psi(r_1,t_1)d^4r_1 \quad \rightarrow \Psi'_{\mathrm{L}} = M\Psi_{\mathrm{L}} \qquad \Psi'_{\mathrm{R}} = \Psi_{\mathrm{R}}M^*$$

$$(\Psi_L\Psi_R)' = M(\Psi_L\Psi_R)M^*$$

Putting back together

[Feynman Rev. Mod Phys. 20 (1948) 367]

Light Rays and Light Cones.

Take the Pauli algebra. Choose $\epsilon = (1+e_3)/2$

Then

$$\Psi_L(\mathbf{r},t) = \psi_L(\mathbf{r},t)(1+e_3)/2 = g_0(\mathbf{r},t) + g_1(\mathbf{r},t)e_{23} + g_2(\mathbf{r},t)e_{31} + g_3(\mathbf{r},t)e_{12}$$

Now form $\boxed{\mathcal{V} = \psi_L(1+e_3)\psi_R = v_0 1 + v_1 e_1 + v_2 e_2 + v_3 e_3}$

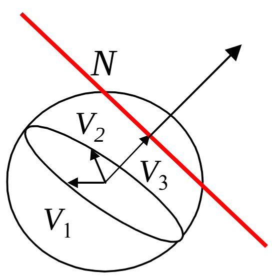
We find

$$v_0 = g_0^2 + g_1^2 + g_2^2 + g_3^2 \qquad v_1 = 2(g_1 g_3 - g_0 g_2)$$
$$v_2 = 2(g_0 g_1 + g_2 g_3) \qquad v_3 = g_0^2 - g_1^2 - g_2^2 + g_3^2.$$

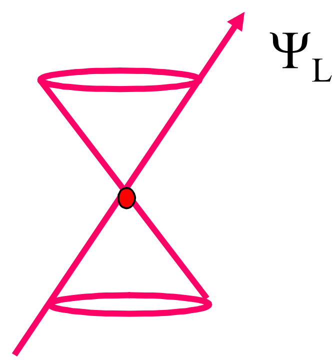
$\eta : \mathcal{C}_{1,3} \to V_{1,3}$

$$\eta(\mathcal{V}) = v_0\hat{t} + v_1\hat{x} + v_2\hat{y} + v_3\hat{z}$$

Null Ray



Fix origin in *V*(1,3) then form $\mathcal{V}' = g\mathcal{V}g^{-1} \quad g \in SU(2)$



Generates the light cone in *V*(1,3)

The Matrix Method.

Start with

$$|\Psi\rangle \rightarrow \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Then form

$$|\Psi\rangle\langle\Psi| = \begin{pmatrix} |\psi_1|^2 & \psi_1\psi_2^* \\ \psi_2\psi_1^* & |\psi_2|^2 \end{pmatrix} \qquad \rightarrow \qquad X = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$

Then we find the null ray:

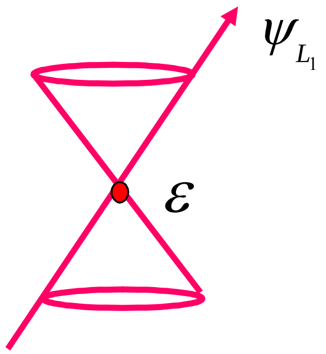
$$t = |\psi_1|^2 + |\psi_2^*|^2 \qquad x = \psi_1\psi_2^* + \psi_1^*\psi_2 \qquad y = i(\psi_1\psi_2^* - \psi_1^*\psi_2) \qquad z = |\psi_1|^2 - |\psi_2^*|^2$$

Looks like quantum mechanics but has little to do with QM!

