

Aspects of duality in 2-categories

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Introduction

Basic Principle (Baez, Dolan, Coecke, Abramsky, ...)

Spacetime and quantum theory make more sense when expressed in their natural language — not the language of sets and functions, but the language of (higher) categories with duals.

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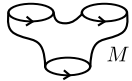


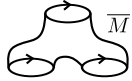
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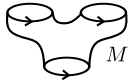
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Stupid is as stupid does. — Forrest Gump's mum.

Introduction

	nCob	Hilb
objects	 $(n-1)$ -dim space	H fin dim Hilbert space
morphisms	 n -dim spacetime	H_1 $\downarrow A$ H_2 linear map
monoidal		$H \otimes H$
duals for objects		\bar{H}
duals for morphisms		H_2 $\downarrow A^*$ H_1 adjoint

Introduction

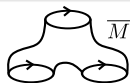
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morphisms	 n -dim spacetime	$\begin{array}{c} H_1 \\ \downarrow A \\ H_2 \end{array}$ linear map

For instance, quantum entanglement:

$$\begin{array}{ccc}
 \text{Diagram 1} & \neq & \text{Diagram 2} + \text{Diagram 3} \\
 \downarrow \psi & \neq & \downarrow \psi_1 + \downarrow \psi_2 \\
 H \otimes H & & H + H
 \end{array}$$

The diagrams show a pair of pants with two bottom boundaries (Diagram 1) versus two separate circles (Diagram 2) and two separate circles (Diagram 3). The arrows in the diagrams indicate the direction of flow.

duals for morphisms



$$\begin{array}{c}
 \dots \\
 \downarrow A^* \\
 H_1
 \end{array}$$

adjoint

Introduction: Reminder on Cobordism Hypothesis

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Baez and Dolan proposed that a unitary extended n -dimensional TQFT is a *unitary representation* of the cobordism n -category on the n -category of n -Hilbert spaces:

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$n\mathcal{Cob}$ is the free weak n -category with duals on one object.

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So understanding duality in higher categories will aid our understanding of spacetime and quantum theory!

» Skip extra stuff

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$$Z(\text{pt}) = 2\text{Rep}(G).$$

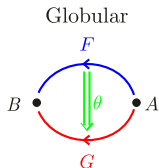
Indeed, at least when G is finite:

Theorem (BB, SW).

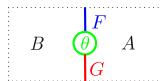
$$\underbrace{\text{Bun}_G(G)}_{Z(S^1)} \overset{\text{braided mon}}{\cong} \underbrace{\text{Dim } 2\text{Rep}(G)}_{\text{category of weak transformations and modifications of identity 2-functor}}.$$

2. String diagram notation for 2-categories

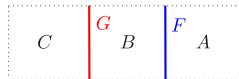
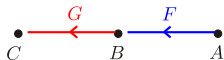
Objects, morphisms
and 2-morphisms



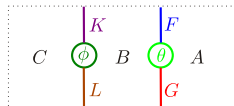
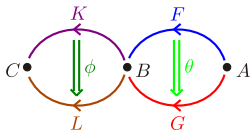
String



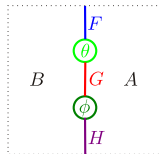
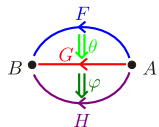
Composition
of 1-morphisms



Horizontal composition
of 2-morphisms

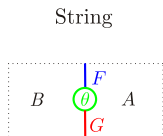
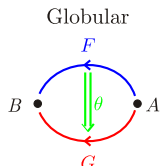


Vertical composition
of 2-morphisms

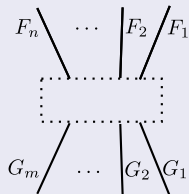


2. String diagram notation for 2-categories

Objects, morphisms
and 2-morphisms



Note — string diagrams work perfectly well for *weak* 2-categories, as long as the the parenthesis scheme of the input and output 1-morphisms are understood in each context, eg.:



unique interpretation
 \mapsto
by coherence

$$[F_n \circ (F_{n-1} \circ F_{n-2})] \circ \cdots \circ [F_2 \circ F_1]$$



$$G_m \circ \cdots \circ [G_3 \circ (G_2 \circ G_1)]$$



Ambijunction groupoid I

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An *ambidextrous adjoint* of a morphism $F: A \rightarrow B$ in a 2-category is a morphism $F^*: B \rightarrow A$ equipped with unit and counit 2-morphisms expressing F^* as a right adjoint of F , and unit and counit 2-morphisms expressing F^* as a left adjoint of F :

