Categories of Spectral Geometries.

Paolo Bertozzini

Department of Mathematics and Statistics - Thammasat University - Bangkok.

Second Workshop on Categories Logic and Physics
Imperial College - London
14 May 2008.

Introduction 1.

In A. Connes' non-commutative geometry, "spaces" are described "dually" as spectral triples. We provide an overview of some of the notions that we deem necessary for the development of a categorical framework in the context of spectral geometry, namely:

- several notions of morphism of spectral geometries,
- a spectral theory for commutative full C*-categories,
- a definition of strict-n-C*-categories,
- spectral geometries over C*-categories.

If time will allow, we will speculate on possible applications to foundational issues in quantum physics:

- categorical covariance,
- spectral quantum space-time,
- modular quantum gravity.

Introduction 2

This is an ongoing joint research with

- Dr. Roberto Conti
  (University of Newcastle - Australia) and
- Dr. Wicharn Lewkeeratiyutkul
  (Chulalongkorn University - Bangkok - Thailand).

MSC-2000: 46L87, 46M15, 16D90, 18F99, 81R60, 81T05, 83C65.

Keywords: Non-commutative Geometry, Spectral Triple, Category, Morphism, C*-category, Quantum Physics.

Outline 1.

- Introduction and Outline.
- Categories.
  - Object/Morphisms.
  - Functors, Natural Transformations, Dualities.
- Non-commutative Geometry.
  - Non-commutative Topology
    (C*-algebras, Gel'fand Theorem, Hilbert C*-modules, Serre-Swan Theorem, Takahashi Theorem).
  - Non-commutative Spin Geometry
    (Non-commutative Manifolds, Connes Spectral Triples, Connes Theorem, Connes, Rennie-Varilly Theorem).
  - Other Spectral Geometries.
    (Riemannian Spectral Triples, Lord-Rennie Theorem)

Outline 2.

- Categories in Non-commutative Geometry.
  - ** Morphisms
    (Totally Geodesic Spin, Metric, Riemannian, Lord-Rennie Duality, Spectral Congruences, Spectral Spans, General, Morita, Other).
  - * Categorification (Vertical/Horizontal, C*-categories, Fell Bundles, Spectral Theorem for Commutative Full C*-categories, Spectral Theorem for Imprimitivity Bimodules).
  - ** Strict Higher C*-categories.
  - ** Spectral Geometries over C*-categories.
  - * Categorical NCG.
- Applications to Physics.
  - Categories in Physics.
  - * Categorical Covariance.
  - Spectral/Quantum Space-Time.
  - * Algebraic Quantum Gravity.

Objects and Morphisms.

A **category** $\mathscr{C}$ consists of

a) a class of **objects** $\mathrm{Ob}_{\mathscr{C}}$,

b) for any two object $A, B \in \mathrm{Ob}_{\mathscr{C}}$ a set of **morphisms** $\mathrm{Hom}_{\mathscr{C}}(A, B)$,

c) for any three objects $A, B, C \in \mathrm{Ob}_{\mathscr{C}}$ a **composition** map

$$\circ : \mathrm{Hom}_{\mathscr{C}}(B, C) \times \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{C}}(A, C)$$

that satisfies the following properties for all morphisms f, g, h that can be composed:

$$(f \circ g) \circ h = f \circ (g \circ h),$$
$$\forall A \in \mathrm{Ob}_{\mathscr{C}} \ \exists \iota_A \in \mathrm{Hom}_{\mathscr{C}}(A, A) \ : \ \iota_A \circ f = f, \ g \circ \iota_A = g.$$

Functors.

Given two categories $\mathscr{C}, \mathscr{D}$, a **covariant functor** $\mathfrak{F} : \mathscr{C} \to \mathscr{D}$ is a pair of maps

$$\mathfrak{F} : \mathrm{Ob}_{\mathscr{C}} \to \mathrm{Ob}_{\mathscr{D}}, \quad \mathfrak{F} : A \mapsto \mathfrak{F}_A, \quad \forall A \in \mathrm{Ob}_{\mathscr{C}},$$
$$\mathfrak{F} : \mathrm{Hom}_{\mathscr{C}} \to \mathrm{Hom}_{\mathscr{D}}, \quad \mathfrak{F} : x \mapsto \mathfrak{F}(x), \quad \forall x \in \mathrm{Hom}_{\mathscr{C}},$$

such that $x \in \mathrm{Hom}_{\mathscr{C}}(A, B) \Rightarrow \mathfrak{F}(x) \in \mathrm{Hom}_{\mathscr{D}}(\mathfrak{F}_A, \mathfrak{F}_B)$, and that, for any two composable morphisms f, g and any object A, satisfies

$$\mathfrak{F}(g \circ f) = \mathfrak{F}(g) \circ \mathfrak{F}(h),$$
$$\mathfrak{F}(\iota_A) = \iota_{\mathfrak{F}_A}.$$

For a **contravariant functor** we require $\mathfrak{F}(x) \in \mathrm{Hom}_{\mathscr{D}}(\mathfrak{F}_B, \mathfrak{F}_A)$.

Natural Transformations.

A morphism $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ is called an **isomorphism** if there exists another morphism $g \in \mathrm{Hom}_{\mathscr{C}}(B, A)$ such that

$$f \circ g = \iota_B \quad \text{and} \quad g \circ f = \iota_A.$$

A **natural transformation** $\eta : \mathfrak{F} \to \mathfrak{G}$ between two functors $\mathfrak{F}, \mathfrak{G} : \mathscr{C} \to \mathscr{D}$, is a map

$$\eta : \mathrm{Ob}_{\mathscr{C}} \to \mathrm{Hom}_{\mathscr{D}}, \quad \eta : A \mapsto \eta_A \in \mathrm{Hom}_{\mathscr{D}}(\mathfrak{F}_A, \mathfrak{G}_A),$$

such that the diagram

$$\begin{array}{ccc} \mathfrak{F}_A & \xrightarrow{\eta_A} & \mathfrak{G}_A \\ {\scriptstyle \mathfrak{F}(x)}\downarrow & & \downarrow{\scriptstyle \mathfrak{G}(x)} \\ \mathfrak{F}_B & \xrightarrow[\eta_B]{} & \mathfrak{G}_B. \end{array}$$

is commutative for all $x \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, $A, B \in \mathrm{Ob}_{\mathscr{C}}$.

Dualities.

The functor $\mathfrak{F} : \mathscr{C} \to \mathscr{D}$ is

- **faithful** if, for all $A, B \in \mathrm{Ob}_{\mathscr{C}}$, its restriction to the sets $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is injective;
- **full** if its restriction to $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is surjective;
- **representative** if for all $X \in \mathrm{Ob}_{\mathscr{D}}$ there exists $A \in \mathrm{Ob}_{\mathscr{C}}$ such that $\mathfrak{F}_A$ is isomorphic to X in $\mathscr{D}$.

A **duality** (a contravariant equivalence) of two categories $\mathscr{C}$ and $\mathscr{D}$ is a pair of contravariant functors $\Gamma : \mathscr{C} \to \mathscr{D}$ and $\Sigma : \mathscr{D} \to \mathscr{C}$ such that $\Gamma \circ \Sigma$ and $\Sigma \circ \Gamma$ are naturally equivalent to the respective identity functors $\mathcal{I}_{\mathscr{D}}$ and $\mathcal{I}_{\mathscr{C}}$. A duality is actually specified by two functors, but given any one of the two functors in the dual pair, the other one is unique up two natural isomorphism. A functor Γ is in a duality pair if and only if it is full, faithful and representative.

Non-commutative Geometry 0 - Introduction.

Non-commutative Geometry

||

[Operator Algebra] ∩ [Differential Geometry]

Operator Algebra

⋂

Functional Analysis

||

[Linear Algebra] ∩ [General Topology]

Non-commutative Geometry 1.

Non-commutative geometry, created by A. Connes, is the name of a very young and fast developing mathematical theory that is making use of operator algebras (itself a branch of functional analysis created by J. von Neumann in 1929) to find algebraic generalizations of most of the structures currently available in mathematics: measurable, topological, differential, metric etc.

From the algebraic point of view, mathematicians have been dealing with "non-commutative" algebraic structures since a relatively long time ago (for example under the form of groups, matrices in linear algebra and Hamilton's quaternions).

Non-commutative Geometry 2.

The first seeds of non-commutative geometry (i.e. the idea to substitute a commutative algebra with a non-commutative one) is strictly linked with physics' developments and can be traced back to the beginning of quantum mechanics in the form of Heisenberg's matrix mechanics in 1925, where non-commuting matrices (operators) take the place of classical commuting observables of a physical system (hence the fashionable name of "quantum mathematics" often cited in the recent literature).

Anyway, it is only starting in 1980, with the extraordinary work of A. Connes, the real founder of non-commutative geometry, that a systematic theory capable of describing differential and metric structures becomes available.

Non-commutative Geometry 3.

The fundamental idea, implicitly used in A. Connes' non-commutative geometry is a powerful extension of R. Decartes' analytic geometry:

- to "trade" "geometrical spaces" X of points with their Abelian algebras of (say complex valued) functions $f : X \to \mathbb{C}$;
- to "translate" the geometrical properties of spaces into algebraic properties of the associated algebras (a line of thought already present in J.L. Koszul algebraization of differential geometry);

Non-commutative Geometry 4.

- to "reconstruct" the original geometric space X as a derived entity (the spectrum of the algebra), a technique that appeared for the first time in the work of I. Gel'fand on Abelian C*-algebras in 1939 (although similar ideas, previously developed by D. Hilbert, are well known and used also in algebraic geometry in P. Cartier-A. Grothendieck's definition of schemes).

Non-commutative Geometry 5.

In order to develop non-commutative geometries, we usually proceed as follows:

1) First we find a suitable way to "codify" or translate the geometric properties of a space X (topology, measure, differential structure, metric, ...) in algebraic terms, using a commutative algebra of functions over X.
2) Then we try to see if this codification "survives" generalizing to the case of non-commutative algebras.

Non-commutative Geometry 6.

3) Finally the generalized properties are taken as axioms defining what a "dual" of a "non-commutative" (topological, measurable, differential, metric, ...) space is, without referring to any underlying point space.

Of course the process of generalization of the properties from the commutative to the non-commutative algebra case is highly non trivial and, as a result, several alternative possible axiomatizations arise in the non-commutative case, corresponding to a unique "commutative limit".

Non-commutative Geometry 7.

Here is a short "dictionary" with some of the "translations" from "commutative" to "quantum" mathematics:

Commutative	Non-commutative
Set	Algebra
Point	Pure State
Topology (locally compact, T_2)	C*-algebra
Measure (Radon)	State on a C*-algebra
Vector Bundle	Module over a C*-algebra
Riemannian Manifold (compact, spin)	Connes' Spectral Triple $(\mathcal{A}, \mathcal{H}, D)$

▸ Gel'fand ▸ Serre-Swan ▸ Connes, Rennie-Varilly

Non-commutative Geometry 8.

The existence of dualities between categories of "geometrical spaces" and categories "constructed from Abelian algebras" is the starting point of any generalization of geometry to the non-commutative situation.
Here we will present Gel'fand duality and Takahashi duality a generalization of Gel'fand that also subsumes Serre-Swan equivalence. ▸ Duality ▸ Gel'fand ▸ Serre-Swan ▸ Takahashi ▸ Connes, Rennie-Varilly

Non-commutative Geometry 9 - References.

For general introductions to the subject see:

- A. Connes,
  Noncommutative Geometry, Academic Press (1994).
- G. Landi,
  An Introduction to Noncommutative Spaces and Their Geometries, Springer (1997).
- H. Figueroa-J. Gracia-Bondia-J. Varilly,
  Elements of Noncommutative Geometry, Birkhäuser (2000).

C*-algebras.

A complex unital **algebra** $\mathcal{A}$ is a vector space over $\mathbb{C}$ with an associative unital bilinear multiplication.
$\mathcal{A}$ is **Abelian** (commutative) if $ab = ba$, for all $a, b \in \mathcal{A}$.
An **involution** on $\mathcal{A}$ is a conjugate linear map $* : \mathcal{A} \to \mathcal{A}$ such that $(a^*)^* = a$ and $(ab)^* = b^* a^*$, for all $a, b \in \mathcal{A}$. An involutive complex unital algebra is $\mathcal{A}$ called a **C*-algebra** if $\mathcal{A}$ is a Banach space with a norm $a \mapsto \|a\|$ such that $\|ab\| \leq \|a\| \cdot \|b\|$ and $\|a^* a\| = \|a\|^2$, for all $a, b \in \mathcal{A}$.
Notable examples are the algebras of continuous complex valued functions $C(X;\mathbb{C})$ on a compact topological space with the "sup norm" and the algebras of linear bounded operators $\mathcal{B}(H)$ on a given Hilbert space H.

Operator Algebras - References.

For all the details on operator algebras, the reader may refer to

- R. Kadison-J. Ringrose,
  Fundamentals of the Theory of Operator Algebras, vol. 1-2, AMS (1998).
- M. Takesaki,
  The Theory of Operator Algebras I-II-III, Springer (2001-2002).
- B. Blackadar,
  Operator Algebras, Springer (2006).

For an elementary introduction to functional analysis for operator algebras:

- G. Pedersen,
  Analysis Now, Springer (1998).