It works because we can identify

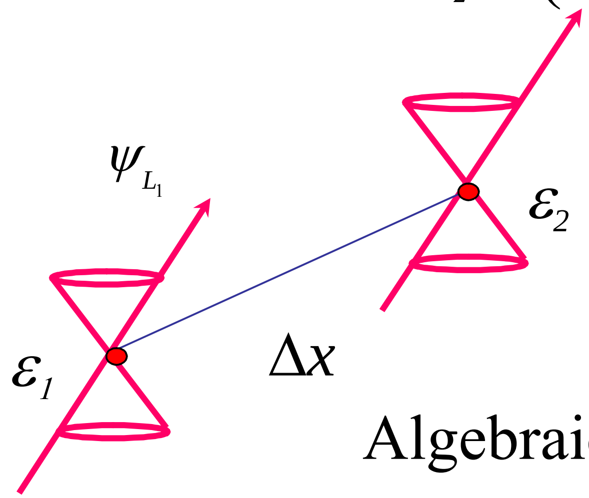
$$g_0 = (\psi_1^* + \psi_1)/2 \qquad g_1 = i(\psi_2^* - \psi_2)/2$$
$$g_2 = (\psi_2^* + \psi_2)/2 \qquad g_3 = i(\psi_1^* - \psi_1)/2$$

Algebraic Spinors and Twistors



Algebraic light cone

$$\varepsilon_2 = i\left(\Delta x^0 + \sigma.\Delta x\right)\varepsilon_1$$

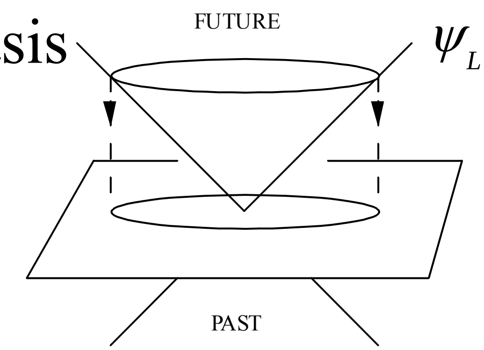


Algebraic twistor

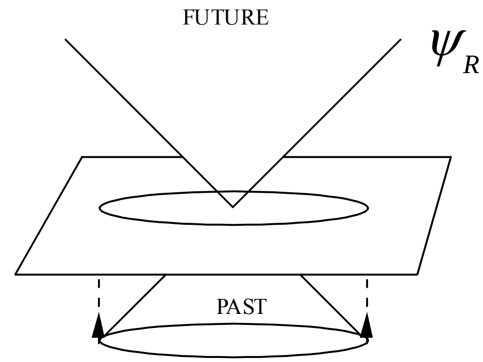
[Bohm & Hiley, Rev. Briz Fis, 1-26 (1984)]

Algebraic *L*-spinor basis

$$(1 - i\gamma^5)\,(1 - \gamma^{03})$$
$$(1 - i\gamma^5)(\gamma^{13} + \gamma^{01})$$
$$(1 + i\gamma^5)\,(\gamma^0 - \gamma^3)$$
$$(1 + i\gamma^5)(\gamma^2 - \gamma^{023})$$



$v_A(\psi_L)$ transforms like the future light cone



$u^{\dot{A}}(\psi_R)$ transforms like the past light cone

[Frescura & Hiley, Found. Phys. 10, 7-31 (1980)]

Quantum Kinematics.

Need to go back to Heisenberg and introduce a symplectic structure.

We need to distinguish between two types of process $X[P_1P_2]$ and $P[\mathrm{P}_1\mathrm{P}_2]$

Heisenberg $X[P_nP_k]\cdot P[P_kP_n]-P[P_nP_k]\cdot X[P_kP_n]=i\hbar$ Put $\hbar=1$

We do this through the discrete Weyl algebra, $C^2{}_n$.

$X[P_nP_k]\cdot P[P_kP_n]=\omega P[P_nP_k]\cdot X[P_kP_n].$ where $\omega^n=1$

n

$\boxed{\text{n}^{\text{th}} \text{ root of unity}}$

Write as $UV=\omega VU;\quad U^n=1;\quad V^n=1.$

$\boxed{\text{The quantum doughnut.}}$

As n $\rightarrow \infty$ Weyl algebra $\rightarrow$ Heisenberg algebra.

i.e. we have a discrete phase space structure but it is **non-commutative**.