$$\langle F^* \rangle \equiv \left(\begin{array}{c} \downarrow \\ F^* \\ \uparrow \end{array}, \begin{array}{c} \uparrow \\ F^* \\ \downarrow \\ F \end{array}, \begin{array}{c} \downarrow \\ F \\ \uparrow \\ F^* \end{array}, \begin{array}{c} \downarrow \\ F \\ \uparrow \\ F^* \end{array}, \begin{array}{c} \downarrow \\ F^* \\ \uparrow \\ F \end{array} \right)$$

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$$\langle F^* \rangle \equiv \left(\left| \begin{array}{c} \uparrow F^* \\ \downarrow F \end{array} \right., F^* \curvearrowright F, F \curvearrowleft F^*, F \curvearrowright F^*, F^* \curvearrowleft F \right)$$

Taken together these form the *ambijunction groupoid* $\text{Amb}(F)$ of F , where a morphism $\gamma: \langle F^* \rangle \rightarrow \langle (F^*)' \rangle$ is defined to be an invertible 2-morphism

$$\begin{array}{c} \uparrow F^* \\ \circlearrowleft \gamma \\ \downarrow (F^*)' \end{array}$$

such that

$$\left(\left| \begin{array}{c} \downarrow \\ \uparrow \end{array} \right., \circlearrowleft \gamma, \circlearrowright \gamma^{-1}, \circlearrowleft \gamma, \circlearrowright \gamma^{-1} \right) = \left(\left| \begin{array}{c} \downarrow \\ \uparrow \end{array} \right., \curvearrowright, \curvearrowleft, \curvearrowright, \curvearrowleft \right).$$

Ambijunction groupoid II

We write $[\text{Amb}(F)]$ for the isomorphism classes in $\text{Amb}(F)$.

Properties of the ambijunction groupoid

If $\text{Amb}(F)$ is not empty, then

- There is at most one arrow between any two ambidextrous adjunctions in $\text{Amb}(F)$.
- The group $\text{Aut}(F)$ of automorphisms of F acts freely and transitively on $[\text{Amb}(F)]$.

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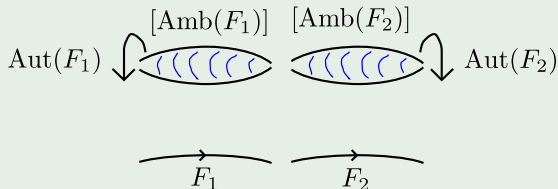
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An automorphism $\alpha: F \rightarrow F$ acts on an ambidextrous adjoint by twisting the right unit and counit maps,

$$\alpha \cdot [F^*] = \left[\begin{array}{c} \uparrow \\ F^* \\ \downarrow \end{array}, \begin{array}{c} \text{arc} \\ \uparrow \downarrow \\ F^* \quad F \end{array}, \begin{array}{c} \text{arc} \\ \downarrow \uparrow \\ F \quad F^* \end{array}, \begin{array}{c} \text{arc} \\ \uparrow \downarrow \\ F \quad F^* \\ \circlearrowleft \alpha \end{array}, \begin{array}{c} \text{arc} \\ \downarrow \uparrow \\ F^* \quad F \\ \circlearrowright \alpha^{-1} \end{array} \right].$$

Even-handed structures

So... an even-handed structure on a 2-category is an 'even-handed trivialization of the *ambijunction gerbe*':



Analagous to Murray and Singer's reformulation of a spin structure on a manifold as a trivialization of the *spin gerbe*.

Definition

An *even-handed structure* on a 2-category with ambidextrous adjoints is a choice $F^{[*]} \in [\text{Amb}(F)]$ for every morphism F , such that:

- 1 $\text{id}^{[*]} =$ class of trivial ambidextrous adjunction for all identity 1-cells,
- 2 $(G \circ F)^{[*]} = F^{[*]} \circ G^{[*]}$ for all composable pairs of morphisms, and
- 3 $\theta^\dagger = \dagger \theta$ for every 2-morphism $\theta: F \Rightarrow G$, provided they are computed using ambidextrous adjoints from the classes $F^{[*]}$ and $G^{[*]}$.