Gel'fand Theorem 1

There exists a duality (Γ, Σ) between the category $\mathscr{T}^{(1)}$, of continuous maps between compact Hausdorff topological spaces, and the category $\mathscr{A}^{(1)}$, of unital homomorphisms of commutative unital C*-algebras, where

- Γ is the functor that to every compact Hausdorff topological space $X \in \mathrm{Ob}_{\mathscr{T}^{(1)}}$ associates the unital commutative C*-algebra $\mathrm{C}(X; \mathbb{C})$ of complex valued continuous functions on X (with pointwise multiplication and conjugation and supremum norm) and that to every continuous map $f : X \to Y$ associates the unital $*$-homomorphism $f^\bullet : C(Y; \mathbb{C}) \to C(X; \mathbb{C})$ given by the pull-back of continuous functions by f;

Gel'fand Theorem 2

- Σ is the functor that to every unital commutative C*-algebra $\mathcal{A}$ associates its spectrum

$$\mathrm{Sp}(\mathcal{A}) := \{\omega \mid \omega : \mathcal{A} \to \mathbb{C}, \text{ is a unital } *\text{-homomorphism}\}$$

(as a topological space with the weak topology induced by the evaluation maps $\omega \mapsto \omega(x)$, for all $x \in \mathcal{A}$) and that to every unital $*$-homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ of algebras associates the continuous map $\phi^\bullet : \mathrm{Sp}(\mathcal{B}) \to \mathrm{Sp}(\mathcal{A})$ given by the pull-back under ϕ.

Serre-Swan Theorem | Takahashi Theorem | Connes Rennie-Varilly Theorem

Gel'fand Theorem 3

- The natural isomorphism $\mathfrak{G} : \mathcal{I}_{\mathscr{A}^{(1)}} \to \Gamma \circ \Sigma$ is given by the **Gel'fand transforms** $\mathfrak{G}_{\mathcal{A}} : \mathcal{A} \to C(\mathrm{Sp}(\mathcal{A}))$ defined by $\mathfrak{G}_{\mathcal{A}} : a \mapsto \hat{a}$ where $\hat{a} : \mathrm{Sp}(\mathcal{A}) \to \mathbb{C}$ is the Gelf'and transform of a i.e. $\hat{a} : \omega \mapsto \omega(a)$.
- Similarly the natural isomorphism $\mathfrak{E} : \mathcal{I}_{\mathscr{T}^{(1)}} \to \Sigma \circ \Gamma$ is given by the **evaluation** homeomorphisms $\mathfrak{E}_X : X \to \mathrm{Sp}(C(X))$ defined by $\mathfrak{E}_X : p \mapsto \mathrm{ev}_p$, where $\mathrm{ev}_p : C(X) \to \mathbb{C}$ is the p-evaluation i.e. $\mathrm{ev}_p : f \mapsto f(p)$.

Hilbert C*-modules 1.

A **left pre-Hilbert C*-module** ${}_{\mathcal{A}}M$ over $\mathcal{A}$,[1] is a unital left module M over the unital ring $\mathcal{A}$ that is equipped with an $\mathcal{A}$-valued inner product $M \times M \to \mathcal{A}$ denoted by $(x, y) \mapsto {}_{\mathcal{A}}\langle x \mid y \rangle$ such that:[2]

$$\langle x + y \mid z \rangle = \langle x \mid z \rangle + \langle y \mid z \rangle, \quad \forall x, y, z \in M,$$
$$\langle a \cdot x \mid z \rangle = a \langle x \mid z \rangle, \quad \forall x, y \in M, \quad \forall a \in \mathcal{A},$$
$$\langle y \mid x \rangle = \langle x \mid y \rangle^*, \quad \forall x, y \in M,$$
$$\langle x \mid x \rangle \in \mathcal{A}_+, \quad \forall x \in M,$$
$$\langle x \mid x \rangle = 0_{\mathcal{A}}, \Rightarrow x = 0_M.$$

${}_{\mathcal{A}}M$ is a **left Hilbert C*-module** if it is complete in the norm defined by $x \mapsto \sqrt{\| {}_{\mathcal{A}}\langle x \mid x \rangle \|}$.

[1] A unital C*-algebra whose positive part is denoted by $\mathcal{A}_+$.

[2] The definition for **right module** requires linearity in the second variable.

Hilbert C*-modules 2.

${}_{\mathcal{A}}M$ is **full** if $\overline{\mathrm{span}\{\langle x \mid y\rangle \mid x, y \in M\}} = \mathcal{A}$, where the closure is in the norm topology of the C*-algebra $\mathcal{A}$. A **pre-Hilbert C*-bimodule** ${}_{\mathcal{A}}M_{\mathcal{B}}$ over the unital C*-algebras $\mathcal{A}, \mathcal{B}$, is a left pre-Hilbert module over $\mathcal{A}$ and a right pre-Hilbert C*-module over $\mathcal{B}$ such that: $(a \cdot x) \cdot b = a \cdot (x \cdot b), \quad \forall a \in \mathcal{A},\ x \in M,\ b \in \mathcal{B}$.
A full pre-Hilbert C*-bimodule is said to be an **imprimitivity bimodule** or a **Morita equivalence bimodule** if:

$$ {}_{\mathcal{A}}\langle x \mid y\rangle \cdot z = x \cdot \langle y \mid z\rangle_{\mathcal{B}}, \quad \forall x, y, z \in M. $$

A bimodule ${}_{\mathcal{A}}M_{\mathcal{A}}$ (over an Abelian $\mathcal{A}$) is called **symmetric** if $ax = xa$ for all $x \in M$ and $a \in \mathcal{A}$.
A module ${}_{\mathcal{A}}M$ is **free** if it is isomorphic to a module of the form $\oplus_J \mathcal{A}$ for some index set J. A module ${}_{\mathcal{A}}M$ is **projective** if there exists another module ${}_{\mathcal{A}}N$ such that $M \oplus N$ is a free module.

Serre-Swan Theorem 1.

An "equivalence result" strictly related to Gel'fand theorem, is the following "Hermitian" version of Serre-Swan theorem (see for example Theorem 7.1 in M. Frank[3] or Theorem 9.1.6 in N. Weaver[4]) that provides a "spectral interpretation" of symmetric projective finite bimodules over a commutative unital C*-algebra as finite rank Hermitian vector bundles over the spectrum of the algebra.[5]

[3] M. Frank Geometrical Aspects of Hilbert C*-modules, Positivity, **3**, n. 3, 215-243 (1999).

[4] N. Weaver, Mathematical Quantization, Chapmann and Hall, 2001.

[5] The result is actually true also without the finiteness condition (with Hilbert bundles in place of Hermitian bundles).

Serre-Swan Theorem 2.

Let X be a compact Hausdorff topological space. Let $\mathscr{M}_{C(X)}$ denote the weak monoidal $*$-category[6] of symmetric projective finite Hilbert C*-bimodules over the commutative C*-algebra $C(X;\mathbb{C})$ with $C(X)$-bimodule morphisms. Let $\mathscr{E}_X$ be the weak monoidal $*$-category[7] of finite rank Hermitian vector bundles over X with bundle morphisms[8].

The functor $\Gamma : \mathscr{E}_X \to \mathscr{M}_{C(X)}$, that to every Hermitian vector bundle associates its symmetric $C(X)$-bimodule of sections, is an equivalence of weak monoidal $*$-categories.

▸ Takahashi's Theorem ▸ Connes Rennie-Varilly Theorem

[6] Were the monoidal structure is the usual tensor product of bimodules and the $*$ is the usual Rieffel dual of bimodules.

[7] Were the monoidal structure comes from the fiberwise tensor product of Hermitian bundles and the $*$ is the dualization of Hermitian bundles.

[8] Continuous, fiberwise linear maps, preserving the base points.

Problems.

- Serre-Swan theorem deals only with categories of bundles over a fixed topological space (categories of modules over a fixed algebra, respectively).
- Serre-Swan theorem, in its actual form, gives an equivalence of categories (and not a duality), this will create problems of "covariance" for any generalization of the well-known covariant functors between categories of manifolds and categories of their associated vector (tensor, Clifford) bundles.

Work on these issues (considering "conguences" of bimodules and reformulating Serre-Swan theorem in terms of "relators") is in progress. A first immediate solution is provided by Takahashi duality theorem here below.

Takahashi Theorem 1

Serre-Swan theorem is actually a particular case of the following general (and surprisingly almost unnoticed) Gel'fand duality result obtained in 1971 by A. Takahashi in his Ph.D. thesis[9] under the supervision of K. Hofmann.

Note that our Gel'fand duality result for commutative full C*-categories (that we will present later) can be seen as "strict"-$*$-monoidal version of Takahashi duality.

[9] A. Takahashi, Hilbert Modules and their Representation, Rev. Colombiana Mat., **13**, 1-38 (1979).
A. Takahashi, A Duality between Hilbert Modules and Fields of Hilbert Spaces, Rev. Colombiana Mat., **13**, 93-120 (1979).

Takahashi Theorem 2

There is a (weak $*$-monoidal) category ${}_{\bullet}\mathscr{M}$ of left Hilbert C*-modules ${}_{\mathcal{A}}M, {}_{\mathcal{B}}N$ over unital commutative C*-algebras, whose morphisms are given by pairs (ϕ, Φ) where $\phi : \mathcal{A} \to \mathcal{B}$ is a unital $*$-homomorphism of C*-algebras and $\Phi : M \to N$ is a continuous map such that $\Phi(ax) = \phi(a)\Phi(x)$, for all $a \in \mathcal{A}$ and $x \in M$.

There is a (weak $*$-monoidal) category $\mathscr{E}$ of Hilbert bundles $(\mathcal{E}, \pi, \mathcal{X}), (\mathcal{F}, \rho, \mathcal{Y})$ over compact Hausdorff topological spaces with morphisms given by pairs $(f, \mathcal{F})$ with $f : \mathcal{X} \to \mathcal{Y}$ a continuous map and $\mathcal{F} : f^{\bullet}(\mathcal{F}) \to \mathcal{E}$ satisfies $\pi \circ \mathcal{F} = \rho^f$, where $(f^{\bullet}(\mathcal{F}), \rho^f, \mathcal{X})$ denotes the pull-back of the bundle $(\mathcal{F}, \rho, \mathcal{Y})$ under f.

There is a duality of (weak $*$-monoidal) categories given by the functor Γ that associates to every Hilbert bundle $(\mathcal{E}, \pi, \mathcal{X})$ the set of sections $\Gamma(\mathcal{X}; \mathcal{E})$ and that to every section $\sigma \in \Gamma(\mathcal{Y}; \mathcal{F})$ associates the section $\mathcal{F} \circ f^{\bullet}(\sigma) \in \Gamma(\mathcal{X}; \mathcal{E})$.

▸ Connes Rennie-Varilly Theorem

Non-commutative Manifolds.

What are non-commutative manifolds?
In order to define "non-commutative manifolds", we have to find a categorical duality between a category of manifolds and a suitable category constructed out of Abelian C*-algebras of functions over the manifolds. The complete answer to the question is not known, but (at least in the case of compact finite dimensional orientable Riemannian spin manifolds), the notion of Connes' spectral triples and Connes-Rennie-Varilly reconstruction theorem provide and adeguate starting point, specifying the objects of our non-commutative category.

Connes Spectral Triples 1.

A (compact) **spectral triple** $(\mathcal{A}, \mathcal{H}, D)$ is given by:

- a unital pre-C*-algebra $\mathcal{A}$ closed under holomorphic functional calculus;
- a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$;
- a (non-necessarily bounded) self-adjoint operator D on $\mathcal{H}$, called the Dirac operator, such that:
  a) the resolvent $(D - \lambda)^{-1}$ is a compact operator, $\forall \lambda \in \mathbb{C} \setminus \mathbb{R}$,
  b) $[D, \pi(a)]_- \in \mathcal{B}(\mathcal{H})$, for every $a \in \mathcal{A}$, where $[x, y]_- := xy - yx$ denotes the commutator of $x, y \in \mathcal{B}(\mathcal{H})$.

Several additional technical conditions (grading, real structure, orientability, regularity, summability, finiteness, Poincaré duality) should be imposed on a spectral triple in order to formulate the following results.

Connes Spectral Triples 2.

Given an orientable compact Riemannian spin m-dimensional differentiable manifold M, with a given complex spinor bundle $S(M)$, a given spinorial charge conjugation C_M and a given volume form μ_M, define by

- $\mathcal{A}_M := C^\infty(M; \mathbb{C})$ the algebra of complex valued regular functions on the differentiable manifold M,
- $\mathcal{H}_M :=\mathrm{L}^2(M; S(M))$ the Hilbert space of "square integrable" sections of the given spinor bundle $S(M)$ of the manifold M i.e. the completion of the space $\Gamma^\infty(M; S(M))$ of smooth sections of the spinor bundle $S(M)$ equipped with the inner product $\langle \sigma \mid \tau \rangle := \int_M \langle \sigma(p) \mid \tau(p) \rangle_p \, \mathrm{d}\mu_M$, where $\langle \mid \rangle_p$, with $p \in M$, is the unique inner product on $S_p(M)$ compatible with the Clifford action and the Clifford product.

Connes Spectral Triples 3.

- D_M the Atiyah-Singer Dirac operator i.e. the closure of the operator that on $\Gamma^\infty(M; S(M))$ is obtained by "contracting" with the Clifford multiplication, the unique spinorial covariant derivative $\nabla^{S(M)}$ (induced on $\Gamma^\infty(M; S(M))$ by the Levi-Civita covariant derivative of M);
- J_M the unique antilinear unitary extension $J_M : \mathcal{H}_M \to \mathcal{H}_M$ of the operator determined by the spinorial charge conjugation C_M by $(J_M\sigma)(p) := C_M(\sigma(p))$ for $\sigma \in \Gamma^\infty(M; S(M))$, $p \in M$;
- Γ_M the unique unitary extension on $\mathcal{H}_M$ of the operator given by fiberwise grading on $S_p(M)$, with $p \in M$.[10]

[10]The grading is actually the identity in odd dimension.

Connes, Rennie-Varilly Theorems 1.

Theorem (Connes)

The data $(\mathcal{A}_M, \mathcal{H}_M, D_M)$ define an Abelian regular finite m-dimensional spectral triple that is real, with real structure J_M, orientable, with grading Γ_M, and that satisfies Poincaré duality.

Connes, Rennie-Varilly Theorems 2.

