Aspects of duality in 2-categories

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Categories, Logic and Foundations of Physics
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Introduction

Basic Principle (Baez, Dolan, Coecke, Abramsky, ...)

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Stupid **is** *as stupid* **does**. — Forrest Gump's mum.

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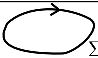
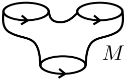
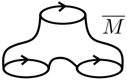
	nCob	Hilb
objects	 Σ $(n-1)$-dim space	H fin dim Hilbert space
morphisms	 M n-dim spacetime	$H_1 \xrightarrow{A} H_2$ linear map
monoidal	 $\sqcup$ 	$H \otimes H$
duals for objects	 $\overline{\Sigma}$	$\overline{H}$
duals for morphisms	 $\overline{M}$	$H_2 \xrightarrow{A^*} H_1$ adjoint

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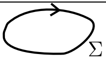
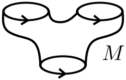
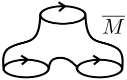
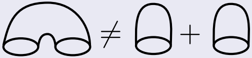
objects	 $(n-1)$-dim space	H fin dim Hilbert space
morphisms	 n-dim spacetime	$H_1 \xrightarrow{A} H_2$ linear map
duals for morphisms		$H_2 \xrightarrow{A^*} H_1$ adjoint

For instance, quantum entanglement:



$$\mathbb{C} \xrightarrow{\psi} H \otimes H \quad \neq \quad \mathbb{C} \xrightarrow{\psi_1} H \; + \; \mathbb{C} \xrightarrow{\psi_2} H$$

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Baez and Dolan proposed that a unitary extended n-dimensional TQFT is a *unitary representation* of the cobordism n-category on the n-category of n-Hilbert spaces:

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So understanding duality in higher categories will aid our understanding of spacetime and quantum theory!

Skip extra stuff

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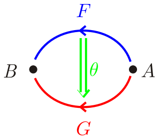
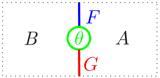
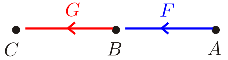
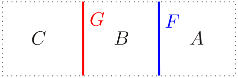
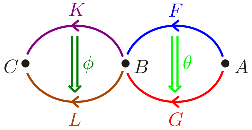
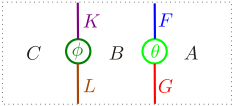
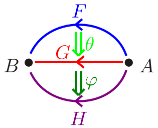
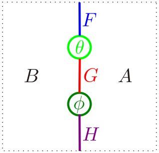
Indeed, at least when G is finite:

Theorem (BB, SW).

$$\underbrace{\mathrm{Bun}_G(G)}_{Z(S^1)} \overset{\text{braided mon}}{\simeq} \underbrace{\mathrm{Dim}\, 2\mathcal{R}\mathrm{ep}(G)}_{\substack{\text{category of weak transformations} \\ \text{and modifications of identity 2-functor}}}.$$

2. String diagram notation for 2-categories

	Globular	String
Objects, morphisms and 2-morphisms		
Composition of 1-morphisms		
Horizontal composition of 2-morphisms		
Vertical composition of 2-morphisms		

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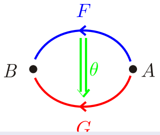
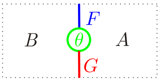
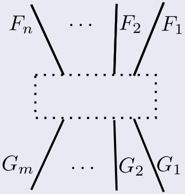




Note — string diagrams work perfectly well for *weak* 2-categories, as long as the the parenthesis scheme of the input and output 1-morphisms are understood in each context, eg.:

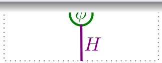


unique interpretation $\longmapsto$ by coherence

$$\left[F_n \circ (F_{n-1} \circ F_{n-2})\right] \circ \cdots \circ [F_2 \circ F_1]$$

$$\downarrow$$

$$G_m \circ \cdots \circ \left[G_3 \circ (G_2 \circ G_1)\right]$$



Ambiijunction groupoid I

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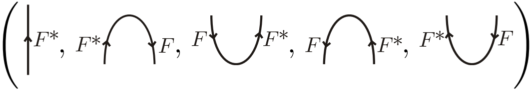
An *ambidextrous adjoint* of a morphism $F\colon A \to B$ in a 2-category is a morphism $F^*\colon B \to A$ equipped with unit and counit 2-morphisms expressing F^* as a right adjoint of F, and unit and counit 2-morphisms expressing F^* as a left adjoint of F:

$\langle F^* \rangle \equiv$



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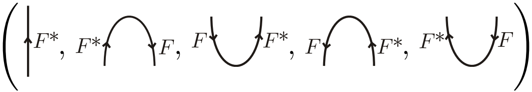
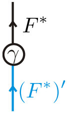
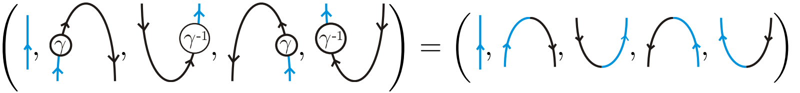
$\langle F^* \rangle \equiv$ 

Taken together these form the *ambijunction groupoid* $\mathrm{Amb}(F)$ of F, where a morphism $\gamma \colon \langle F^* \rangle \to \langle (F^*)' \rangle$ is defined to be an invertible 2-morphism



such that



Ambijunction groupoid II

We write $[\mathrm{Amb}(F)]$ for the isomorphism classes in $\mathrm{Amb}(F)$.

Properties of the ambijunction groupoid

If $\mathrm{Amb}(F)$ is not empty, then

- There is at most one arrow between any two ambidextrous adjunctions in $\mathrm{Amb}(F)$.
- The group $\mathrm{Aut}(F)$ of automorphisms of F acts freely and transitively on $[\mathrm{Amb}(F)]$.

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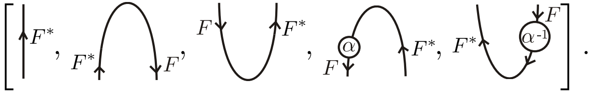
An automorphism $\alpha\colon F \to F$ acts on an ambidextrous adjoint by twisting the right unit and counit maps,

$$\alpha \cdot [F^*] =$$



Even-handed structures

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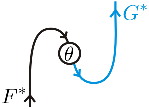
Suppose $\theta \colon F \Rightarrow G$ is a 2-morphism, and that choices of ambidextrous adjoints $\langle F^* \rangle$ and $\langle G^* \rangle$ have been made. The *right and left daggers* of θ are:

$$\theta^\dagger =$$



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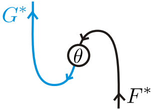

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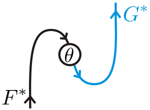
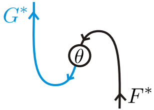
$\theta^\dagger =$ 

${}^\dagger\theta =$ 

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We say that a 2-category $\mathcal{C}$ *has ambidextrous adjoints* if every morphism has an (unspecified) ambidextrous adjoint.

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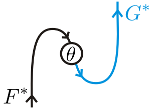
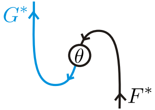
$\theta^\dagger =$ 

$^\dagger\theta =$ 

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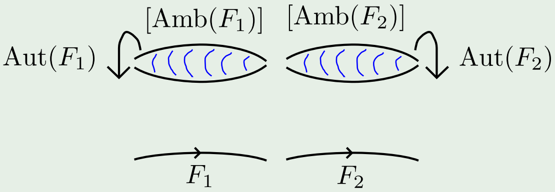
Definition

An *even-handed structure* on a 2-category with ambidextrous adjoints is a choice $F^{[*]} \in [\mathrm{Amb}(F)]$ for every morphism F, such that:

1. $\mathrm{id}^{[*]} =$ class of trivial ambidextrous adjunction for all identity 1-cells,
2. $(G \circ F)^{[*]} = F^{[*]} \circ G^{[*]}$ for all composable pairs of morphisms, and
3. $\theta^\dagger =^\dagger \theta$ for every 2-morphism $\theta \colon F \Rightarrow G$, provided they are computed using ambidextrous adjoints from the classes $F^{[*]}$ and $G^{[*]}$.

So... an even-handed structure on a 2-category is an 'even-handed trivialization of the *ambijunction gerbe*':



Analagous to Murray and Singer's refomulation of a spin structure on a manifold as a trivialization of the *spin gerbe*.

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Lemma

For each choice of system of right duals $\star$ on C, there is a canonical bijection

$$\left\{ \begin{array}{c} \textit{Even-handed structures on } C, \\ \textit{considered as a one object 2-category} \end{array} \right\} \cong \left\{ \begin{array}{c} \textit{Pivotal structures on} \\ C \textit{ with respect to } \star \end{array} \right\}.$$

Moreover, this bijection is natural with respect to changing the system of right duals $\star$.

Examples: Monoidal categories, remarks

Consider a monoidal category C (a one-object 2-category) with ambidextrous duals, eg. $\mathrm{Rep}(G)$. Some preliminary remarks:

- The set of even-handed structures on C is a torsor for the group $\mathrm{Aut}_{\otimes}(\mathrm{id}_C)$, which acts by twisting the right unit and counit maps as before.
- The conventional paradigm for duals in monoidal categories is to choose a specific right dual $V^\star$ for each object, and define a *pivotal structure* on C as a monoidal natural isomorphism $\gamma\colon \mathrm{id} \Rightarrow \star\star$.