Let us see how it works in toy space.

[Morris, Quart. J. Math. 18 (1967) 7-12]

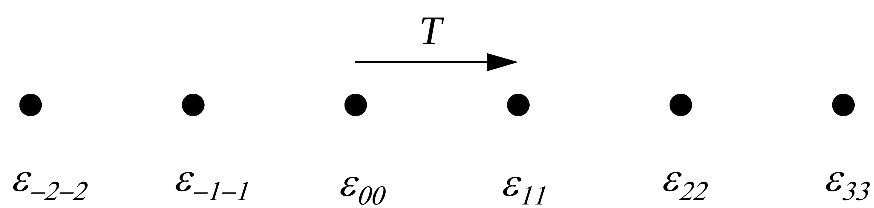
Points in the Finite Phase Space.

In $C^2{}_n$ $\exists$ a set of IDEMPOTENTS $\{\varepsilon_{jj}\}$. They are the points of our space.

$$\varepsilon_{jj} = \tfrac{1}{n}\sum_k \omega^{-jk} R(0,k) \qquad \text{where} \qquad R(j,k) = \omega^{-jk/2} U^j V^k, \quad j,k = 0,1,\ldots n-1.$$

How do we relate the idempotents?

Use $\varepsilon_{j+1\,j+1} = T\varepsilon_{jj}T^{-1}$ where $T = U = \exp\left[\dfrac{2\pi i\,\delta x}{n}P\right]$



Position points

$$X = \delta x \sum_j j\varepsilon_{jj} \qquad \text{so that} \qquad X\varepsilon_{jj} = \delta x j \varepsilon_{jj} \qquad [\delta x j \varepsilon_{jj} \in I_L.]$$

Again the algebra contains within itself a set of points.

[Hiley & Monk, , Mod. Phys. Lett., **A8,** 3225-33.1993]

The Momentum Space.

We form idempotents $\quad \varepsilon'_{ii} = \frac{1}{n}\sum_{j}\omega^{-ij}R(j,0)$

This generates a new set of points. $\quad T' = V = \exp\left[\frac{2\pi i\delta p}{n}X\right]$

$$\xrightarrow{T'}$$

$\varepsilon'_{-2-2} \quad \varepsilon'_{-1-1} \quad \varepsilon'_{00} \quad \varepsilon'_{11} \quad \varepsilon'_{22} \quad \varepsilon'_{33}$

Momentum space

Then $\quad P = \delta p\sum_{j} j\varepsilon'_{jj} \quad$ which gives $\quad P\varepsilon'_{jj} = \delta p j\varepsilon'_{jj} \quad [\delta p j\varepsilon'_{jj} \in I'_L]$

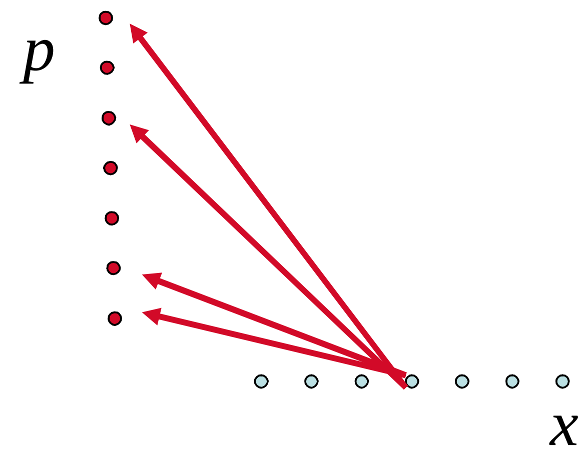
Furthermore

$\boxed{\varepsilon'_{jj} = Z^{-1}\varepsilon_{jj}Z}$ with $\quad Z = \frac{1}{\sqrt{n^3}}\sum_{ijk}\omega^{j(i-)}R(j-i,k)$

Finite Fourier transform.

[Hiley, CASYS 2000 4th Int. Conf.,77-88, 2001.]

Important Lesson.



What does $\varepsilon' = Z^{-1}\varepsilon Z$ mean?