Theorem (Connes, Rennie-Varilly)

Let $(\mathcal{A}, \mathcal{H}, D)$ be an irreducible Abelian real (with real structure J and grading Γ) regular m-dimensional orientable finite spectral triple satisfying Poincaré duality (and, in the Rennie-Varilly formulation, some additional "metric/regularity conditions").
The spectrum of (the norm closure of) $\mathcal{A}$ can be endowed, with the structure of an m-dimensional connected compact spin Riemannian manifold M with an irreducible complex spinor bundle $S(M)$, a charge conjugation J_M and a grading Γ_M such that:
$\mathcal{A} \simeq C^\infty(M; \mathbb{C})$, $\mathcal{H} \simeq L^2(M, S(M))$, $D \simeq D_M$, $J \simeq J_M$, $\Gamma \simeq \Gamma_M$.

Connes-Rennie Theorems 3.

A. Connes proved the previous theorem under the additional condition that $\mathcal{A}$ is already given as the algebra $\mathcal{A} = C^\infty(M; \mathbb{C})$ of smooth functions over a differentiable manifold M and conjectured[11] the result for general commutative pre-C*-algebras.

A proof of this fact, under a slightly different set of assumptions, has been presented by A. Rennie-J. Varilly.[12]

---

[11] A. Connes, Brisure de Symétrie Spontanée et Géométrie du Pont de Vue Spectral, J. Geom. Phys., 23, 206-234 (1997).

[12] A. Rennie, Commutative Geometries are Spin Manifolds, Rev. Math. Phys., **13**, 409 (2001), `math-ph/9903021`.
Some gaps that were pointed out in the original argument and a different revised proof has appeared: A. Rennie, J. Varilly, Reconstruction of Manifolds in Noncommutative Geometry, `math.OA/0610418`.

Connes-Rennie-Varilly Theorems 4.

It has been announced that A. Connes is providing a complete proof of the theorem under the original assumptions.[13]

As a consequence, a one-to-one correspondence should exist between unitary equivalence classes of these Abelian spectral triples and connected compact oriented Riemannian spin manifolds, up to spin-preserving isometric diffeomorphisms. morphisms

[13] A. Connes: On the Spectral Characterization of Manifolds, Lectures, Sixth Annual Spring Institute on NCG and OA, Vanderbilt University, 7-13 May 2008.

Other Spectral Geometries 1.

Although NCG, following A. Connes, has been mainly developed in the axiomatic framework of spectral triples, that essentially generalize the structures available for the Atiyah-Singer theory of first order differential elliptic operators of the Dirac type, it is very likely that suitable "spectral geometries" might be developed using operators of higher order (the Laplacian type being the first notable example). Since "topological obstructions" (such us non-orientability, non-spinoriality) are expected to survive essentially unaltered in the transition from the commutative to the non-commutative world, these "higher-order non-commutative geometries" will deal with more general situations compared to usual spectral triples. In this direction we are working [14] in the hope to obtain Connes-Rennie type theorems also in these cases.

[14] P.B., R. Conti, W. Lewkeeratiyutkul, Second Order Non-commutative Geometry, work in progress.

Other Spectral Geometries 2.

In the last few years several other variants for the axioms of spectral triples have been considered or proposed:

- non-compact spectral triples,[15]
- spectral triples for quantum groups,[16]
- Lorentzian spectral triples,[17]

---

[15] V. Gayral, J. M. Gracia-Bondia, B. Iochum, T. Schüker, J. C. Varilly, Moyal Planes are Spectral Triples, Commun. Math. Phys. **246**, no. 3 (2004), 569-623, `hep-th/0307241`.

[16] L. Dabrowski, G. Landi, A, Sitarz, W. van Suijlekom, J. Varilly, The Dirac Operator on $SU_q(2)$, Commun. Math. Phys., **259**, 729-759 (2005), `math.QA/0411609`.

[17] A. Strohmaier, On Noncommutative and semi-Riemannian Geometry, `math-ph/0110001`.

Other Spectral Geometries 3.

- Riemannian non-spin geometries,[18],[19]
- "von Neumann" spectral triples.[20],[21],[22]

---

[18] J. Fröhlich, O. Grandjean, A. Recknagel, Supersymmetric Quantum Theory and Differential Geometry, Commun. Math. Phys., 193, 527-594 (1998), hep-th/9612205; J. Fröhlich, O. Grandjean, A. Recknagel, Supersymmetric Quantum Theory and Non-commutative Geometry, Commun. Math. Phys., **203**, 119-184 (1999), mat-ph/9807006.

[19] S. Lord, Riemannian Geometries, math-ph/0010037.

[20] M-T. Benameur, T. Fack, On von Neumann Spectral Triples, math.KT/0012233.

[21] M-T. Benameur, A. Carey, J. Phillips, A. Rennie, F. Sukochev, K. Wojciechowski, An Analytic Approach to Spectral Flow in von Neumann Algebras, math.OA/0512454.

[22] A. Carey, J. Phillips, A. Rennie, Semifinite Spectral Triples Associated with Graph C*-algebras, arXiv:0707.3853.

Riemannian Spectral Triples 1.

Let (M, g) be a compact orientable m-dimensional Riemannian manifold (not necessarily spinorial) and define by

- $\mathcal{A}_M := C^\infty(M; \mathbb{C})$ the algebra of smooth complex-valued functions on the differentiable manifold M.
- $\mathcal{H}_M^\Lambda$ the Hilbert space obtained completing the pre-Hilbert space

$$\Gamma^\infty(M; \Lambda(M)) = \oplus_{q=0}^m \Gamma^\infty(M; \Lambda_q(M))$$

of smooth sections of the Grassmann bundle $\Lambda(M)$ of M (i.e. the smooth differential forms) with respect to the inner product defined (on each $\Gamma^\infty(M; \Lambda_q(M))$) by

$$\langle \sigma \mid \tau \rangle := \int_M \sigma(p) \wedge *\tau(p) \, d\mu_M, \quad \forall \sigma, \tau \in \Lambda_q(M).$$

Riemannian Spectral Triples 2.

- $\pi_M : \mathcal{A}_M \to \mathcal{B}(\mathcal{H}_M^\Lambda)$ the representation of the algebra $\mathcal{A}_M$ onto the Hilbert space $\mathcal{H}_M$ obtained by unique linear extension from the action by multiplication of $\mathcal{A}_M$ on $\Gamma^\infty(M; \Lambda(M))$.
- D_M^Λ the Dirac operator on forms i.e. the densely defined linear map given by $D_M^\Lambda := d + *d*$ where

$$d : \Gamma^\infty(M; \Lambda(M)) \to \Gamma^\infty(M; \Lambda(M))$$

is the exterior differential and
$* : \Gamma^\infty(M; \Lambda(M)) \to \Gamma^\infty(M; \Lambda(M))$ is the Hodge-dual of differential forms.

Lord-Rennie Theorem.

In this context of Riemannian (non spinorial manifolds) we have:

Theorem

For any compact orientable m-dimensional Riemannian manifold (M, g), the data $(\mathcal{A}_M, \mathcal{H}_M, D_M)$ give an Abelian spectral triple.

S. Lord[23] has proposed axioms for "Riemannian spectral triples" and a reconstruction theorem based on Rennie's original proof of Connes' reconstruction theorem. A detailed proof of this result is under investigation[24]

[23] S. Lord, Riemannian Geometries, math-ph/0010037.

[24] A. Rennie, personal communication, La Trobe, September 2007.

Categories in Non-commutative Geometry - Introduction.

[Non-commutative Geometry] $\cap$ [Category Theory]

$||$

Categorical Non-commutative Geometry

Morphisms.

Having described A. Connes spectral triples and somehow justified the fact that spectral triples are a possible definition for "non-commutative" compact finite dimensional orientable Riemannian spin manifolds, our next goal here is to discuss definitions of "morphisms" between spectral triples and to construct categories of spectral triples that might support dualities with categories of manifolds.

There are actually several possible notions of morphism, according to the amount of "background structure" of the manifold that we would like to see preserved (and also depending on the kind of topological properties that we would like to "attach" to our morphisms: orientation, spinoriality, . . .).

Totally-Geodesic-Spin Morphisms 1.

In our first paper[25], we proposed this notion of morphism: given two spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j)$, with $j = 1, 2$, a **morphism of spectral triples** is a pair (ϕ, Φ), where $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is a $*$-morphism between the pre-C*-algebras $\mathcal{A}_1, \mathcal{A}_2$ and $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear map in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ that "intertwines" the representations $\pi_1, \pi_2 \circ \phi$ and the Dirac operators D_1, D_2:

$$\pi_2(\phi(x)) \circ \Phi = \Phi \circ \pi_1(x), \quad \forall x \in \mathcal{A}_1,$$
$$D_2 \circ \Phi(\xi) = \Phi \circ D_1(\xi), \quad \forall \xi \in \mathrm{Dom}\, D_1.$$

With this definition of morphism, isomorphisms are (unitary) equivalences of spectral triples.

[25] P.B., R. Conti, W. Lewkeeratiyutkul, A Category of Spectral Triples and Discrete Groups with Length Function, Osaka J. Math., **43**, n. 2, (2006).

Totally-Geodesic-Spin Morphisms 2.

This definition of morphism implies quite a strong relationship between the spectra of the Dirac operators of the two spectral triples. Loosely speaking, for ϕ epi and Φ coisometric (respectively mono and isometric), in the commutative case, one should expect such definition to become relevant only for maps that "preserve the geodesic structures" (totally geodesic immersions and respectively totally geodesic submersions)[26].
Furthermore these morphisms depend, at least in some sense, on the spin structures: this "spinorial rigidity" (at least in the case of morphisms of real even spectral triples) require that such morphisms between spectral triples of different dimensions might be possible only when the difference in dimension is a multiple of 8.

[26] See P.B., R. Conti, W. Lewkeeratiyutkul, Non-commutative Totally Geodesic Submanifolds and Quotient Manifolds, in preparation.

Metric Morphisms.

Another notion of morphism that is essentially blind to the spin structures has been proposed.[27]

Given two spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j)$, with $j = 1, 2$, denote by

$$d_{D_j}(\omega_1, \omega_2) := \sup\{|\omega_1(x) - \omega_2(x)| \mid x \in \mathcal{A},\ \|[D_j, \pi(x)]\| \leq 1\}$$

the quasi-distance induced on the sets $\mathcal{P}(\mathcal{A}_j)$ of pure states of $\mathcal{A}_j$.

A **metric morphism** of spectral triples is a unital epimorphism[28] $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ of pre-C*-algebras whose pull-back $\phi^\bullet : \mathcal{P}(\mathcal{A}_2) \to \mathcal{P}(\mathcal{A}_1)$, $\phi^\bullet(\omega) := \omega \circ \phi$ is an isometry, i.e.

$$d_{D_1}(\phi^\bullet(\omega_1), \phi^\bullet(\omega_2)) = d_{D_2}(\omega_1, \omega_2), \quad \forall \omega_1, \omega_2 \in \mathcal{P}(\mathcal{A}_2).$$

---

[27] P.B., R. Conti, W. Lewkeeratiyutkul, A Remark on Gel'fand Duality for Spectral Triples, preprint (2005).

[28] Note that if ϕ is an epimorphism, its pull-back $\phi^\bullet$ maps pure states into pure states.

Riemannian Morphisms.

Work is in progress[29] on a weaker notion of morphisms[30]: given two spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j)$, with $j = 1, 2$, a **Riemannian morphism of spectral triples** is a pair (ϕ, Φ) where $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is a $*$-morphism between the pre-C*-algebras $\mathcal{A}_1, \mathcal{A}_2$ and $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear map in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ that "intertwines" the representations $\pi_1, \pi_2 \circ \phi$ and the commutators of the Dirac operators D_1, D_2:

$$\pi_2(\phi(x)) \circ \Phi = \Phi \circ \pi_1(x), \quad \forall x \in \mathcal{A}_1,$$
$$[D_2, \phi(x)] \circ \Phi = \Phi \circ [D_1, x], \quad \forall x \in \mathcal{A}_1.$$

[29] P.B., R. Conti, W. Lewkeeratiyutkul, Categories of Spectral Triples and Morita Equivalence, in progress.

[30] In this case, isomorphisms reduce to the unitary maps considered in M. Paschke-R. Verch, Local Covariant Quantum Field Theory over Spectral Geometries, gr-qc/0405057.

Lord-Rennie Duality.

Modulo Lord-Rennie reconstruction theorem for Riemannian spectral triples, we already have duality results[31]:

- the pull-back of functions $\phi := f^\bullet : C^\infty(N) \to C^\infty(M)$ and forms $\Phi = f^\bullet : L^2(\Lambda(N)) \to L^2(\Lambda(M))$ give a duality from the category of Riemannian immersions $f : M \to N$ to the category of spectral triples with morphisms (ϕ, Φ) such that $\Phi[D_N, x]_- \Phi^* = [D_M, \phi(x)]_-$, with $x \in C^\infty(N)$;
- similarly, the pull back of functions and forms give a duality from the category of Riemannian submersions $f : N \to M$ to the category of spectral triples with morphisms (ϕ, Φ) such that $\Phi^*[D_N, \phi(y)]_- \Phi = [D_M, y]_-$, with $y \in C^\infty(M)$.

[31] P.B., R. Conti, W. Lewkeeratiyutkul, Morphisms of Non-commutative Riemannian Manifolds, in preparation.

Spectral Congruences.[32]

Let $(\mathcal{A}_j, H_j, D_j)$, for $j = 1, 2$, be two spectral triples.
A **spectral conguence** is a triple (ϕ, Φ, D_Φ) where:

- $\phi \subset \mathcal{A}_1 \times \mathcal{A}_2$ is a conguence of involutive unital C*-algebras i.e. a unital C*-subalgebra of $\mathcal{A}_1 \oplus \mathcal{A}_2$,
- $\Phi \subset H_1 \times H_2$ is a closed congruence of Hilbert spaces i.e. a closed subpace of $H_1 \oplus H_2$,
- D_Φ is subspace of $D_1 \oplus D_2 \subset (H_1 \oplus H_1) \oplus (H_2 \oplus H_2)$,

such that Φ is stable under componentwise action of ϕ and D_Φ and (ϕ, Φ, D_Φ) satisfies the additional axioms imposed on spectral triples, for example, for every $(x, y) \in \phi$, $[D_\Phi, (x, y)]_-$ extends to a bounded operator on the Hilbert space $\Phi \subset H_1 \oplus H_2$.