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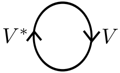
- In good cases, an even-handed structure $[*]$ gives rise to a ring homomorphism

$$\dim_{[*]}\colon K(C) \to \mathrm{End}(1)$$

$$[V] \mapsto$$



.

If C is a semisimple linear category, $[*]$ is characterized by $\dim_{[*]}$.

Examples: Monoidal categories (one-object 2-categories)

1. For the monoidal category (G, ω) coming from a 3-cocycle $\omega \in Z^3(G, U(1))$,

$$\left\{ \begin{array}{c} \text{Even-handed structures} \\ \text{on } (G, \omega) \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Group homomorphisms} \\ f \colon G \to U(1) \end{array} \right\}.$$

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2. Suppose C is a monoidal category where every object has a right dual. If C can be equipped with a braiding σ, then

$$\left\{ \begin{array}{c} \text{Even-handed structures} \\ \text{on } C \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Pretwists on } C \\ \text{with respect to } \sigma \end{array} \right\}.$$

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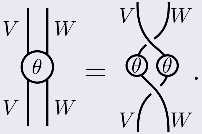
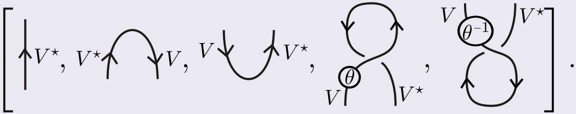
   ω … ns }.
2. Su … right
   du … }.

$\theta \in \mathrm{Aut}(\mathrm{id})$ is a *pretwist* with respect to σ if



Gives even-handed structure

$V^{[*]} =$



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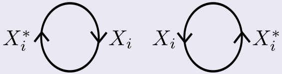
3. For *fusion categories* (semisimple rigid monoidal k-linear categories), some mysteries remain:
   - What are the equations which have the symmetry group

   $$\mathrm{Aut}_{\otimes}(\mathrm{id}) = \{\theta_i \in k^{\times} : \theta_i = \theta_j\theta_k \text{ whenever } X_i \text{ appears in } X_j \otimes X_k\}?$$

   - Will they ensure that an even-handed structure always exists?

Examples: Monoidal categories (one-object 2-categories)

The manifest invariants of a fusion category are the fusion ring $K(C)$ and Müger's *squared norms* d_i of the simple objects:

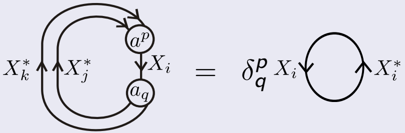
$d_i =$ 

...this is independent of choice of $\langle X_i^* \rangle$.

If an even-handed structure exists, then $\dim_{[*]} \colon K(C) \to k$ satisfies:

- It is a ring homomorphism taking nonzero values on simple objects,
- $\dim[X_i] \dim[X_i^*] = d_i$ for all simple objects.

Do *these* equations have $\mathrm{Aut}_{\otimes}(\mathrm{id})$ as their symmetry group? To make an even-handed structure, crucially need:



Even-handed structures on sub-2-categories of $\mathcal{C}$at

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If $F\colon A \to B$ is a functor between categories, express right and left adjunctions via

$$\phi\colon \mathrm{Hom}(Fx, y) \xrightarrow{\cong} \mathrm{Hom}(x, F^*y)$$

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For $\theta\colon F \Rightarrow G$, can express $\theta^\dagger, {}^\dagger\theta\colon G^* \Rightarrow F^*$ as follows:

$$\text{post}(\theta^\dagger_y) = \phi_F \circ \text{pre}(\theta_x) \circ \phi^{-1}_G$$

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In this way, an even-handed structure on a sub-2-category $\mathcal{C} \subseteq \mathcal{C}$at with ambidextrous adjoints translates into a system of bijective maps

$$\Psi_{F,F^*}\colon \ \mathrm{Adj}(F \dashv F^*) \to \mathrm{Adj}(F^* \dashv F)$$

which transform correctly under isomorphism 2-cells, respect composition, and satisfy the *even-handed equation*:

$$\boxed{\mathrm{post}^{-1}\big(\phi_F \circ \mathrm{pre}(\theta) \circ \phi_G^{-1}\big) = \mathrm{pre}^{-1}\big(\Psi(\phi_G)^{-1} \circ \mathrm{post}(\theta) \circ \Psi(\phi_F)\big)}$$

Even-handed structures from traces on linear categories

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Definition

A *trace* on a k-linear category is a linear map $\mathrm{Tr}_x \colon \mathrm{End}(x) \to k$ for each object satisfying:

- $\mathrm{Tr}_x(gf) = \mathrm{Tr}_y(fg)$.
- Nondegeneracy: $\quad s \colon \mathrm{Hom}(x, y) \xrightarrow{\cong} \mathrm{Hom}(y, x)^\vee$.

