Dynamical Logic

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Plan of Talk

- History not State
- Classical and Quantum stochastic theories (Quantum Measure Theory)
- Typicality and Preclusion
- The Three Sit Experiment
- What is real in a Quantum Measure Theory?
- Non-classical, dynamical logic (Rules of Inference)
- Summary

References:

- Rafael D. Sorkin (1994) “Quantum Mechanics as Quantum Measure Theory” [gr-qc/9401003]
- Rafael D. Sorkin (2006) “Quantum Dynamics without the Wave Function” [quant-ph/0610204]
- Rafael D. Sorkin (2007) “An Exercise in Anhomomorphic Logic” [quant-ph/0703276]

History not State

The general covariance of General Relativity implies that there is no physical meaning to the notion of a state at a moment of time. Reality in GR has, instead, a fully four-dimensional character. We do not yet know how quantum theory and gravity will be reconciled but if we take the spacetime nature of reality to heart then “state” should play no fundamental role in quantum mechanics.

Relatively little effort has been devoted to pursuing the idea that quantum mechanics should be founded on the concept of “history” and not that of “state.” A major step was taken by Dirac and Feynman in showing that the dynamics of quantum mechanics can be expressed in terms of a “sum over histories”. Few physicists since then have taken up the torch for histories as a basis for the interpretation of quantum mechanics. Notable exceptions include Griffiths, Omnés, Gell-Mann, and Hartle who developed the approach known as the decoherent/consistent histories approach. Even among these 4 workers, Hartle was the only one to set out new axioms for a quantum theory which did not require the existence of a Hilbert space of states.

Classical and Quantum Measure Theory

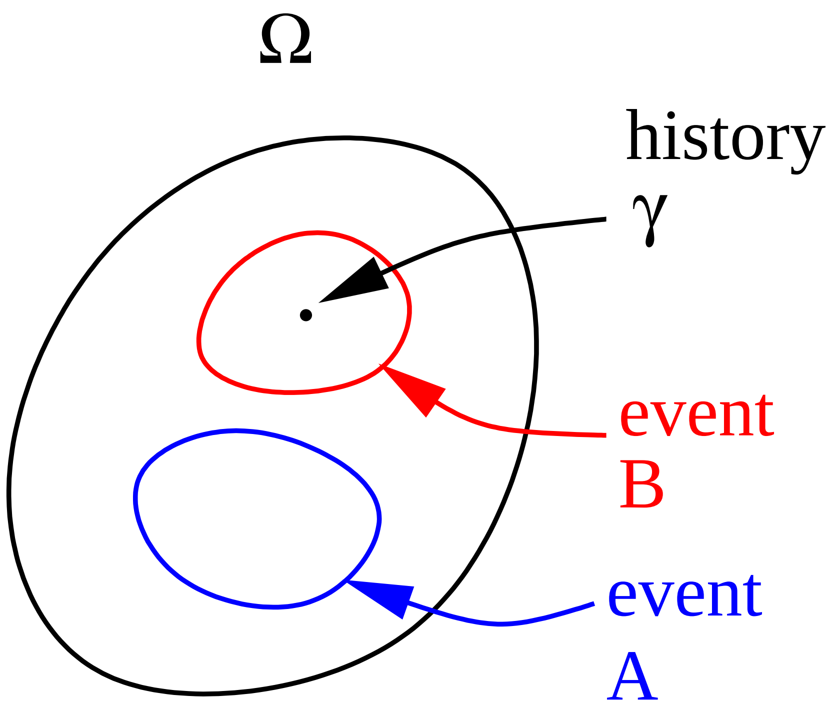
Motivated by trying to base quantum theory on histories, and avoiding the problems that arise in the decoherent histories approach, Sorkin suggested the beginnings of an axiomatic basis for Quantum Mechanics which he called Quantum Measure Theory. [Indeed Sorkin proposed an even more general setting “Generalised Measure Theory” which includes classical, quantum and trans-quantum theories in a hierarchy characterised by the amount of interference the theory permits between histories.]

To understand Quantum Measure Theory let us review the salient features of a classical stochastic theory.

A Common Structure

Every classical stochastic theory has the following structure:

- A sample space, Ω, of possible spacetime histories (e.g. sequences of outcomes of 1000 coin tosses, Weiner paths for a Brownian particle).
- A event algebra, $\mathfrak{A}$, whose elements are the possible questions that can be asked about the system, which is a Boolean algebra of subsets of Ω.
- A measure, μ, on $\mathfrak{A}$ which encodes the dynamics and initial conditions.
- A single reality, which is one element γ of Ω, which contains the answer, yes or no, to every question that can be asked.