[32]P.B., R. Conti, W. Lewkeeratiyutkul, Spectral Geometries over C*-categories and Morphisms of Spectral Geometries, in preparation.

Spectral Spans.[33]

Given two spectral triples $(\mathcal{A}_j, H_j, D_j)$, for $j = 1, 2$, a **spectral span** is a spectral triple $(\mathcal{A}, H, D)$ equipped with a pair of morphisms of spectral triples (ϕ_1, Φ_1), (ϕ_2, Φ_2):

$$\begin{array}{ccccc} & & (\mathcal{A}, H, D) & & \\ & \swarrow^{(\phi_1,\Phi_1)} & & \searrow^{(\phi_2,\Phi_2)} & \\ (\mathcal{A}_1, H_1, D_1) & & & & (\mathcal{A}_2, H_2, D_2) \end{array}$$

Every spectral span is associated to a spectral congruence and such map is surjective (every spectral correspondence comes from a spectral span). Two spectral spans are equivalent if they induce the same spectral congruence.

[33] P.B., R. Conti, W. Lewkeeratiyutkul, Spectral Geometries over C*-categories and Morphisms of Spectral Geometries, in preparation.

Categories of Spectral Congruences and Spans.[34]

Spectral congruences form a category under composition:

$$(\psi, \Psi, D_\Psi) \circ (\phi, \Phi, D_\Phi) := (\psi \circ \phi, \Psi \circ \Phi, D_\Psi \circ D_\Phi)$$

where all the compositions are compositions of relations and where the identity of the triple $(\mathcal{A}, H, D)$ is given by $(\iota_\mathcal{A}, \iota_H, D \oplus D)$.

This category can also be described as the category of equivalence classes of spectral spans.

[34] P.B., R. Conti, W. Lewkeeratiyutkul, Spectral Geometries over C*-categories and Morphisms of Spectral Geometries, in preparation.

General Morphisms.

The several notions of morphism of spectral triples described above are not as general as possible. In a wider perspective under study[35], a morphism of spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j)$, for $j = 1, 2$, might be formalized as a "suitable" functor $\mathcal{F} : {}_{\mathcal{A}_2}\mathscr{M} \to {}_{\mathcal{A}_1}\mathscr{M}$, between the categories ${}_{\mathcal{A}_j}\mathscr{M}$ of $\mathcal{A}_j$-modules, having "appropriate intertwining" properties with the Dirac operators D_j.
Under some "mild" hypothesis, by Eilenberg-Gabriel-Watt theorem, any such functor is given by "tensorization" by a bimodule. These bimodules, suitably equipped with spectral data (as in the case of spectral triples), provide, in our opinion, the natural setting for a general theory of morphisms of non-commutative spaces. categorification

[35] P.B., R. Conti, W. Lewkeeratiyutkul, Categories of Spectral Triples and Morita Equivalence, in progress.

Morita Morphisms 1.

- Y. Manin[36] has been the first to propose such "Morita morphisms" (tensorizations with Hilbert C*-bimodules) as the natural notion of morphism of non-commutative spaces.
- A. Connes[37] already discussed how to transfer a given Dirac operator using Morita equivalence bimodules and compatible connections on them.

[36] Y. Manin, Real Multiplication and Noncommutative Geometry, in: The Legacy of Niels Henrik Abel, Springer (2004), 685-727. math.AG/0202109.

[37] A. Connes, Gravity coupled with Matter and the Foundations of Noncommutative Geometry, Commun. Math. Phys., 182, 155-176 (1996).

Morita Morphisms 2.

- In a forthcoming paper[38], we define a strictly related category of **Morita-Connes morphism** of spectral triples (that contains "inner deformations" as isomorphisms) as a pair (X, ∇) where X is a $\mathcal{A}_1$-$\mathcal{A}_2$-bimodule that is a Hilbert-C*-module over $\mathcal{A}_2$, ∇ is a Riemannian connection on the bimodule X and the composition is given by $X^3 := X^2 \otimes_{\mathcal{A}_1} X^1$ with connection:

$$\nabla^3(\xi_1 \otimes \xi_2)(h_3) := \xi_1 \otimes (\nabla^2 \xi_2)(h_3) + (\nabla^1 \xi_1)(\xi_2 \otimes h_3).$$

[38] P.B., R. Conti, W. Lewkeeratiyutkul, Categories of Spectral Triples and Morita Equivalence, in progress.

Morita Morphisms 3.

- In a remarkable paper, A. Connes-C. Consani-M. Marcolli[39] have been pushing even further the notion of "Morita morphism" defining morphisms between two algebras $\mathcal{A}, \mathcal{B}$ as "homotopy classes" of bimodules in G. Kasparov KK-theory $KK(\mathcal{A}, \mathcal{B})$. In this way, every morphism is an equivalence class determined by a bimodule that is further equipped with additional structure (Fredholm module).
- In the same paper, A. Connes and collaborators provide ground for considering "cyclic cohomology" as an "absolute cohomology of non-commutative motives" and the category of modules over the "cyclic category" (already defined by A. Connes-H. Moskovici) as a "NC motivic cohomology".

[39] A. Connes, C. Consani, M. Marcolli, Noncommutative Geometry and Motives: the Thermodynamics of Endomotives, math.QA/0512138.

Morita Morphisms 4.

- A. Connes-M. Marcolli (in chapter 8.4 of their book[40]) and M. Marcolli-A. Zainy[41] give a definition of "spectral correspondences" as Hilbert C*-bimodules providing a "bivariant version" of spectral triple.

[40] A. Connes, M. Marcolli, Noncommutative Geometry, Quantum Fields and Motives, AMS, 2008.

[41] M. Marcolli, A. Zainy, Covering, Correspondences and Noncommutative Geometry, preprint, 2007.

Morita Morphisms 5.

- S. Mahanta[42] is trying to relate "spectral correspondences" with the "geometric morphisms" of derived categories of the differential graded categories already used in the non-commutative algebraic geometry approach to non-commutative spaces[43].

[42] S. Mahanta, Noncommutative Correspondence Categories, work in progress.

[43] See S. Mahanta, On Some Approaches to Non-commutative Algebraic Geometry, `arXiv:0501166` for a survey.

Other Approaches 1.

Work is in progress on several other variants of morphisms of spectral triples, their mutual relations and their duality with geometrical categories:

- Modifications of the notion of spin morphisms that satisfy some "graded intertwining relations"[44] with the relevant operators, according to sign rules (depending on the dimension of the triple modulo 8) as proposed by A. Sitarz[45].

[44] P.B., R. Conti, W. Lewkeeratiyutkul, Morphism of Spectral Triples and Spin Manifolds, in progress.

[45] A. Sitarz, Habilitation Thesis, Jagellonian University (2002).

Other Approaches 2.

- Morphisms of AF-algebras and categories for spectral triples arising from AF-algebras[46] with C. Antonescu-E. Christensen construction[47].

[46] P.B., R. Conti, W. Lewkeeratiyutkul, Morphisms of Spectral Triples and AF-Algebras, in progress.

[47] C. Antonescu, E. Christensen, Spectral Triples for AF C* Algebras and Metrics on the Cantor Set, `math.OA/0309044`.

Categorification 1.

Categorification (L. Crane-D. Yetter[48], J. Baez-J. Dolan[49]) denotes the generic process in which ordinary algebraic set theoretic structures are replaced with categorical counterparts.

Vertical categorification is performed by promoting sets to categories, functions to functors, . . . , hence replacing a category with a bi-category and so on, increasing the "depth" of morphisms.

A kind of vertical categorification is a compulsory step in NCG: spaces are defined "dually" by "spectra" i.e. categories of representations of their algebras of functions and morphisms of non-commutative spaces are particular functors between "spectra".

[48] L. Crane, D. Yetter, Examples of Categorification, Cahiers de Topologie et Géometrie Différentielle Categoriques, **39** n. 1, 3-25 (1998).

[49] J. Baez, J. Dolan, Categorification, in: Higher Category Theory, eds. E. Getzler, M. Kapranov, Contemp. Math., **230**, 1-36 (1998), `math/9802029`.

Categorification 2.

In **horizontal categorification**[50], ordinary algebraic associative structures are interpreted as categories with only one object and suitable analog categories with more than one object are defined. In this case the passage is from endomorphisms of a single object to morphisms between different objects:

Monoids	Small Categories (Monoidoids)
Groups	Groupoids
Associative Unital Rings	Ringoid
Associative Unital Algebras	Algebroids
Unital C*-algebras	C*-categories (C*-algebroids)

[50] We use here the term "horizontal" in order to stress the difference from the proper "vertical categorification" process in which n-categories are substituted with $n+1$-categories. J. Baez prefers to use the term **oidization** for this case.

Horizontal Categorification of Gel'fand Duality 1.

It is our purpose here to find:

- suitable horizontal categorifications $\mathscr{T}$ of $\mathscr{T}^{(1)}$ and $\mathscr{A}$ of $\mathscr{A}^{(1)}$;
- to extend the categorical duality $(\Gamma^{(1)}, \Sigma^{(1)})$ between $\mathscr{T}^{(1)}$ and $\mathscr{A}^{(1)}$ of Gel'fand Theorem, to a natural categorical equivalence between $\mathscr{T}$ and $\mathscr{A}$:

$$\begin{array}{ccc} \mathscr{T}^{(1)} & \underset{\Sigma^{(1)}}{\overset{\Gamma^{(1)}}{\rightleftarrows}} & \mathscr{A}^{(1)} \\ \downarrow & & \downarrow \\ \mathscr{T} & \underset{\Sigma}{\overset{\Gamma}{\rightleftarrows}} & \mathscr{A}. \end{array}$$

Since $\mathscr{A}^{(1)}$ is a full subcategory of the category of C*-algebras, we expect to identify the horizontal categorification of $\mathscr{A}^{(1)}$ as a subcategory of a category of small C*-categories.

Horizontal Categorification of Gel'fand Duality 2.

In our forthcoming paper[51], in the setting of C*-categories, we provide a definition of "spectrum" of a commutative full C*-category as a one dimensional saturated unital Fell-bundle over a suitable groupoid (equivalence relation) and we prove a categorical Gel'fand duality theorem generalizing the usual Gel'fand duality between the categories of Abelian C*-algebras and compact Hausdorff spaces. ▸ categorical gelfand theorem

As a byproduct, we also obtain a spectral theorem for imprimitivity bimodules over Abelian C*-algebras: every such bimodule is obtained by "twisting" (by the 2 projection homeomorphisms) the symmetric bimodule of sections of a unique Hermitian line bundle over the graph of a unique homeomorphism between the spectra of the two C*-algebras. ▸ spectral theorem for bimodules ▸ generalizations ▸ gravity

[51] P.B., R. Conti, W. Lewkeeratiyutkul, Horizontal Categorification of Gel'fand's Theory, preprint (2007).

C*-categories 1.

A **C*-category**[52],[53] is a category $\mathcal{C}$ such that:

- the sets $\mathcal{C}_{AB} := \mathrm{Hom}_{\mathcal{C}}(B, A)$ are complex Banach spaces,
- the compositions are bilinear maps,
- the norm satisfies:

$$\|xy\| \leq \|x\| \cdot \|y\|, \quad \forall x \in \mathcal{C}_{AB}, \ \forall y \in \mathcal{C}_{BC},$$
$$\|\iota_A\| = 1, \quad \forall A \in \mathrm{Ob}_{\mathcal{C}},$$

[52] P. Ghez, R. Lima, J. Roberts, W*-categories, Pacific J. Math., **120**, 79-109 (1985).

[53] P. Mitchener, C*-categories, Proceedings of the London Mathematical Society, **84**, 375-404 (2002).

C*-categories 2.

- there is an involution $* : \mathrm{Hom}_{\mathcal{C}} \to \mathrm{Hom}_{\mathcal{C}}$ such that:

$$x^* \in \mathrm{Hom}_{\mathcal{C}}(B, A), \quad \forall x \in \mathrm{Hom}_{\mathcal{C}}(A, B),$$
$$(x + y)^* = x^* + y^*, \quad \forall x, y \in \mathcal{C}_{AB},$$
$$(\alpha \cdot x)^* = \overline{\alpha} \cdot x^*, \quad \forall \alpha \in \mathbb{C}, \ \forall x \in \mathcal{C}_{AB},$$
$$(xy)^* = y^* x^*, \quad \forall y \in \mathcal{C}_{BC}, \ \forall x \in \mathcal{C}_{AB},$$
$$(x^*)^* = x, \quad \forall x \in \mathcal{C}_{AB},$$
$$\|x^* x\| = \|x\|^2, \quad \forall x \in \mathcal{C}_{BA},$$
$$x^* x \in \mathcal{C}_{AA+}, \quad \forall x \in \mathcal{C}_{BA}.$$

C*-categories 3.

- In a C*-category $\mathcal{C}$, the sets $\mathcal{C}_{AA} := \mathrm{Hom}_{\mathcal{C}}(A, A)$ are unital C*-algebras for all $A \in \mathrm{Ob}_{\mathcal{C}}$.
- The sets $\mathcal{C}_{AB} := \mathrm{Hom}_{\mathcal{C}}(B, A)$ have a natural structure of unital Hilbert C*-bimodule on the C*-algebras $\mathcal{C}_{AA}$ on the right and $\mathcal{C}_{BB}$ on the left.