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Lemma

For each choice of system of right duals \star on C , there is a canonical bijection

$$\left\{ \begin{array}{l} \text{Even-handed structures on } C, \\ \text{considered as a one object 2-category} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Pivotal structures on} \\ C \text{ with respect to } \star \end{array} \right\}.$$

Moreover, this bijection is natural with respect to changing the system of right duals \star .

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- In good cases, an even-handed structure $[*]$ gives rise to a ring homomorphism

$$\dim_{[*]}: K(C) \rightarrow \text{End}(1)$$

$$[V] \mapsto V^* \circlearrowleft V.$$

If C is a semisimple linear category, $[*]$ is characterized by $\dim_{[*]}$.

Examples: Monoidal categories (one-object 2-categories)

- ① For the monoidal category (G, ω) coming from a 3-cocycle $\omega \in Z^3(G, U(1))$,

$$\left\{ \begin{array}{c} \text{Even-handed structures} \\ \text{on } (G, \omega) \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Group homomorphisms} \\ f: G \rightarrow U(1) \end{array} \right\}.$$

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- ② Suppose C is a monoidal category where every object has a right dual. If C can be equipped with a braiding σ , then

$$\left\{ \begin{array}{c} \text{Even-handed structures} \\ \text{on } C \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Pretwists on } C \\ \text{with respect to } \sigma \end{array} \right\}.$$

Examples: Monoidal categories (one-object 2-categories)

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$\omega \in \text{Aut}(\text{id})$ is a *pretwist* with respect to σ if

$$\begin{array}{c} V \\ | \\ | \\ | \\ W \\ | \\ | \\ | \\ W \end{array} \circlearrowleft \theta = \begin{array}{c} V \\ \diagdown \\ \diagup \\ W \end{array} \circlearrowleft \theta \quad \begin{array}{c} W \\ \diagup \\ \diagdown \\ V \end{array} \circlearrowleft \theta$$

- 2 Su
du

ns }.

right

Gives even-handed structure

$$V[*] = \left[\begin{array}{c} \uparrow V^*, V^* \uparrow \downarrow V, V \downarrow \uparrow V^* \\ \uparrow V^* \downarrow V \uparrow \downarrow V, V \downarrow \uparrow V^* \\ \uparrow V^* \downarrow V \uparrow \downarrow V, V \downarrow \uparrow V^* \\ \uparrow V^* \downarrow V \uparrow \downarrow V, V \downarrow \uparrow V^* \end{array} \right] = \left[\begin{array}{c} \uparrow V^* \downarrow V \uparrow \downarrow V, V \downarrow \uparrow V^* \\ \uparrow V^* \downarrow V \uparrow \downarrow V, V \downarrow \uparrow V^* \end{array} \right]$$

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- 3 For *fusion categories* (semisimple rigid monoidal k -linear categories), some mysteries remain:

- What are the equations which have the symmetry group

$$\text{Aut}_{\otimes}(\text{id}) = \{\theta_i \in k^\times : \theta_i = \theta_j \theta_k \text{ whenever } X_i \text{ appears in } X_j \otimes X_k\}?$$

- Will they ensure that an even-handed structure always exists?

Examples: Monoidal categories (one-object 2-categories)

The manifest invariants of a fusion category are the fusion ring $K(C)$ and Müger's *squared norms* d_i of the simple objects:

$$d_i = X_i^* \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} X_i \quad X_i \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} X_i^* \quad \dots \text{this is independent of choice of } \langle X_i^* \rangle.$$

If an even-handed structure exists, then $\dim_{[*]}: K(C) \rightarrow k$ satisfies:

- It is a ring homomorphism taking nonzero values on simple objects,
- $\dim[X_i] \dim[X_i^*] = d_i$ for all simple objects.