Each *x* point 'explodes' into all *p* points

and vice-versa.

The *p*-points are not 'hidden', they are not 'manifest' but 'enfolded'.

The *p*-space is 'hologrammed' in the *x*-space and vice-versa

In the continuum limit idempotents are delta functions.

Note there are many sets of idempotents in $C^2{}_n$.

$$\varepsilon' = Z_i^{-1}\varepsilon Z_i$$

Equivalent to fractional Fourier transformations.

[Hiley & Monk, Found. Phys. Lett. **11**, 371-377 (1998)]

[M. Brown, PhD Thesis 2004]

The Continuum Limit.

Recall

$$R(j,k)=\omega^{-\frac{jk}{2}}U^jV^k; \qquad U^j=\exp[ij\xi P] \qquad V^k=\exp[ik\eta Q] \qquad \begin{cases} \xi=\dfrac{2\pi i\delta x}{n} \\ \eta=\dfrac{2\pi i\delta P}{n} \end{cases}$$

where $\omega=\exp[i\xi\eta]$

$$U^s: x'_j=x_{j-s} \qquad V^t: x'_j=\omega^{jt}x_j$$

[x are elements of a left ideal]

Consider the limit $n\to\infty$

$$s\xi\to\sigma \qquad t\eta\to\tau \qquad \omega^{kt}=\exp[i\xi k\tau\eta]=\exp[iq\tau]; \quad (k\xi=q)$$

k integer, but $\xi\propto 1/n$ $\quad\therefore$ for n large k runs from $-\infty$ to ∞

[k is mod n; $k\xi$ is mod $n\xi$, but $n\xi=2\pi/\eta\to\infty$ as $\eta\to 0$]

x* becomes the wave function *ψ (q) $\quad\rightarrow\quad x_k=\sqrt{\xi}\,\psi(q)$

$$U^s:\psi(q)\to\exp[i\sigma P]\psi(q)\to\psi(q-\sigma) \qquad V^t:\psi(q)\to\exp[iq\tau]\psi(q)$$

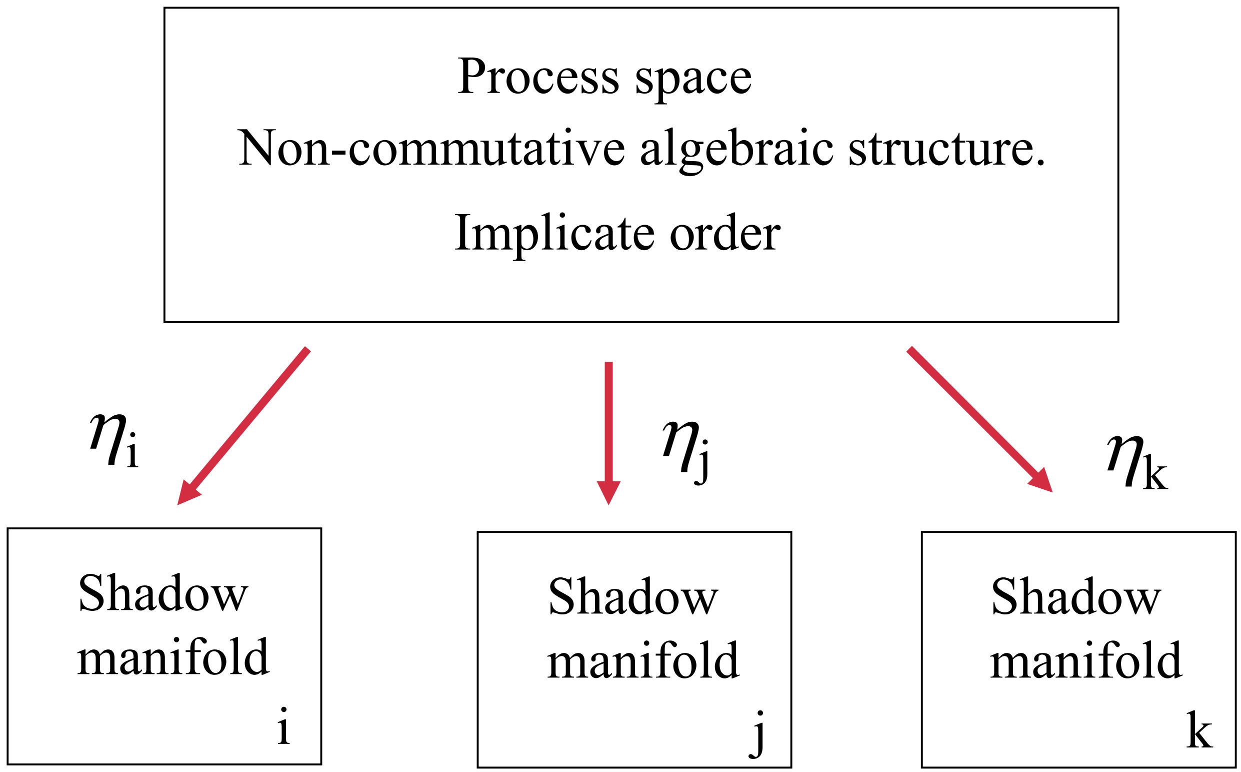
Heisenberg group $\quad H(q,\sigma,\varsigma)=\exp[iqQ+i\sigma P+i\varsigma Z]$