A C*-category is **commutative** if the C*-algebras $\mathcal{C}_{AA}$ are Abelian for all $A \in \mathrm{Ob}_{\mathcal{C}}$.
The C*-category $\mathcal{C}$ is **full** if all the bimodules $\mathcal{C}_{AB}$ are full[54].
A standard example is the C*-category of linear bounded maps between Hilbert spaces.

[54] In this case $\mathcal{C}_{AB}$ are imprimitivity bimodules.

Fell Bundles 1.

A **Banach bundle** is a $\mathbb{C}$-vector bundle (E, p, X), such that each fiber $E_x := p^{-1}(x)$ is a Banach space for all $x \in X$ in such a way that the maps $x \mapsto \|\sigma(x)\|$ are continuous for every section $\sigma \in \Gamma(X; E)$.

If the topological space X is equipped with the algebraic structure of category (let X^o be the set of its units, by $r, s : X \to X^o$ its range and source maps and by $X^n := \{(x_1, \dots, x_n) \in \times_{j=1}^n X \mid s(x_j) = r(x_{j+1})\}$ its set of n-composable morphisms), we further require that the composition $\circ : X^2 \to X$ is a continuous map.

If X is an involutive category i.e. there is a map $* : X \to X$ with the properties $(x^*)^* = x$ and $(x \circ y)^* = y^* \circ x^*$, for all $(x, y) \in X^2$, we also require $*$ to be continuos.

Fell Bundles 2.

A **Fell bundle**[55],[56],[57] over the (involutive) category X is a Banach bundle (E, p, X) whose total space E is equipped with

i) a multiplication defined on the set $E^2 := \{(e, f) \mid (p(e), p(f)) \in X^2\}$, denoted by $(e, f) \mapsto ef$, that satisfies the following properties:

$$e(fg) = (ef)g, \quad \forall (p(e), p(f), p(g)) \in X^3,$$
$$p(ef) = p(e) \circ p(f), \quad \forall e, f \in E^2,$$
$$\forall x, y \in X^2, \text{ the restriction of } (e, f) \mapsto ef \text{ to } E_x \times E_y \text{ is bilinear},$$
$$\|ef\| \leq \|e\| \cdot \|f\|, \forall e, f \in E^2,$$

[55] J. Fell, R. Doran, Representations of C*-algebras, Locally Compact Groups and Banach $*$-algebraic Bundles, Vol. 1, 2, Academic Press (1998).
[56] A. Kumjian, Fell Bundles over Groupoids, `math.OA/9607230`.
[57] R. Martins, Double Fell Bundles and Spectral Triples, `arXiv:0709.2972v2`.

Fell Bundles 3.

ii) an involution $*: E \to E$ that satisfies:

$$(e^*)^* = e, \quad \forall e \in E,$$
$$p(e^*) = p(e)^*, \forall e \in E,$$
$$\forall x \in X, \text{ the restriction of } e \mapsto e^* \text{ to } E_x \text{ is conjugate linear,}$$

iii) and moreover such that:

$$(ef)^* = f^* e^*, \quad \forall e, f \in E^2,$$
$$\|e^* e\| = \|e\|^2, \forall e \in E,$$
$$e^* e \geq 0, \quad \forall e \in E,$$

where, in the last line we mean that $e^* e$ is a positive element in the C*-algebra E_x with $x = p(e^* e)$

Fell Bundles 4.

It is in fact easy to see that for every $x \in X^o$, E_x is a C*-algebra.
A Fell bundle (E, p, X) is said to be **unital** if the C*-algebras E_x, for $x \in X^o$, are unital.
Note that the fiber E_x has a natural structure of Hilbert C*-bimodule over the C*-algebras $E_{r(x)}$ on the left and $E_{s(x)}$ on the right.
A Fell bundle is said to be **saturated** if the above Hilbert C*-bimodules E_x are full.
Note also that in a saturated Fell bundle, the Hilbert C*-bimodules E_x are imprimitivity bimodules.

Spaceoids

Let $\mathcal{O}$ be a set and X a compact Hausdorff topological space.
We denote by

$$\mathcal{R}_{\mathcal{O}} := \{(A, B) \mid A, B \in \mathcal{O}\}$$

the "total" equivalence relation in $\mathcal{O}$ and by

$$\Delta_X := \{(p, p) \mid p \in X\}$$

the "diagonal" equivalence relation in X.

Definition

A **topological spaceoid** $(\mathcal{E}, \pi, \mathcal{X})$ is a saturated unital rank-one Fell bundle over the product involutive topological category $\mathcal{X} := \Delta_X \times \mathcal{R}_{\mathcal{O}}$.

categorical gelfand theorem

Morphisms of Spaceoids.

Let $(\mathcal{E}_j, \pi_j, \mathcal{X}_j)$, for $j = 1, 2$, be two spaceoids[58]

Definition

A morphism of spaceoids $(\mathcal{E}_1, \pi_1, \mathcal{X}_1) \xrightarrow{(f, \mathcal{F})} (\mathcal{E}_2, \pi_2, \mathcal{X}_2)$ is a pair $(f, \mathcal{F})$ where

- $f := (f_\Delta, f_\mathcal{R})$ with $f_\Delta : \Delta_1 \to \Delta_2$ a continuous map of topological spaces and $f_\mathcal{R} : \mathcal{R}_1 \to \mathcal{R}_2$ an isomorphism of equivalence relations;
- $\mathcal{F} : f^\bullet(\mathcal{E}_2) \to \mathcal{E}_1$ is a fiberwise linear $*$-functor such that $\pi_1 \circ \mathfrak{F} = (\pi_2)^f$, where $(f^\bullet(\mathcal{E}_2), \pi_2^f, \mathcal{X}_1)$ denotes an f-pull-back of $(\mathcal{E}_2, \pi_2, \mathcal{X}_2)$.

58 Where $\mathcal{X}_j = \Delta_{X_j} \times \mathcal{R}_{\mathcal{O}_j}$, with $\mathcal{O}_j$ sets and X_j compact Hausdorff topological spaces for $j = 1, 2$.

Category of Spaceoids 1.

Topological spaceoids constitute a category if composition is defined by

$$(g, \mathcal{G}) \circ (f, \mathcal{F}) := (g \circ f, \mathcal{F} \circ f^\bullet(\mathcal{G}))$$

with identities given by

$$\iota(\mathcal{E}, \pi, \mathcal{X}) := (\iota_\mathcal{X}, \iota_\mathcal{E}).$$

Note that $f^\bullet(g^\bullet(\mathcal{E}_3))$ is naturally a $(g \circ f)$-pull-back of $(\mathcal{E}_3, \pi_3, \mathcal{X}_3)$ and that $(\mathcal{E}, \pi, \mathcal{X})$ is a natural $\iota_\mathcal{X}$-pull-back of itself.

Category of Spaceoids 2.

The category $\mathscr{T}^{(1)}$ of continuous maps between compact Hausdorff spaces can be naturally identified with the full subcategory of the category $\mathscr{T}$ of spaceoids with index set $\mathcal{O}$ containing a single element.

To every object $X \in \mathrm{Ob}_{\mathscr{T}^{(1)}}$ we associate the trivial $\mathbb{C}$-line bundle $\mathcal{X}_X \times \mathbb{C}$ over the involutive category $\mathcal{X}_X := \Delta_X \times \mathcal{R}_{\mathcal{O}_X}$ with $\mathcal{O}_X := \{X\}$ the one point set.

To every continuous map $f : X \to Y$ in $\mathscr{T}^{(1)}$ we associate the morphism $(g, \mathcal{G})$ with $g_\Delta(p,p) := (f(p), f(p))$, $g_\mathcal{R} : (X,X) \mapsto (Y,Y)$ and $\mathcal{G} := \iota_{\mathcal{X}_X \times \mathbb{C}}$.

Note that the trivial bundle over $\mathcal{X}_X$ is naturally a f-bull-back of the trivial bundle over $\mathcal{X}_Y$ hence $\mathcal{G}$ can be taken as the identity map.

The Category of Small C*-categories.

Let $\mathcal{C}$ and $\mathcal{D}$ be two full commutative small C*-categories (with the same cardinality of the set of objects). Denote by $\mathcal{C}_o$ and $\mathcal{D}_o$ their sets of identities.

A morphism $\Phi : \mathcal{C} \to \mathcal{D}$ is an object bijective $*$-functor i.e. a map such that

$$\Phi(x+y) = \Phi(x) + \Phi(y), \ \forall x, y \in \mathcal{C}_{AB},$$
$$\Phi(a \cdot x) = a \cdot \Phi(x), \ \forall x \in \mathcal{C}, \ \forall a \in \mathbb{C},$$
$$\Phi(x \circ y) = \Phi(x) \circ \Phi(y), \ \forall x \in \mathcal{C}_{CB}, \ y \in \mathcal{C}_{BA}$$
$$\Phi(x^*) = \Phi(x)^*, \ \forall x \in \mathcal{C}_{AB},$$
$$\Phi(\iota) \in \mathcal{D}_o, \ \forall \iota \in \mathcal{C}_o,$$
$$\Phi_o := \Phi|_{\mathcal{C}_o} : \mathcal{C}_o \to \mathcal{D}_o \quad \text{is bijective.}$$

The Section Functor Γ on Objects.

To every spaceoid $(\mathcal{E}, \pi, \mathcal{X})$, with $\mathcal{X} := \Delta_X \times \mathcal{R}_{\mathcal{O}}$, we can associate a full commutative C*-category $\Gamma(\mathcal{E})$ as follows:

- $\mathrm{Ob}_{\Gamma(\mathcal{E})} := \mathcal{O}$;
- $\forall A, B \in \mathrm{Ob}_{\Gamma(\mathcal{E})}, \quad \mathrm{Hom}_{\Gamma(\mathcal{E})}(B, A) := \Gamma(\Delta_X \times \{(A, B)\}; \mathcal{E})$, where $\Gamma(\Delta_X \times \{(A, B)\}; \mathcal{E})$ denotes the set of continuous sections $\sigma : \Delta_X \times \{(A, B)\} \to \mathcal{E}$, $\sigma : p_{AB} \mapsto \sigma^{AB}_p \in \mathcal{E}_{p_{AB}}$ of the restriction of $\mathcal{E}$ to the base space $\Delta_X \times \{(A, B)\} \subset \mathcal{X}$.
- for all $\sigma \in \mathrm{Hom}_{\Gamma(\mathcal{E})}(A, B)$ and $\rho \in \mathrm{Hom}_{\Gamma(\mathcal{E})}(B, C)$:

$$\rho \circ \sigma : p_{AC} \mapsto (\rho \circ \sigma)^{AC}_p := \rho^{AB}_p \circ \sigma^{BC}_p,$$
$$\sigma^* : p_{BA} \mapsto (\sigma^*)^{BA}_p := (\sigma^{AB}_p)^*,$$
$$\|\sigma\| := \sup_{p \in \Delta_X} \|\sigma^{AB}_p\|_{\mathcal{E}},$$

with operations taken in the total space $\mathcal{E}$ of the Fell bundle.

The Section Functor Γ on Morphisms.

We extend now the definition of Γ to the morphism of $\mathscr{T}$ in order to obtain a contravariant functor.

Let $(f, \mathcal{F})$ be a morphism in $\mathscr{T}$ from $(\mathcal{E}_1, \pi_1, \mathcal{X}_1)$ to $(\mathcal{E}_2, \pi_2, \mathcal{X}_2)$. Given $\sigma \in \Gamma(\mathcal{E}_2)$, by we consider the unique section $f^\bullet(\sigma) : \mathcal{X}_1 \to f^\bullet(\mathcal{E}_2)$ such that $f^{\pi_2} \circ f^\bullet(\sigma) = \sigma \circ f$ and the composition $\mathcal{F} \circ f^\bullet(\sigma)$. In this way we get a map

$$\Gamma_{(f,\mathcal{F})} : \Gamma(\mathcal{E}_2) \to \Gamma(\mathcal{E}_1), \quad \Gamma_{(f,\mathcal{F})} : \sigma \mapsto \mathcal{F} \circ f^\bullet(\sigma), \quad \forall \sigma \in \Gamma(\mathcal{E}_2).$$

Proposition

For any morphism $(\mathcal{E}_1, \pi_1, \mathcal{X}_1) \xrightarrow{(f,\mathcal{F})} (\mathcal{E}_2, \pi_2, \mathcal{X}_2)$ *in* $\mathscr{T}$, *the map* $\Gamma_{(f,\mathcal{F})} : \Gamma(\mathcal{E}_2) \to \Gamma(\mathcal{E}_1)$ *is a morphism in* $\mathscr{A}$.
The pair of maps $\Gamma : (\mathcal{E}, \pi, \mathcal{X}) \mapsto \Gamma(\mathcal{E})$ *and* $\Gamma : (f, \mathcal{F}) \mapsto \Gamma_{(f,\mathcal{F})}$ *gives a contravariant functor from the category* $\mathscr{T}$ *of spaceoids to the category* $\mathscr{A}$ *of small full commutative C*-categories.*

The Spectrum functor Σ on Objects 1.

We proceed to associate to every commutative full C*-category $\mathcal{C}$ its spectral spaceoid $\Sigma(\mathcal{C}) := (\mathcal{E}^{\mathcal{C}}, \pi^{\mathcal{C}}, \mathcal{X}^{\mathcal{C}})$.

- The set $[\mathcal{C}; \mathbb{C}]$ of $\mathbb{C}$-valued $*$-functors $\omega : \mathcal{C} \to \mathbb{C}$, with the weakest topology making all evaluations continuous, is a compact Hausdorff topological space.
- By definition two $*$-functors $\omega_1, \omega_2 \in [\mathcal{C}; \mathbb{C}]$ are **unitarily equivalent** if there exists a "unitary" natural trasformation $A \mapsto \nu_A \in \mathbb{T}$ between them. This is true iff $\omega_1|_{\mathcal{C}_{AA}} = \omega_2|_{\mathcal{C}_{AA}}$ for all $A \in \mathrm{Ob}_{\mathcal{C}}$.
- Let $\mathrm{Sp}_b(\mathcal{C}) := \{[\omega] \mid \omega \in [\mathcal{C}; \mathbb{C}]\}$ denote the **base spectrum** of $\mathcal{C}$, defined as the set of unitary equivalence classes of $*$-functors in $[\mathcal{C}; \mathbb{C}]$. It is a compact Hausdorff space with the quotient topology induced by the map $\omega \mapsto [\omega]$.