For example, Ω could be the set of all weather patterns and A could be the event "It is raining" and γ could be one weather pattern in which it is not raining. Each subset of Ω is an event/question. So $\mathfrak{A} = 2^{\Omega}$ (Ω finite).

If the event B is "It is dry and windy" and γ is the realised history then we answer the question "B?" with "Yes" and the question "A?" with "No."

Reality, this single history, is thus equivalent to a map, ϕ,

$$\phi : \mathfrak{A} \rightarrow Z_2 \tag{1}$$

where $\phi(A) = 0$ means the event A doesn't happen (the answer to question "A?" is "no") and $\phi(A) = 1$ means the event A does happen (the answer to question "A?" is "yes").

Z_2 is the set of "truth values", if you like. And the map ϕ that corresponds to a single realised history is a homomorphism:

$$\phi(A) = 1 \text{ iff } \gamma_{realised} \in A \tag{2}$$

This is ordinary, classical, Boolean, *homomorphic* logic.

The Role of the Measure

In a classical theory, the dynamics and initial state are encoded in a measure, μ which is a positive real function on the event algebra

$$\mu : \mathfrak{A} \to \mathbb{R}$$

$$\mu(A) \geq 0, \ \ \forall A \in \mathfrak{A}$$

$$\mu(A \sqcup B) = \mu(A) + \mu(B), \ \ \forall A, B \in \mathfrak{A}$$

$$\mu(\Omega) = 1$$

μ is interpreted as a probability measure: $\mu(A)$ is the probability of event A.

Quantum Theory from a Spacetime Perspective

Following Dirac and Feynman, we can express the formalism of quantum mechanics in fully spacetime terms: in terms of histories. In this form, quantum mechanics has much of the structure reviewed above:

- A sample space, Ω, of possible spacetime histories.
- A event algebra, $\mathfrak{A}$, whose elements are subsets of Ω.
- A measure, μ, on $\mathfrak{A}$ which encodes the dynamics and initial state.

The sample space is the set of histories that go into the Dirac-Feymnan Sum-Over-Histories. The measure $\mu(A)$ is less familiar: roughly speaking it is the mod squared of the sum of the quantum amplitudes of all the histories in A. (It is the diagonal term of the "decoherence functional.")

This way of thinking about a quantum theory is not the textbook one, but these structures exist (more or less) for every quantum theory. And indeed they furnish an alternative foundation from which the usual Hilbert space may be constructed (for ordinary quantum mechanics we have a theorem (FD, Steven Johnston, Rafael Sorkin)).

Can we also take over the axiom that One History Happens?

The Quantum Measure

The quantum measure μ is a positive real function on the event algebra (normalised)

$$\mu : \mathfrak{A} \to \mathbb{R}$$
$$\mu(A) \geq 0, \quad \forall A \in \mathfrak{A}$$
$$(\mu(\Omega) = 1)$$

but it does not, in general, satisfy the Kolmogorov sum rule, because of quantum interference:

$$\mu(A) = |\sum_{\gamma \in A} a(\gamma)|^2$$
$$\mu(A \sqcup B) = |\sum_{\gamma \in A \sqcup B} a(\gamma)|^2$$
$$= \mu(A) + \mu(B) + \text{ interference terms}$$

(μ does, however, satisfy a generalised sum rule for the disjoint union of three events.)

Typicality and Preclusion

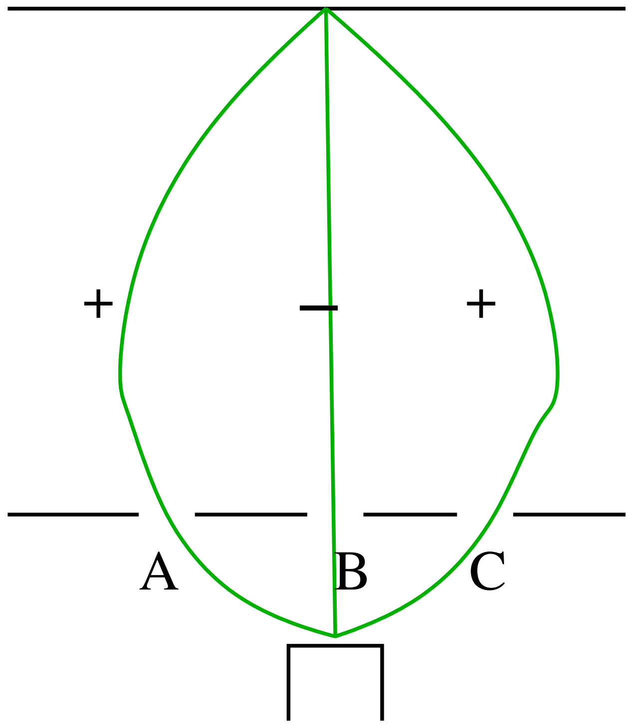
This interference means that μ CANNOT be interpreted as a probability measure. How then are we to use it for scientific purposes? Going back to the classical case, we can ask the same question: how do we, in fact, use μ to do science?