$$\exp\left[i\sigma.\hat{P}\right]\psi(x)=\psi(x+\sigma) \qquad \exp\left[iq.\hat{Q}\right]\psi(x)=\exp[iqx]\psi(x)$$

U $\qquad\qquad$ ***V***

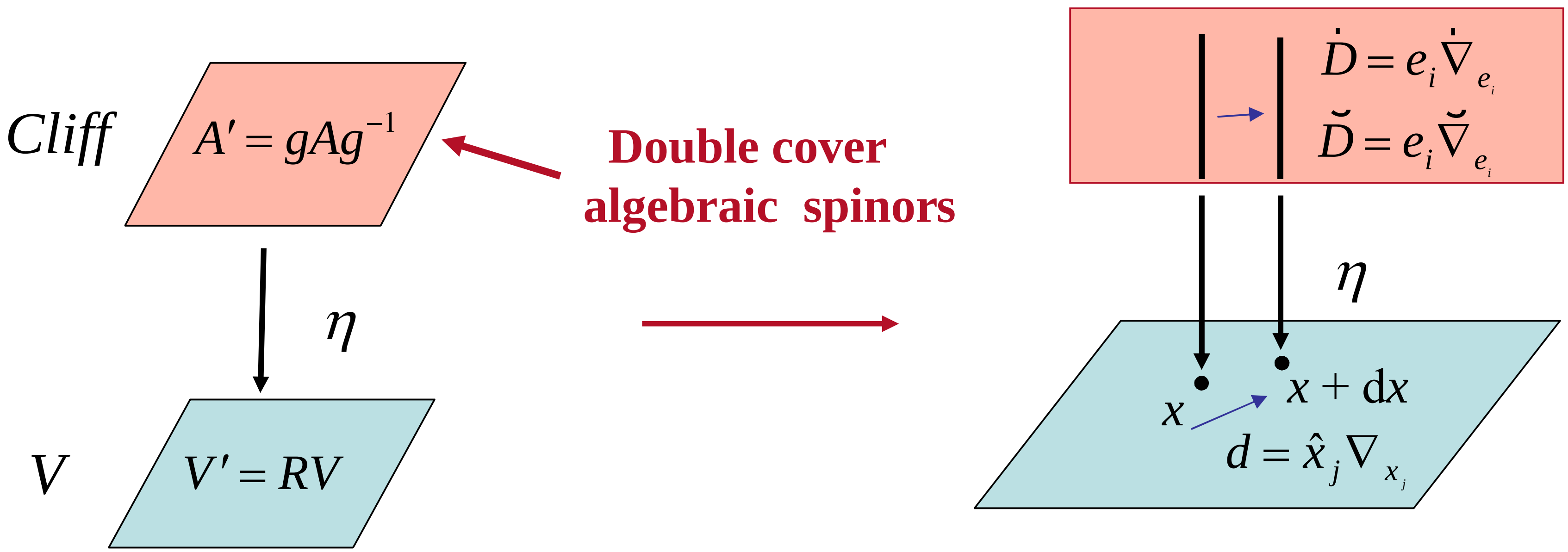
[Weyl, Theory of Groups and quantum mechanics]

General Algebraic Structure.