The Spectrum functor Σ on Objects 2.

- Let $\mathcal{X}^{\mathcal{C}} := \Delta^{\mathcal{C}} \times \mathcal{R}^{\mathcal{C}}$ be the direct product of the compact Hausdorff $*$-category $\Delta^{\mathcal{C}} := \Delta_{\mathsf{Sp}_b(\mathcal{C})}$ and the topologically discrete $*$-category $\mathcal{R}^{\mathcal{C}} := \mathcal{C}/\mathcal{C} \simeq \mathcal{R}_{\mathsf{Ob}_{\mathcal{C}}}$.
- For $\omega \in [\mathcal{C}; \mathbb{C}]$, the set $\mathcal{J}_\omega := \{x \in \mathcal{C} \mid \omega(x) = 0\}$ is an ideal in $\mathcal{C}$ and $\mathcal{J}_{\omega_1} = \mathcal{J}_{\omega_2}$ if $[\omega_1] = [\omega_2]$.
- Denoting $[\omega]_{AB}$ the point $([\omega], (A, B)) \in \mathcal{X}^{\mathcal{C}}$, we define:

$$\mathcal{J}_{[\omega]_{AB}} := \mathcal{J}_\omega \cap \mathcal{C}_{AB}, \quad \mathcal{E}^{\mathcal{C}}_{[\omega]_{AB}} := \frac{\mathcal{C}_{AB}}{\mathcal{J}_{[\omega]_{AB}}}, \quad \mathcal{E}^{\mathcal{C}} := \biguplus_{[\omega]_{AB} \in \mathcal{X}^{\mathcal{C}}} \mathcal{E}^{\mathcal{C}}_{[\omega]_{AB}}.$$

Proposition

The map $\pi^{\mathcal{C}} : \mathcal{E}^{\mathcal{C}} \to \mathcal{X}^{\mathcal{C}}$, that sends an element $e \in \mathcal{E}^{\mathcal{C}}_{[\omega]_{AB}}$ to the point $[\omega]_{AB} \in \mathcal{X}^{\mathcal{C}}$ has a natural structure of unital saturated rank one Fell bundle over the topological involutive category $\mathcal{X}^{\mathcal{C}}$.

The Spectrum functor Σ on Morphisms 1.

Let $\Phi : \mathcal{C} \to \mathcal{D}$ be an object-bijective $*$-functor between two small commutative full C*-categories with spaceoids $\Sigma(\mathcal{C}), \Sigma(\mathcal{D}) \in \mathscr{T}$.

We define a morphism $\Sigma^\Phi : \Sigma(\mathcal{D}) \xrightarrow{(\lambda^\Phi, \Lambda^\Phi)} \Sigma(\mathcal{C})$ in the category $\mathscr{T}$:

- $\lambda^\Phi : \mathcal{X}^\mathcal{D} \xrightarrow{(\lambda^\Phi_\Delta, \lambda^\Phi_\mathcal{R})} \mathcal{X}^\mathcal{C}$ where
  $\lambda^\Phi_\mathcal{R}(A, B) := (\Phi_o^{-1}(A), \Phi_o^{-1}(B))$, for all $(A, B) \in \mathcal{R}_{\mathrm{Ob}_\mathcal{D}}$;
  $\lambda^\Phi_\Delta([\omega]) := [\omega \circ \Phi] \in \Delta_{\mathrm{Sp}_b(\mathcal{C})}$, for all $[\omega] \in \Delta_{\mathrm{Sp}_b(\mathcal{D})}$.
- The bundle $\biguplus_{[\omega]_{AB} \in \mathcal{X}^\mathcal{D}} \dfrac{\mathcal{C}_{\lambda^\Phi_\mathcal{R}(AB)}}{\mathcal{J}_{\lambda^\Phi([\omega]_{AB})}}$ with the maps
  $\pi^\Phi : ([\omega]_{AB},\ x + \mathcal{J}_{\lambda^\Phi([\omega]_{AB})}) \mapsto [\omega]_{AB} \in \mathcal{X}^\mathcal{D}, \quad x \in \mathcal{C}_{\lambda^\Phi_\mathcal{R}(AB)}$,
  $\Phi^\pi : ([\omega]_{AB},\ x + \mathcal{J}_{\lambda^\Phi([\omega]_{AB})}) \mapsto (\lambda^\Phi([\omega]_{AB}), x + \mathcal{J}_{\lambda^\Phi([\omega]_{AB})}) \in \mathcal{E}^\mathcal{C}$
  is a λ^Φ-pull-back $(\lambda^\Phi)^\bullet(\mathcal{E}^\mathcal{C})$ of the Fell bundle $(\mathcal{E}^\mathcal{C}, \pi^\mathcal{C}, \mathcal{X}^\mathcal{C})$.

The Spectrum functor Σ on Morphisms 2.

- Since $\Phi(\mathcal{I}_{\lambda^\Phi([\omega]_{AB})}) \subset \mathcal{I}_{[\omega]_{AB}}$ for $[\omega]_{AB} \in \mathcal{X}^{\mathcal{D}}$, we define $\Lambda^\Phi : (\lambda^\Phi)^\bullet(\mathcal{E}^{\mathcal{C}}) \to \mathcal{E}^{\mathcal{D}}$ by $\left([\omega]_{AB},\ x + \mathcal{I}_{\lambda^\Phi([\omega]_{AB})}\right) \mapsto \left([\omega]_{AB},\ \Phi(x) + \mathcal{I}_{[\omega]_{AB}}\right)$.

Proposition

For any morphism $\mathcal{C} \xrightarrow{\Phi} \mathcal{D}$ in $\mathscr{A}$, the map $\Sigma(\mathcal{D}) \xrightarrow{\Sigma^\Phi} \Sigma(\mathcal{C})$ is a morphism of spectral spaceoids. The pair of maps $\Sigma : \mathcal{C} \mapsto \Sigma(\mathcal{C})$ and $\Sigma : \Phi \mapsto \Sigma^\Phi$ give a contravariant functor $\Sigma : \mathscr{A} \to \mathscr{T}$, from the category $\mathscr{A}$ of object-bijective $*$-functors between small commutative full C-categories to the category $\mathscr{T}$ of spaceoids.*

Gel'fand Duality Theorem for C*-categories.

There exists a duality (Γ, Σ) between the category $\mathscr{T}$ of object-bijective morphisms between spaceoids and the category $\mathscr{A}$ of object-bijective $*$-functors between small commutative full C*-categories, where

- Γ is the functor that to every spaceoid $(\mathcal{E}, \pi, \mathcal{X}) \in \mathrm{Ob}_{\mathscr{T}}$ associates the small commutative full C*-category $\Gamma(\mathcal{E})$ and that to every morphism between spaceoids $(f, \mathcal{F}) : (\mathcal{E}_1, \pi_1, \mathcal{X}_1) \to (\mathcal{E}_2, \pi_2, \mathcal{X}_2)$ associates the $*$-functor $\Gamma_{(f,\mathcal{F})}$;
- Σ is the functor that to every small commutative full C*-category $\mathcal{C}$ associates its spectral spaceoid $\Sigma(\mathcal{C})$ and that to every object-bijective $*$-functor $\Phi : \mathcal{C} \to \mathcal{D}$ of C*-categories in $\mathscr{A}$ associates the morphism $\Sigma^{\Phi} : \Sigma(\mathcal{D}) \to \Sigma(\mathcal{C})$ between spaceoids.

C*-categories spectral theorem for bimodules categorification

The Gel'fand Natural Transform $\mathfrak{G}$.

- The natural isomorphism $\mathfrak{G} : \mathcal{I}_{\mathscr{A}} \to \Gamma \circ \Sigma$ is provided by the **horizontally categorified Gel'fand transforms** $\mathfrak{G}_{\mathcal{C}} : \mathcal{C} \to \Gamma(\Sigma(\mathcal{C}))$ defined by

$$\mathfrak{G}_{\mathcal{C}} : \mathcal{C} \to \Gamma(\mathcal{E}^{\mathcal{C}}), \quad \mathfrak{G}_{\mathcal{C}} : x \mapsto \hat{x} \quad \text{where}$$
$$\hat{x}^{AB}_{[\omega]} := x + \mathcal{J}_{[\omega]_{AB}}, \quad \forall x \in \mathcal{C}_{AB}.$$

In particular:

Proposition

The functor $\Gamma : \mathscr{T} \to \mathscr{A}$ is representative i.e. given a commutative full C-category $\mathcal{C}$, the Gel'fand transform $\mathfrak{G}_{\mathcal{C}} : \mathcal{C} \to \Gamma(\Sigma(\mathcal{C}))$ is a full isometric (hence faithful) $*$-functor.*

The Evaluation Natural Transform $\mathfrak{E}$.

- The natural isomorphism $\mathfrak{E} : \mathcal{I}_{\mathscr{T}} \to \Sigma \circ \Gamma$ is provided by the **horizontally categorified "evaluation" transforms** $\mathfrak{E}_{\mathcal{E}} : (\mathcal{E}, \pi, \mathcal{X}) \xrightarrow{(\eta^{\mathcal{E}}, \Omega^{\mathcal{E}})} \Sigma(\Gamma(\mathcal{E}))$, defined as follows:
  - $\eta^{\mathcal{E}}_{\mathcal{R}}(A, B) := (A, B), \quad \forall (A, B) \in \mathcal{R}_{\mathcal{O}}$.
  - $\eta^{\mathcal{E}}_{\Delta} : p \mapsto [\gamma \circ \mathrm{ev}_p] \quad \forall p \in \Delta_X$, where the evaluation map $\mathrm{ev}_p : \Gamma(\mathcal{E}) \to \biguplus_{(AB) \in \mathcal{R}_{\mathcal{O}}} \mathcal{E}_{p_{AB}}$ given by $\mathrm{ev}_p : \sigma \mapsto \sigma_p^{AB}$ is a $*$-functor with values in a one dimensional C*-category that determines[59] a unique point $[\gamma \circ \mathrm{ev}_p] \in \Delta_{\mathrm{Sp}_b(\Gamma(\mathcal{E}))}$.
  - $\biguplus_{p_{AB} \in \mathcal{X}} \Gamma(\mathcal{E})_{\eta^{\mathcal{E}}_{\mathcal{R}}(AB)} / \mathcal{I}_{\eta^{\mathcal{E}}(p_{AB})}$ equipped with the natural projection $(p_{AB},\ \sigma + \mathcal{I}_{\eta^{\mathcal{E}}(p_{AB})}) \mapsto p_{AB}$, and with the $\mathcal{E}^{\Gamma(\mathcal{E})}$-valued map $(p_{AB},\ \sigma + \mathcal{I}_{\eta^{\mathcal{E}}(p_{AB})}) \mapsto \sigma + \mathcal{I}_{\eta^{\mathcal{E}}(p_{AB})}$, is a $\eta^{\mathcal{E}}$-pull-back $(\eta^{\mathcal{E}})^{\bullet}(\mathcal{E}^{\Gamma(\mathcal{E})})$ of $\Sigma(\Gamma(\mathcal{E}))$.

[59] There is always a $\mathbb{C}$ valued $*$-functor $\gamma : \biguplus_{(AB) \in \mathcal{R}_{\mathcal{O}}} \mathcal{E}_{p_{AB}} \to \mathbb{C}$ and any two compositions of ev_p with such $*$-functors are unitarily equivalent because they coincide on the diagonal C*-algebras $\mathcal{E}_{p_{AA}}$.

The Evaluation Natural Transform $\mathfrak{E}$.

- $\Omega^{\mathcal{E}} : (\eta^{\mathcal{E}})^{\bullet}(\mathcal{E}^{\Gamma(\mathcal{E})}) \to \mathcal{E}$ is defined by
  $\Omega^{\mathcal{E}} : (p_{AB},\ \sigma + \mathcal{I}_{\eta^{\mathcal{E}}(p_{AB})}) \mapsto \sigma_p^{AB}, \quad \forall \sigma \in \Gamma(\mathcal{E})_{AB}, \quad p_{AB} \in \mathcal{X}.$

In particular, with such definitions we can prove:

Proposition

The functor $\Sigma : \mathscr{A} \to \mathscr{T}$ is representative i.e. given a spaceoid $(\mathcal{E}, \pi, \mathcal{X})$, the evaluation transform $\mathfrak{E}_{\mathcal{E}} : (\mathcal{E}, \pi, \mathcal{X}) \to \Sigma(\Gamma(\mathcal{E}))$ is an isomorphism in the category of spaceoids.

Spectral Theorem for Imprimitivity Abelian C*-bimodules.

Theorem (P.B.-R. Conti-W. Lewkeeratiyutkul)

Given an imprimitivity Hilbert C-bimodule* ${}_{\mathcal{A}}M_{\mathcal{B}}$ *over the Abelian unital C*-algebras* $\mathcal{A}, \mathcal{B}$*, there exists a canonical homeomorphism*[60] $R_{BA} : \mathrm{Sp}(\mathcal{A}) \to \mathrm{Sp}(\mathcal{B})$ *and a Hermitian line bundle* E *over* R_{BA} *such that* ${}_{\mathcal{A}}M_{\mathcal{B}}$ *is isomorphic to the (left/right) "twisting"*[61] *of the symmetric bimodule* $\Gamma(R_{BA}; E)_{C(R_{BA};\mathbb{C})}$ *of sections of the bundle* E *by the two "pull-back" isomorphisms* $\pi_A^\bullet : \mathcal{A} \to C(R_{BA}; \mathbb{C})$, $\pi_B^\bullet : \mathcal{B} \to C(R_{BA}; \mathbb{C})$. ▸ categorification ▸ generalizations

[60] R_{BA} is a compact Hausdorff subspace of $\mathrm{Sp}(\mathcal{A}) \times \mathrm{Sp}(\mathcal{B})$ homeomorphic to $\mathrm{Sp}(\mathcal{A})$ via the projection $\pi_A : R_{BA} \to \mathrm{Sp}(\mathcal{A})$ and to $\mathrm{Sp}(\mathcal{B})$ via the projection $\pi_B : R_{BA} \to \mathrm{Sp}(\mathcal{B})$.