Claim: We identify events E such that $\mu(E) << 1$ and we say, typically E does not happen: it is precluded.

This is known as Cournot's Principle, was first articulated by Bernoulli, occurs in Kolmogorov's "Grundbegriffe" as "Principle B", and is the interpretation of probability as applied to the real world held by Markov, Borel, Lévy and other founders of probability theory.

The Three Slit Experiment

Let us adopt the Cournot Principle in Quantum Measure Theory.

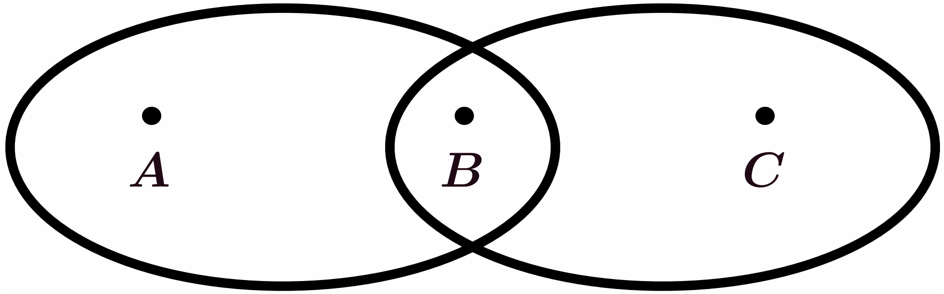


In this experiment, we can arrange the slits and choose a final position on the screen such that the amplitude for the particle to go through the middle slit B and land at the final position is -1 and the amplitude for the particle to go through the outer slit A is +1 and similarly for outer slit C.

The sample space consists of three histories, A, B and C (actually A contains many particle trajectories but this simplification preserves the essential point).

There are two events of measure zero: $\{A, B\}$ and $\{B, C\}$

No single history can happen



If one history γ happens and if an event has measure zero, then γ cannot be in that event.

BUT, the two sets of measure zero cover the whole sample space, so no single history can happen.

This example is not absolutely conclusive because there are other positions on the screen to consider (this is a subset of the full sample space). There are other conclusive examples, based on the Kochen-Specker theorem for example (FD, Yousef Ghazi-Tabatabai).

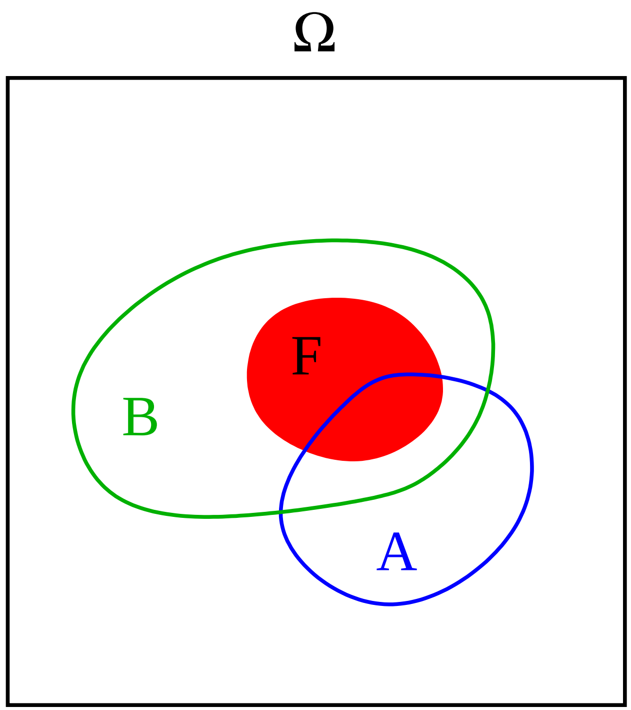
A Resolution

What do we replace the One History Axiom with? Sorkin has proposed an answer that can be given in several equivalent forms. Here is one:

"A subset of histories happens"

and

"That subset is minimal, subject to the preclusion condition"



The set of histories that happens is F. Question "Does event B happen?", Answer: yes. "Does event A happen?", Answer:no. In general if F is a subset of X then X is true, otherwise X is false. F must not be contained in a precluded event (an event of measure zero).