Shadow manifolds are the explicate orders

The Clifford Group and Spinors.



Clifford bundle with connection.

N.B. We need TWO derivatives in the bundle space $\overrightarrow{D}$and $\overleftarrow{D}$.

The Relation to the Quantum Formalism.

QM **Cliff.**

$|\psi\rangle$ $\qquad$ $|\psi\rangle \leftarrow \{\Psi_L\}$ $\qquad$ Ψ_L $\qquad$ **More general**

Hilbert space

Let's do quantum mechanics in the Clifford algebra.

We have TWO derivatives $\overrightarrow{D}$ and $\overleftarrow{D}$.

Therefore we need to use two Schrödinger equations,

$$i\frac{d\Psi_L}{dt} = \overrightarrow{H}\Psi_L \qquad \text{and} \qquad -i\frac{d\Psi_R}{dt} = \Psi_R\overleftarrow{H}$$

Time Evolutions: Differences and Sums.

Differences:-

$$i\,[(\partial_t \Psi_L)\Psi_R + \Psi_L(\partial_t \Psi_R)] = (H\Psi_L)\Psi_R - \Psi_L(H\Psi_R)$$

We can rewrite this as $i\partial_t \rho = [H, \rho]_-$ **Liouville equation.**

ρ is the density operator for a pure state.

Sums:-

$$i\,[(\partial_t \Psi_L)\Psi_R - \Psi_L(\partial_t \Psi_R)] = (H\Psi_L)\Psi_R + \Psi_L(H\Psi_R) = [H, \rho]_+$$ **Anonymous.**

What is this equation?

Could

$$E \to i\,[(\partial_t \Psi_L)\Psi_R - \Psi_L(\partial_t \Psi_R)] \ ?$$

$$P \to i\,[(\nabla \Psi_L)\Psi_R - \Psi_L(\nabla \Psi_R)] \ ?$$

[Brown and Hiley quant-ph/0005026]

Real

The Schrödinger Particle.

Clifford algebra $C(0,1)$ generated by $\{1, e\}$ where $e^2 = -1$

General element $\psi_L = g_0 + g_1 e$ with $g_0, g_1 \in \Re$

Clifford conjugation $\tilde{\psi}_L = \psi_R = g_0 - g_1 e$

Write $\psi_L = \rho^{\frac{1}{2}} U$ with $\rho = g_0^2 + g_1^2$ and $U\tilde{U} = \tilde{U}U = 1$

Then

$$\Psi_R \Psi_L = \varepsilon \psi_R \psi_L \varepsilon = \varepsilon \rho^{\frac{1}{2}} U \tilde{U} \rho^{\frac{1}{2}} \varepsilon = \rho \varepsilon$$

and

$$\hat{\rho} = \Psi_L \Psi_R = \psi_L \varepsilon \psi_R = \rho U \varepsilon \tilde{U}$$

$$\boxed{\begin{array}{l} \Psi_R \Psi_L \Rightarrow \langle \psi_R | \psi_L \rangle \\ \Psi_L \Psi_R \Rightarrow | \psi_L \rangle \langle \psi_R | \end{array}}$$

The Details.