[61] If M is a left module over $\mathcal{C}$ and $\phi : \mathcal{A} \to \mathcal{C}$ is an isomorphism, the left twisting of M by ϕ is the module over $\mathcal{A}$ defined by $a \cdot x := \phi(a)x$ for $a \in \mathcal{A}$ and $x \in M$.

Generalizations and Applications of Gel'fand Duality 1.

We are now working on a number of generalizations of our horizontal categorified Gel'fand duality:

- Gel'fand duality for general $*$-functors and $*$-relators.
- Gel'fand duality for non-full C*-categories.
- Categorification of Dauns-Hofmann spectral theorem and dualities for non-commutative C*-categories or more generally higher rank Fell bundles.
- Gel'fand dualities for commutative higher C*-categories and "higher-spaceoids".[62]
- Spectral triples over C*-categories and horizontal categorification of spectral triples and other spectral geometries.

[62] Very interesting is the possible relation between such "higher" spectra and the notions of stacks and gerbes already used in higher gauge theory.

Generalizations and Applications of Gel'fand Duality 2.

Extremely intriguing for its possible physical implications is the appearence of a natural local gauge structure on the spectra: the spectrum is no more just a (topological) space, but a special fiber bundle.

Every isomorphism class of a full commutative C*-category can be identified with an equivalence relation in the Picard-Morita 1-category of Abelian unital C*-algebras. In practice a C*-category is just a "strict implementation" of an equivalence relation subcategory of Picard-Morita. Since morphism of spectral triples are essentially "special cases" of Morita morphisms, we are now trying to develop a notion of horizontal categorification of spectral triples (and of other spectral geometries) in order to identify a correct definition of morphism of spectral triples that supports a duality with a suitable spectrum (in the commutative case).

Strict Higher C*-categories 1.[63]

Given $n \in \mathbb{N}$, a **globular** n**-set**

$$\mathcal{C}^0 \leftleftarrows \mathcal{C}^1 \leftleftarrows \cdots \mathcal{C}^{m-1} \leftleftarrows \mathcal{C}^m \leftleftarrows \cdots \leftleftarrows \mathcal{C}^n,$$

is given by:

- for all $m = 0, \dots, n$, a collections of classes $\mathcal{C}^m$ whose elements are called m-**arrows**,
- for all $m = 1, \dots, n$, a pair of **source, target** maps $s_m, t_m : \mathcal{C}^m \to \mathcal{C}^{m-1}$ such that for all $m = 1, \dots, n-1$:

$$s_m \circ s_{m+1} = s_m \circ t_{m+1}$$
$$t_m \circ s_{m+1} = t_m \circ t_{m+1}.$$

[63] P.B., Roberto Conti, Wicharn Lewkeeratiyutkul, Noppakhun Suthichitranont, Strict Higher C*-categories, in preparation.

Strict Higher C*-categories 2.

A (globular) **strict** ***n*****-category** is a globular *n*-set such that

- for all $0 \leq p < m \leq n$, there is a partial ***p*****-composition** map

$$\circ^m_p : \mathcal{C}^m \times_{\mathcal{C}^p} \mathcal{C}^m \to \mathcal{C}^m, \quad (x, y) \mapsto x \circ^m_p y,$$

  defined on the set $\mathcal{C}^m \times_{\mathcal{C}^p} \mathcal{C}^m$ of p-composable m-arrows $(x, y) \in \mathcal{C}^m \times_{\mathcal{C}^p} \mathcal{C}^m \Leftrightarrow t_{p+1} \circ \cdots \circ t_m(y) = s_{p+1} \circ \cdots \circ s_m(x)$,

- for all $m = 0, \dots, n-1$ there is an **identity map**

$$\iota_m : \mathcal{C}^m \to \mathcal{C}^{m+1},$$

in such a way that the following axioms are satisfied:

Strict Higher C*-categories 3.

- for all $m = 0, \dots, n$, for all $p = 0, \dots, m-1$, for all $(x, y) \in \mathcal{C}^m \times_{\mathcal{C}^p} \mathcal{C}^m$,

$$s_m(x \circ_p^m y) = s_m(y), \quad t_m(x \circ_p^m y) = t_m(x), \quad \text{if } p = m-1;$$
$$s_m(x \circ_p^m y) = s_m(x) \circ_p^{m-1} s_m(y), \quad \text{if } p = 0, \dots, m-2,$$
$$t_m(x \circ_p^m y) = t_m(x) \circ_p^{m-1} t_m(y), \quad \text{if } p = 0, \dots, m-2;$$

- for all $x \in \mathcal{C}^m$,

$$s_{m+1}(\iota_m(x)) = x, \quad t_{m+1}(\iota_m(x)) = x;$$

- for all $m = 1, \dots, n$ and $p = 0, \dots, m-1$ and for all $x, y, z \in \mathcal{C}^m$ with $(x, y), (y, z) \in \mathcal{C}^m \times_{\mathcal{C}^p} \mathcal{C}^m$,

$$(x \circ_p^m y) \circ_p^m z = x \circ_p^m (y \circ_p^m z);$$

Strict Higher C*-categories 4.

- for all $m = 1, \dots, n$, for all $p = 0, \dots, m-1$,
  for all $x \in \mathcal{C}^m$,
  $$\Big(\iota_{m-1} \circ \cdots \circ \iota_p\big(t_{p+1} \circ \cdots \circ t_m(x)\big)\Big) \circ_p^m x = x,$$
  $$x = x \circ_p^m \Big(\iota_{m-1} \circ \cdots \circ \iota_p\big(s_{p+1} \circ \cdots \circ s_m(x)\big)\Big);$$
- for all $m = 2, \dots, n$, for all $p, q = 0, \dots, m-1$, with $q < p$,
  for all $w, x, y, z \in \mathcal{C}^m$ such that $(w, x), (y, z) \in \mathcal{C}^m \times_{\mathcal{C}^p} \mathcal{C}^m$
  and $(w, y), (x, z) \in \mathcal{C}^m \times_{\mathcal{C}^q} \mathcal{C}^m$,
  $$(w \circ_p^m x) \circ_q^m (y \circ_p^m z) = (w \circ_q^m y) \circ_p^m (x \circ_q^m z);$$
- for all $m = 1, \dots, n-1$, for all $p = 0, \dots, m-1$,
  for all $(x, y) \in \mathcal{C}^m \times_{\mathcal{C}^p} \mathcal{C}^m$,
  $$\iota_m(x \circ_p^m y) = \iota_m(x) \circ_p^{m+1} \iota_m(y).$$

Strict Higher C*-categories 5.

A (globular) **strict involutive** n**-category** is a strict n-category that is equipped with a family of "involutions" $*^m : \mathcal{C}^m \to \mathcal{C}^m$, for $0 < m \leq n$, that satisfy the following properties:

- for all $x \in \mathcal{C}^m$, $\quad s_m(x^{*^m}) = t_m(x), \quad t_m(x^{*^m}) = s_m(x)$,
- for all $x, y \in \mathcal{C}^m \times_{\mathcal{C}^p} \mathcal{C}^m$ with $m = 1, \dots, n$,

$$(x \circ^m_p y)^{*^m} = y^{*^m} \circ^m_p x^{*^m}, \quad \text{for } 0 \leq p = m-1,$$
$$(x \circ^m_p y)^{*^m} = x^{*^m} \circ^m_p y^{*^m}, \quad \text{for } 0 \leq p < m-1,$$

- for all $x \in \mathcal{C}^m$, $\quad (x^{*^m})^{*^m} = x$.

For $m = 0$ we do not require an involution, alternatively we can assume the property of "hermitianity of zero arrows" i.e.

- for all $x \in \mathcal{C}^0$, $x^{*^0} = x$.

Strict Higher C*-categories 6.

It is also possible to require involutions $*^m_q : \mathcal{C}^m \to \mathcal{C}^m$ of depth q for $0 \leq q < m \leq n$ with the properties:

- for all $x \in \mathcal{C}^m$, $s_m(x^{*^m_{m-1}}) = t_m(x), \quad t_m(x^{*^m_{m-1}}) = s_m(x)$, and, for $q < m-1$,
  $s_m(x^{*^m_q}) = (s_m(x))^{*^{m-1}_q}, \quad t_m(x^{*^m_q}) = (t_m(x))^{*^{m-1}_q}$;
- for all $x, y \in \mathcal{C}^m \times_{\mathcal{C}^p} \mathcal{C}^m$ with $m = 1, \dots, n$ and $p = 0, \dots, m-1$,
  $$(x \circ^m_p y)^{*^m_q} = y^{*^m_q} \circ^m_p x^{*^m_q}, \quad \text{if} \quad q = m-1,$$
  $$(x \circ^m_p y)^{*^m_q} = x^{*^m_q} \circ^m_p y^{*^m_q}, \quad \text{if} \quad q \neq m-1;$$
- for all $x \in \mathcal{C}^m$, $(x^{*^m_q})^{*^m_q} = x$.

Strict Higher C*-categories 7.

As "very tentative" proposal, we define a **strict-n-C*-category** to be a strict involutive n-category such that

- for all $m = 1, \dots, n$, for all $x, y \in \mathcal{C}^{m-1}$, the sets $\mathcal{C}^m(x, y) := \{z \in \mathcal{C}^m \mid s_m(z) = y,\ t_m(z) = x\}$ are Banach spaces with norm denoted by $x \mapsto \|x\|_m$.
- for $0 \le p < m$, for all $w, x, y, z \in \mathcal{C}^{m-1}$ such that $\mathcal{C}^m(w, x) \times \mathcal{C}^m(y, z) \subset \mathcal{C}^m \times_{\mathcal{C}^p} \mathcal{C}^m$, the maps $\circ^m_p : \mathcal{C}^m(w, x) \times \mathcal{C}^m(y, z) \to \mathcal{C}^m$ are "bilinear",
- for all $m = 1, \dots, n$, for all $x, y \in \mathcal{C}^{m-1}$, the maps $*^m : \mathcal{C}^m(x, y) \to \mathcal{C}^m$ are conjugate linear;

Strict Higher C*-categories 8.

- for all $m = 1, \dots, n$, for all $p = 0, \dots, m-1$, for all pairs $(x, y) \in \mathcal{C}^m \times_{\mathcal{C}^p} \mathcal{C}^m$,

$$\|x \circ^m_p y\|_m \leq \|x\|_m \cdot \|y\|_m;$$

- for all $m = 1, \dots, n$ and $0 \leq p < m$, for all $(x^{*^m}, x) \in \mathcal{C}^m \times_{\mathcal{C}^p} \mathcal{C}^m$,

$$\|x^{*^m} \circ^m_p x\|_m = \|x\|^2_m, \tag{4.1}$$

Strict Higher C*-categories 9.

Note that the above properties already imply that, for all $m = 1, \dots, n$ and for all $x \in \mathcal{C}_{m-1}$, the set $\mathcal{C}^m(x, x)$ is a C*-algebra with multiplication $\circ^m_{m-1}$ and involution $*^m$ and hence the following final condition is meaningful:

- for all $m = 1, \dots, n$, for all $x \in \mathcal{C}^m(u, v)$,

$$x^{*^m} \circ^m_{m-1} x \in \mathcal{C}^m(u, u)_+$$

  i.e. $x^{*^m} \circ^m_p x$ is a positive element in the C*-algebra $\mathcal{C}^m(u, u)$.

Strict Higher C*-categories 10.

A **left module** ${}_{\mathcal{C}}\mathcal{M}$ **over the** ***n*****-category** $\mathcal{C}$ is given by

$$
\begin{array}{ccccccccc}
\mathcal{C}^0 & \xleftarrow[s]{t} & \mathcal{C}^1 & \xleftarrow[s]{t} & \cdots & \xleftarrow[s]{t} & \mathcal{C}^{n-1} & \xleftarrow[s]{t} & \mathcal{C}^n \\
& \nwarrow^{\tau} & & \nwarrow^{\tau} & & & & \nwarrow^{\tau} & \\
& & \mathcal{M}^1 & & \mathcal{M}^2 & & \cdots & & \mathcal{M}^n
\end{array}
$$

where for all $m = 0, \dots, n$, $\tau : \mathcal{M}^m \to \mathcal{C}^{m-1}$ is a fibered category over the $(m-1)$-category $\mathcal{C}^{m-1}$ and, for all $0 \leq p < m \leq n$, there is a left action $\mu^m_p : \mathcal{C}^m \times \mathcal{M}^m \to \mathcal{M}^m$ of the bi-fibered $(m-1)$-category $\mathcal{C}^m \rightrightarrows \mathcal{C}^{m-1} \times \mathcal{C}^{m-1}$ over $\mathcal{M}^m \to \mathcal{C}^{m-1}$ such that $\mu^m_p(\mathcal{C}^m(x,y) \times \mathcal{M}^m(z) \subset \mathcal{M}^m(x))$ whenever $(y,z) \in \mathcal{C}^{m-1} \times_{\mathcal{C}^p} \mathcal{C}^{m-1}$.[64]

[64] For $p = m-1$ we assume $\mathcal{C}^{m-1} \times_{\mathcal{C}^p} \mathcal{C}^{m-1} = \Delta_{\mathcal{C}^{m-1}}$.