Co-events

Another way to express the proposal is to focus attention on the “answering map” $\phi : \mathfrak{A} \to Z_2$ which we call a “co-event” because it maps events to numbers. We’ve seen that in some quantum set-ups ϕ cannot be a homomorphism. So what weaker condition should be imposed?

Cournot’s Principle aka Preclusion: $\mu(A) \Rightarrow phi(A) = 0$.

I: The set $\phi^{-1}(1) \equiv A \in \mathfrak{A} | phi(A) = 1$ is a filter, and is maximal amongst filters which satisfy the preclusion condition.

II: ϕ preserves multiplication but not necessarily addition in the algebra and is the finest grained such co-event. $\phi(AB) = \phi(A)\phi(B)$.

III: ϕ respects “and” but not necessarily “or” and is the finest grained such co-event.

Note, these all include a “Principle of Maximal Detail.”

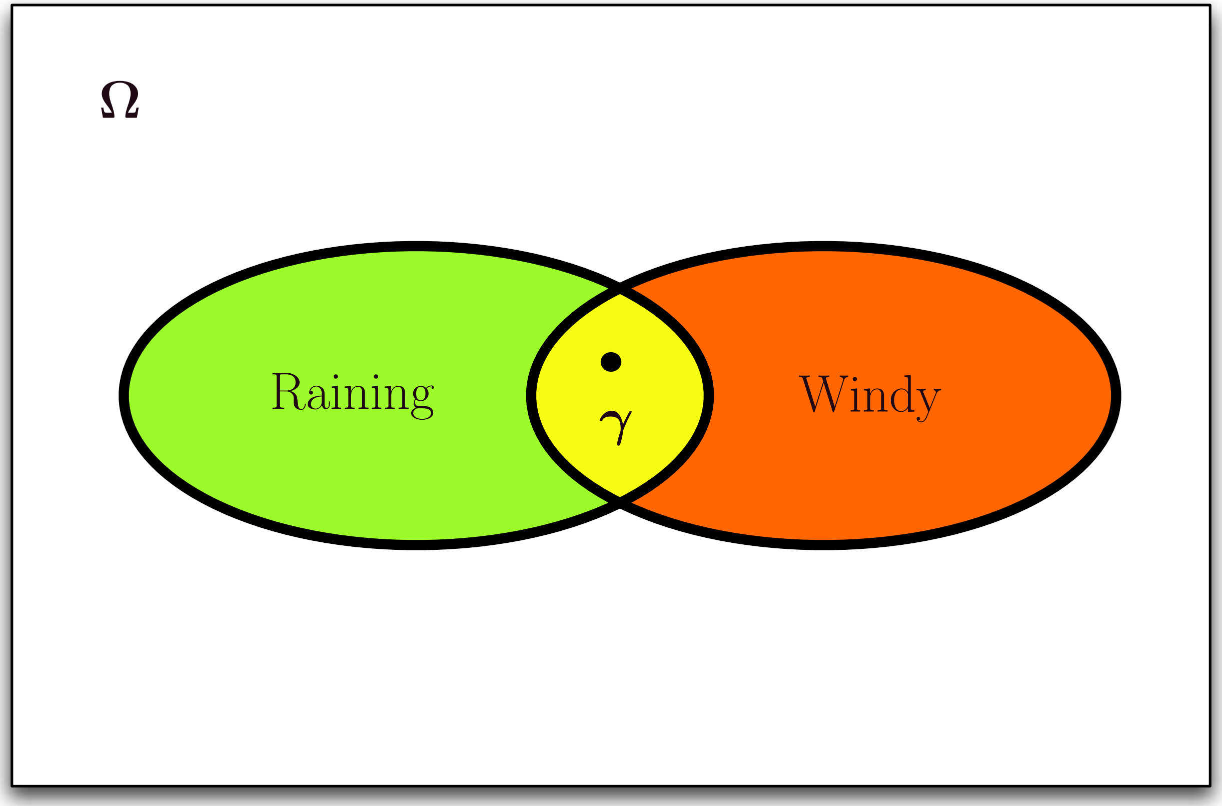
The Un-noticed Background

Classical logic is a background assumption that is so basic we do not notice we are making it. "Logic" here means "Rules of inference regarding the truth and falsity of statements about the physical world."