Energy.

$$2\rho E(t) = i\,[(\partial_t \Psi_L)\Psi_R - \Psi_L(\partial_t \Psi_R)]$$

or

$$2E(t) = \frac{i}{2}[\Omega_t \Sigma + \Sigma \Omega_t] = i\Omega_t \cdot \Sigma$$

$$\boxed{\begin{array}{c} \Sigma = U\varepsilon\tilde{U} \\ \Omega_t = 2(\partial_t U)\tilde{U} = -2U(\partial_t \tilde{U}) \end{array}}$$

If we write $U = e^{iS}$ and $\varepsilon = 1$ then $\Sigma = 1$ $\Rightarrow 2E(t) = i\,\Omega_t = -2\partial_t S$

$$\boxed{E = -\partial_t S}$$

The Bohm energy

Momentum.

$$2\rho P_j(t) = -i\,[(\partial_j \Psi_L)\Psi_R - \Psi_L(\partial_j \Psi_R)] = -i\rho\Omega_j \cdot \Sigma$$

$$\boxed{P_j = \partial_j S}$$

The Bohm momentum.

Also known as the ‘guidance’ condition but nothing is being ‘guided’ here.

The Quantum Hamilton-Jacobi Equation.

$$i\,[(\partial_t \Psi_L)\Psi_R - \Psi_L(\partial_t \Psi_R)] = (H\Psi_L)\Psi_R + \Psi_L(H\Psi_R) = [H,\rho]_+$$

The LHS is $2\rho E(t) = -2\rho\partial_t S$

$$\boxed{\rho\partial_t S + \tfrac{1}{2}[H,\hat{\rho}]_+ = 0}$$

Quantum Hamilton-Jacobi

Since we have written $\psi_L = \rho^{\frac{1}{2}}U = Re^{iS}$ with $\varepsilon = 1$,

using $\hat{H} = \dfrac{\hat{p}^2}{2m} + V(x)$

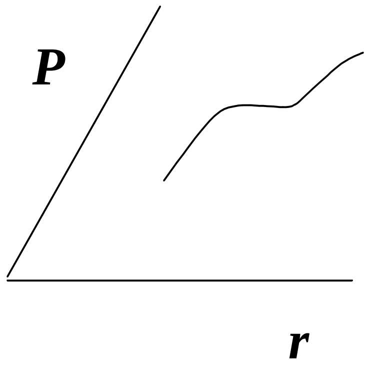
$$\boxed{\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 - \frac{1}{2mR}\left(\nabla^2 R\right) + V(x) = 0}$$

Conservation of energy.

Quantum Potential

The Shadow Phase Space

Since $\hat{\rho}(\boldsymbol{r},t) = \psi_L(\boldsymbol{r},t)\varepsilon\psi_R(\boldsymbol{r},t)$



We construct ***P*** and E

$$P_j(\boldsymbol{r},t) = \partial_j S(\boldsymbol{r},t)$$

$$E = -\partial_t S$$

The dynamics is defined by

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 - Q + V(x) = 0$$

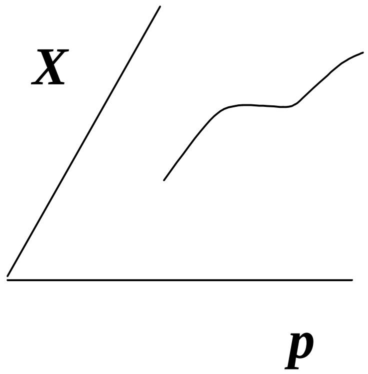
Probability conserved via Liouville equation

$$\frac{\partial P}{\partial t} + \nabla.\left(P\frac{\nabla S}{m}\right) = 0$$

Another Shadow Phase Space.

Consider a Clifford algebra of MOMENTUM space.

$$\hat{\rho}(\boldsymbol{p},t)=\phi_L(\boldsymbol{p},t)\varepsilon\phi_R(\boldsymbol{p},t)$$



Now construct X and E

$$X=-\nabla_p S_p$$

$$E=-\partial_t S_p$$

The dynamics is defined by

$$\frac{\partial S_p}{\partial t}+\frac{p^2}{2m}+Q(p)+V\left(\nabla_p S_p\right)=0$$