Do *these* equations have $\text{Aut}_{\otimes}(\text{id})$ as their symmetry group? To make an even-handed structure, crucially need:

$$X_k^* \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} X_j^* \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} X_i = \delta_q^p X_i \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} X_i^*$$

Even-handed structures on sub-2-categories of $\mathcal{C}at$

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$$\phi: \text{Hom}(Fx, y) \xrightarrow{\cong} \text{Hom}(x, F^*y)$$

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For $\theta: F \Rightarrow G$, can express $\theta^\dagger, \dagger\theta: G^* \Rightarrow F^*$ as follows:

$$\text{post}(\theta_y^\dagger) = \phi_F \circ \text{pre}(\theta_x) \circ \phi_G^{-1}$$

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In this way, an even-handed structure on a sub-2-category $\mathcal{C} \subseteq \mathcal{C}at$ with ambidextrous adjoints translates into a system of bijective maps

$$\Psi_{F, F^*}: \text{Adj}(F \dashv F^*) \rightarrow \text{Adj}(F^* \dashv F)$$

which transform correctly under isomorphism 2-cells, respect composition, and satisfy the *even-handed equation*:

$$\text{post}^{-1}(\phi_F \circ \text{pre}(\theta) \circ \phi_G^{-1}) = \text{pre}^{-1}(\Psi(\phi_G)^{-1} \circ \text{post}(\theta) \circ \Psi(\phi_F))$$

Even-handed structures from traces on linear categories

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Definition

A *trace* on a k -linear category is a linear map $\text{Tr}_x: \text{End}(x) \rightarrow k$ for each object satisfying:

- $\text{Tr}_x(gf) = \text{Tr}_y(fg)$.
- Nondegeneracy: $s: \text{Hom}(x, y) \xrightarrow{\cong} \text{Hom}(y, x)^\vee$.

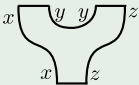

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Remark

A trace on C is the same thing as a symmetric monoidal functor

$\text{PlanarCob}_{\text{Ob}(C)} \rightarrow$	Vect_k
$x \text{---} y \quad \mapsto$	$\text{Hom}(x, y)$
	$\text{Hom}(x, y) \otimes \text{Hom}(y, z)$ $\downarrow \circ$ $\text{Hom}(x, z)$
$x \text{---} x \quad \mapsto$	$\text{Hom}(x, x)$ $\downarrow \text{Tr}_x$ k
	$\frac{1}{\sum_i f_i \otimes f^i}$ etc.

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Turn right adjoints into left adjoints:

$$\begin{array}{ccc} \text{Hom}(F^*y, x) & \xrightarrow{\phi^T} & \text{Hom}(y, Fx) \\ s_A \downarrow & & \downarrow s_B \\ \text{Hom}(x, F^*y)^\vee & \xrightarrow{\phi^\vee} & \text{Hom}(Fx, y)^\vee \end{array} \quad \phi^T := s_B^{-1} \circ \phi^\vee \circ s_A$$

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Theorem

If each category in $\mathcal{C} \subseteq \text{LinearCat}_k$ comes equipped with a trace, then sending $\phi \mapsto \phi^T$ gives an even handed structure on \mathcal{C} .

Example: The 2-category of 2-Hilbert spaces

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A 2-Hilbert space is an abelian Hilb-category H equipped with antilinear maps $*$: $\text{Hom}(x, y) \rightarrow \text{Hom}(y, x)$ compatible with composition and inner products. They form a 2-category $2\mathcal{H}ilb$.

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The resulting even-handed structure on $2\mathcal{Hilb}$ can be expressed as

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where ϕ^* is the adjoint linear map of ϕ .

Indeed, can prove that in the semisimple context,

$$\left\{ \begin{array}{l} \text{Even-handed structures on} \\ \text{a full sub-2-category} \\ \mathcal{S} \subset \mathcal{SLCat}_k \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Weightings on the} \\ \text{simple objects in } \mathcal{S} \\ \text{up to a global scale factor} \end{array} \right\}.$$

This is similar to

$\dim\{\text{harmonic spinors on a spin manifold}\}$ is a conformal invariant.

- Every 2-Hilbert space comes equipped with a natural even-handed structure $f \mapsto (\text{id}_x, f)$.

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X an n -dimensional Calabi-Yau manifold. Have the *graded derived category* $\mathbf{D}(X)$:

- An object is a bounded complex \mathcal{E} of coherent sheaves on X .
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Corollary

The 2-category \mathcal{CYau} comes equipped with a canonical even-handed structure arising from Serre duality on each $\mathbf{D}(X)$.

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- The concept of an even-handed structure, and the ‘moduli space’ of even-handed structures on a given 2-category, have geometric overtones.
- Personal motivation: needed an even-handed structure on $2\mathcal{Hilb}$ to make the 2-character into a *functor*

$$\chi: \underbrace{2\text{Rep}(G)}_{\text{homotopy category}} \rightarrow \text{Bun}_G(G)$$

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