CLAIM: We use classical logic as an unquestioned background because, in everyday life,

One History Happens.

Wind and Rain

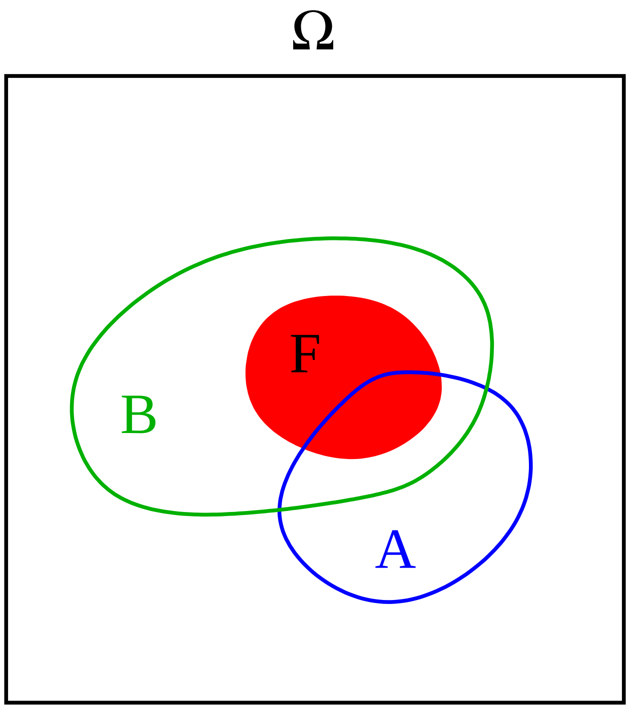


- If “it is raining” is true and “it is windy” is true then “it is raining and windy” is true
- If “it is raining” is true then “it is dry” is false
- If “it is raining” is false then “it is dry” is true

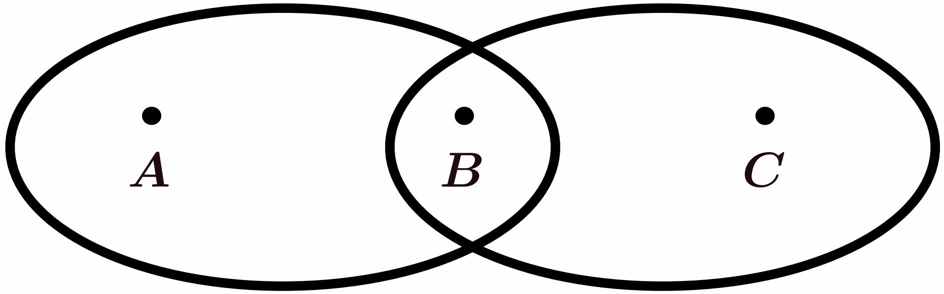
Coarse grained Reality $\equiv$ Non-Classical Logic

If what happens is a set of histories, let's look again at our rules of inference:

- If "R" is true and "W" is true then "R and W" is true HOLDS
- If "R" is true then "not R" is false HOLDS
- If "R" is false then "not R" is true FAILS



Three Slits revisited



The real set of histories cannot be contained in a set of measure zero. So it cannot be a subset of $\{A, B\}$ or $\{B, C\}$. There are only two possible sets satisfying this condition: $\{A, C\}$ and the whole set $\{A, B, C\}$. The latter is not minimal because it contains the former. So the unique minimal preclusive reality is $\{A, C\}$ (it is the base of the filter). We can answer any question: e.g.

Does the particle go though slit A? No

Does the particle go through slit B? No

Does the particle go through slit C? No

Does the particle go through slit A or B? No

Does the particle go through slit A or C? Yes

Real properties are common properties of all the histories in the "real set". This is a true coarse graining – finer details are unreal.

Summary and Comments

- Sum-over-histories quantum mechanics shares much of the same framework as classical stochastic theories: sample space, event algebra, measure.
- Quantum interference and the law that events of very small measure typically do not happen means that the One History axiom must be given up.
- Replaced by “A set of histories happens” and “maximal detail is achieved in reality”
- Classical rules of inference are a consequence of the One History Happens axiom of classical physical theories.
- With the new axioms, classical logic is recovered when the measure is classical (maximal detail drives the “real set” to be a singleton).
- For viability, we need to show that, in a fundamentally quantum theory, classical logic is recovered for macroscopic events.
- Logic itself has become dynamical. For example, for some quantum measures – *e.g.* if there are no sets of measure zero – all physical co-events may be homomorphisms. Which rules of inference can use, therefore, becomes a question of dynamics.