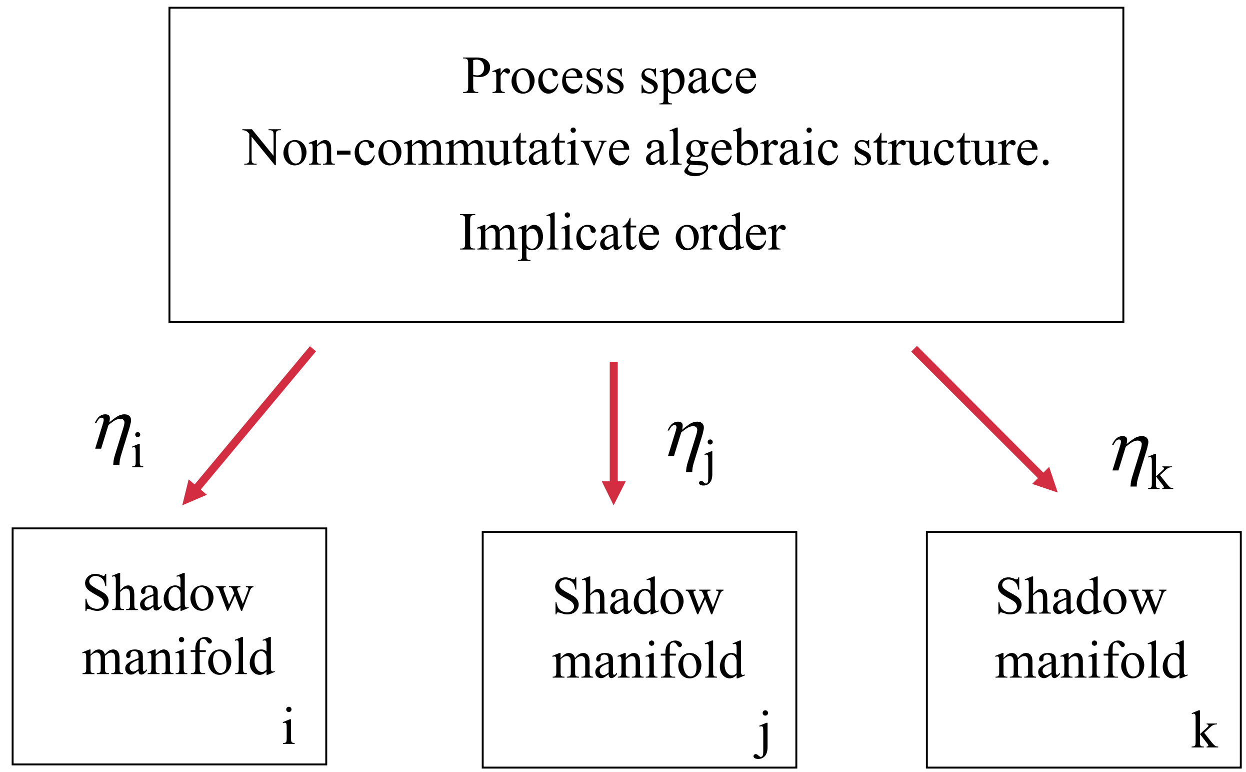
Probability conserved via Liouville equation.

$$\partial_t P_p+\nabla_p j_p=0$$

where

$$j_p=-\langle p|\frac{\partial(\hat{\rho}V(\hat{x}))}{\partial x}|p\rangle$$

General Algebraic Structure.



Shadow manifolds are the explicate orders

Conclusions.

1. Process $\Rightarrow$ **orthogonal** and **symplectic** groupoids $\Rightarrow$ Generalized Clifford
2. Larger Clifford algebras contain the Pauli, Dirac and Conformal structures.
3. Non-commutative structure containing shadow manifolds.
4. Quantum and relativistic processes 'live' in the covering space.
5. The Bohm construction is the first example of these shadow manifolds in QM.
6. The Bohm construction works for Pauli AND DIRAC.
   $\therefore$ consistent with relativity.

Prospects

1. Extension to many particle systems. Quantum non-locality.
2. Curved shadow manifolds.
3. Different algebras.
4. A more careful mathematical analysis using category theory.