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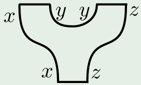
Remark

A trace on C is the same thing as a symmetric monoidal functor

$$\mathrm{PlanarCob}_{\mathrm{Ob}(C)} \longrightarrow \mathrm{Vect}_k$$

$$x \text{ — } y \quad \longmapsto \quad \mathrm{Hom}(x, y)$$



$$\longmapsto \quad \mathrm{Hom}(x,y) \otimes \mathrm{Hom}(y,z) \xrightarrow{\circ} \mathrm{Hom}(x,z)$$

$$\longmapsto \quad \mathrm{Hom}(x,x) \xrightarrow{\mathrm{Tr}_x} k$$

$$\longmapsto \quad 1 \mapsto \sum_i f_i \otimes f^i \quad \text{etc.}$$

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Turn right adjoints into left adjoints:

$$\begin{array}{ccc} \mathrm{Hom}(F^*y, x) & \xrightarrow{\phi^T} & \mathrm{Hom}(y, Fx) \\ \downarrow s_A & & \downarrow s_B \\ \mathrm{Hom}(x, F^*y)^\vee & \xrightarrow[\phi^\vee]{} & \mathrm{Hom}(Fx, y)^\vee \end{array} \qquad \phi^T := s_B^{-1} \circ \phi^\vee \circ s_A$$

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Theorem

If each category in $\mathcal{C} \subseteq \mathcal{L}\mathrm{inearCat}_k$ comes equipped with a trace, then sending $\phi \overset{\Psi}{\mapsto} \phi^T$ gives an even handed structure on $\mathcal{C}$.

Example: The 2-category of 2-Hilbert spaces

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Definition (Baez)

A 2-Hilbert space is an abelian Hilb-category H equipped with antilinear maps $* : \mathrm{Hom}(x, y) \to \mathrm{Hom}(y, x)$ compatible with composition and inner products. They form a 2-category $2\mathcal{H}$ilb.

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Indeed, can prove that in the semisimple context,

$$\left\{ \begin{array}{c} \text{Even-handed structures on} \\ \text{a full sub-2-category} \\ \mathcal{S} \subset \mathcal{SL}\mathrm{Cat}_k \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Weightings on the} \\ \text{simple objects in } \mathcal{S} \\ \text{up to a global scale factor} \end{array} \right\}.$$

This is similar to

$$\dim\{\text{harmonic spinors on a spin manifold}\} \text{ is a conformal invariant.}$$

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X an n-dimensional Calabi-Yau manifold. Have the *graded derived category* $\mathbf{D}(X)$:

- An object is a bounded complex $\mathscr{E}$ of coherent sheaves on X.
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Corollary

The 2-category $\mathcal{CY}$au comes equipped with a canonical even-handed structure arising from Serre duality on each $\mathbf{D}(X)$.

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- The concept of an even-handed structure, and the 'moduli space' of even-handed structures on a given 2-category, have geometric overtones.
- Personal motivation: needed an even-handed structure on $2\mathcal{H}$ilb to make the 2-character into a *functor*

$$\chi\colon \underbrace{\mathsf{2Rep}(G)}_{\text{homotopy category}} \to \mathsf{Bun}_G(G)$$

Reference

- J. Baez and J. Dolan, Higher dimensional algebra and topological quantum field theory, `arXiv:q-alg/9503002`.
- J. Baez, Higher dimensional algebra II: 2-Hilbert spaces, `arXiv:q-alg/9609018`.
- M. Müger, From Subfactors to Categories and Topology I. Frobenius algebras in and Morita equivalence of tensor categories, `arXiv:math/0111204`.
- P. Etingof, D. Nikshych and V. Ostrik, On fusion categories, `arXiv:math/0203060v9`.
- A. Caldararu and S. Willerton, The Mukai Pairing, I : A categorical approach, `arXiv/0707.2052`.
- M. Boyarchenko, notes for the series of lectures 'Introduction to modular categories', `www.math.uchicago.edu/m̃itya/langlands.html`.
- N. Ganter, M. Kapranov, Representation and character theory in 2-categories, `arXiv/math.KT/0602510`.
- J. Roberts and S. Willerton, On the Rozansky-Witten weight systems, `arXiv.org/math/0602653`