Strict Higher C*-categories 11.

Similar definitions can be given for right modules $\mathcal{M}_{\mathcal{C}}$ and bimodules ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}}$ over the n-C*-category $\mathcal{C}$.

The notion of **left Hilbert C*-module** ${}_{\mathcal{C}}\mathcal{M}$ **over a strict** n**-C*-category** $\mathcal{C}$ can be given imposing that for all $m = 1, \dots n$, $\tau : \mathcal{M}^m \to \mathcal{C}^{m-1}$ is a "Fell bundle"(for all the compositions and involutions in $\mathcal{C}^{m-1}$) equipped with an inner product $\langle \cdot \mid \cdot \rangle_m : \mathcal{M}^m \times \mathcal{M}^m \to \mathcal{C}^m$ such that $\langle \mathcal{M}^m(x) \mid \mathcal{M}^m(y) \rangle_m \subset \mathcal{C}^m(y, x)$.

Similar notions can be given for right Hilbert C*-modules and right/left bimodules over a strict n-C*-category.[65].

[65] It is necessary to distiguish right and left structures also for bimodules.

Strict Higher C*-categories 12.

Examples of **rank-one strict-*n*-C*-categories**
i.e. strict-n-C*-categories such that for every $m = 1, \dots, n$, the Banach space $\mathcal{C}^m(x, y)$ is one-dimensional, can be manually constructed by recursion.
In the theory of higher C*-categories they play the role of $\mathbb{C}$.
Rank-one Hilbert C*-modules play the role of n-Hilbert spaces.
Examples of non-commutative strict-n-C*-categories can be constructed via the following:

Theorem

Every left Hilbert module ${}_{\mathcal{C}}\mathcal{M}$ over the strict-n-C-category $\mathcal{C}$ determines a strict-n-C*-category $\mathcal{B}({}_{\mathcal{C}}\mathcal{M})$ of fiberwise adjointable maps. The strict n-C*-category $\mathcal{C}$ is represented (i.e. there is strict $*$-n-functor) into $\mathcal{B}({}_{\mathcal{C}}\mathcal{M})$ via the left covariant actions $\mu : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$.*

Spectral Geometries over C*-categories 1.

In a forthcoming paper[66] we give the following extremely tentative definitions and we will also examine their relation with the notions of morphisms of spectral geometries already presented.

As always, in "horizontal categorification", modules over a category are indexed by the objects of the category, whether bimodules over a category are indexed by pairs of objects of the category and hence we have to distinguish carefully between "monovariant" and several "bivariant" versions of the axioms.

[66] P.B., R. Conti, W. Lewkeeratiyutkul, Spectral Geometries over C*-categories and Morphisms of Spectral Geometries, in preparation.

Spectral Geometries over C*-categories 2.

A **categorical spectral geometry** $(\mathcal{C}, \mathcal{H}, \mathcal{D})$ is given by:

- a pre-C*-category $\mathcal{C}$;
- a module $\mathcal{H}$ over $\mathcal{C}$ that is also a Hilbert C*-module over $\mathbb{C}$; in other terms a family of Hilbert spaces $\mathcal{H}$ equipped with an object bijective $*$-functor $\pi : \mathcal{C} \to \mathcal{B}(\mathcal{H})$ with values in the C*-category of bounded linear maps between the Hilbert spaces in the family $\mathcal{H}$;
- the generator $\mathcal{D}$ of a unitary one-parameter group on $\mathcal{H}$ (i.e. the generator of a one-parameter group whose adjoint action in the eveloping C*-algebra of $\mathcal{B}(\mathcal{H})$ leaves $\mathcal{B}(\mathcal{H})$ invariant) such that, for all $x \in \mathcal{C}$,

$$[\mathcal{D}, \pi(x)]_- \quad \text{is extendable to an operator in } \mathcal{B}(\mathcal{H}).$$

Spectral Geometries over C*-categories 3.

A **bivariant spectral geometry** over two pre-C*-categories (with the same objects) $\mathcal{A}$ and $\mathcal{B}$ is given by a quintuple $(\mathcal{A}, \mathcal{B}, \mathcal{H}, \mathcal{D}_{\mathcal{A}}, \mathcal{D}_{\mathcal{B}})$, where

- $\mathcal{H}$ is a bimodule over $\mathcal{A}$-$\mathcal{B}$ that is also a Hilbert C*-bimodule over $\mathbb{C}$ and hence it is equipped with two $*$-representations $\rho : \mathcal{A} \to \mathcal{B}_\rho(\mathcal{H})$ and $\lambda : \mathcal{B} \to \mathcal{B}_\lambda(\mathcal{H})$ into the right, and respectively left, C*-category of the bimodule;
- $\mathcal{D}_{\mathcal{A}}$ (acting on the left) and $\mathcal{D}_{\mathcal{B}}$ (acting on the right) are two (generally unbounded) self-adjoint operators on $\mathcal{H}$ that generate on the enveloping algebras of $\mathcal{B}_\rho(\mathcal{H})$, and respectively of $\mathcal{B}_\lambda(\mathcal{H})$, one-parameter groups leaving $\mathcal{B}_\rho(\mathcal{H})$, and respectively $\mathcal{B}_\lambda(\mathcal{H})$, invariant and such that $[\mathcal{D}_{\mathcal{A}}, \rho(x)]_-$ and $[\mathcal{D}_{\mathcal{B}}, \lambda(y)]_-$ are extendable to bounded operators in $\mathcal{B}_\rho(\mathcal{H})$, $\mathcal{B}_\lambda(\mathcal{H})$, for all $x \in \mathcal{A}$ and $y \in \mathcal{B}$.

Spectral Congruences and Bivariant Spectral Geometries.

As we already conjectured, the notion of morphism of spectral geometries is strictly related to that of bivariant spectral geometry.[67]

Every spectral congruence (ϕ, Φ, D_Φ) between two spectral geometries $(\mathcal{A}_j, \mathcal{H}_j, D_j)$, $j =, 2$, is naturally associated to a bivariant spectral geometry given by:

$$(\mathcal{A}_1, \mathcal{A}_2, \Phi \otimes \Phi, D_1 \otimes I, I \otimes D_2).$$

[67] P.B., R. Conti, W. Lewkeeratiyutkul, Spectral Geometries over C*-categories and Morphisms of Spectral Geometries, in preparation.

Categorical Non-commutative Geometry and Topoi 1.

One of the main goals of our investigation is to discuss the interplay between ideas of categorification and NCG. Here we can present only a few suggestions. Work is in progress.

- Following higher category and multicategory theory[68],[69], we would like to define "vertical categorifications" for spectral triples or multicategories of non-commutative spaces[70].

[68] J. Baez, An Introduction to *n*-Categories, in: 7[th] Conference on Category Theory and Computer Science, eds. E. Moggi, G. Rosolini, Lecture Notes in Computer Science, **1290**, 1-33, Springer (1997), q-alg/9705009,

[69] T. Leinster, Higher Operads, Higher Categories, Cambridge (2004).

[70] P.B. C* Polycategories, in progress.

Categorical Non-commutative Geometry and Topoi 2.

- Extremely intriguing is the possible relation between the notions of (category of) spectral triples and A. Grothendiek topoi. Speculations in this direction have been given by P. Cartier[71] and are also discussed by A. Connes[72].

A full (categorical) notion of non-commutative space (NC-Klein program / NC-Grothendiek topos) is still waiting to be defined[73].

quantum gravity

[71] P. Cartier, A Mad Day's Work: from Grothendiek to Connes and Kontsevich, The Evolution of Concepts of Space and Symmetry, Bull. Amer. Math. Soc., **38**, n. 4, 389-408 (2001).

[72] A. Connes, A View of Mathematics, on-line on: `www.alainconnes.org`

[73] P.B., R. Conti, W. Lewkeeratiyutkul, Non-commutative Klein-Cartan Program, in progress.

A Panorama of Mathematical-Physics 1.

Finite Degrees of Freedom

Covariance	Classical Physics	Quantum Physics	?
Aristotle	Aristotle Mechanics	-	?
Galilei	Classical Mechanics	Quantum Mechanics	?
Poincaré	Special Relativistic Mechanics	Special Relativistic Quantum Mechanics*	?
Einstein	General Relativistic Mechanics	Generally Covariant Quantum Mechanics ?	?
?	?	?	?

A Panorama of Mathematical-Physics 2.

Infinite Degrees of Freedom

Covariance	Classical Physics	Quantum Physics	?
Aristotle	-	-	?
Galilei	Classical Continuum Statistical Mechanics	Quantum Statistical Mechanics*	?
Poincaré	Classical Field Theory	Relativistic Quantum Field Theory*	?
Einstein	General Relativistic Field Theory	Quantum Gravity ?	?
?	?	?	?

A Panorama of Mathematical-Physics 3.

Some explanations of terms and notations in the tables are in order:

Classical = Commutative,

Quantum = Non-commutative,

Aristotle (= rotation group), Galilei, Poincaré, Einstein (= diffeomorphisms group) refer to the covariance groups of the theory,

$*$ means that, at present, we do NOT have a sound mathematical theory,

? means that we do not have any serious theory yet,

- means that the theory is missing for historical reasons.

Categories in Physics 1.

- S. Doplicher-J. Roberts' theory of superselection sectors in algebraic quantum field theory provides a general Tannaka-Kreĭn duality for compact groups, where the dual is a particular monoidal W^*-category.[74],[75]
- G. Segal[76] and M. Atiyah[77] in conformal/topological QFT.

---

[74] S. Doplicher, J. Roberts, A New Duality Theory for Compact Groups, Inventiones Mathematicae, **98** (1), 157-218 (1989).

[75] S. Doplicher, J. Roberts, Why there Is a Field Algebra with Compact Gauge Group Describing the Superselection Structure in Particle Physics, Commun. Math. Phys, **131**, 51-107 (1990).

[76] G. Segal, The Definition of Conformal Field Theory, in: Topology, Geometry and Quantum Field Theory, Cambridge University Press, 421-577, (2004).

[77] M. Atiyah, Topological Quantum Field Theories, Inst. Hautes Études Sci. Publ. Math., **68**, 175-186 (1988).

Categories in Physics 2.

- C. Isham-J. Butterfield and A. Döring-C. Isham[78],[79],[80],[81] suggest topoi as basic structures for the construction of physical theories in which ordinary set theoretic concepts (including real/complex numbers and classical two valued logic) are replaced by more general topos theoretic notions.

[78] See for example: C. Isham, J. Butterfield, Some Possible Roles for Topos Theory in Quantum Theory and Quantum Gravity, Found. Phys., **30**, 1707-1735 (2000), `gr-qc/9910005`.

[79] A. Döring-C. Isham, A Topos Foundation for Theories of Physics I-II-III-IV, `quant-ph/0703060-62-64-66`.

[80] A. Döring, C. Isham, 'What is a Thing?': Topos Theory in the Foundations of Physics, `arXiv:0803.0417v1`.

[81] A. Döring, Topos Theory and 'Neo-realist' Quantum Theory, `arXiv:0712.4003v1`.

Categories in Physics 3.

- Along similar lines, C. Heunen-N. Landsman-B. Spitters[82] have recently introduced a topos theoretic basis for quantum theory in the C*-algebraic approach.
- S. Abramsky-B. Coecke[83],[84] are developing a categorical axiomatic for quantum mechanics, via monoidal categories, with intriguing links to knot theory and computer science.[85]

[82] C. Heunen-N. Landsman-B. Spitters, A Topos for Algebraic Quantum Theory, `arXiv:0709.4364v2`.

[83] S. Abramsky-B. Coecke, A Categorical Semantic of Quantum Protocols, `quant-ph/0402130`.

[84] B. Coecke, Kindergarten Quantum Mechanics, `quant-ph/0510072`.

[85] S. Abramsky, Temperly-Lieb Algebra: from Knot Theory to Logic and Computation via Quantum Mechanics, 2007.

Categories in Physics 3.

- J. Baez[86],[87] advocates the usage of categorical methods (higher category theory, categorification) in quantum physics and in quantum gravity as well as in connection with logic and computation. [88] A new field of "categorical quantum gravity" is emerging (see L. Crane[89],[90] for an intriguing overview).

---

[86] J. Baez, Higher-Dimensional Algebra and Planck-Scale Physics, in: Physics Meets Philosophy at the Planck Length, eds. C. Callender, N. Huggett, Cambridge University Press, 177-195 (2001). gr-qc/9902017.

[87] J. Baez, Quantum Quandaries: A Category Theoretic Perspective, quant-ph/0404040.

[88] J. Baez, M. Stay, Physics, Topology, Logic and Computation: a Rosetta Stone, http://math.ucr.edu/baez/rosetta.pdf.

[89] L. Crane, Categorical Geometry and the Mathematical Foundations of Quantum Gravity, gr-qc/0602120.

[90] L. Crane, What is the Mathematical Structure of Quantum Spacetime, arXiv:0706.4452.

Covariance.

Covariance of physical theories has been always discussed in the limited domain of groups acting on spaces:

- Aristoteles covariance: the group of rotations in $\mathbb{R}^3$.
- Galilei covariance: the ten parameters symmetry group of the Newton space-time generated by 3 space translations, 1 time translation, 3 rotations and 3 boosts.
- Poincaré covariance: the symmetry group of the four dimensional Minkowski space i.e. the semidirect product of Lorentz group with the group of translations in $\mathbb{R}^4$.
- Einstein covariance: the group of diffeomorphisms of four dimensional Lorentzian manifolds.

Different observers are "related" through transformations in the given covariance group.

Categorical Covariance 1.

There is no deep physical or operational reason to think that only groups (or quantum groups) might be the right mathematical structure to capture the "translation" between different observers and actually, in our opinion, categories provide a much more suitable environment in which also the concept of "partial translations" between observers can be described. Work is in progress on these issues[91].

[91] P.B., Hypercovariant Theories and Spectral Space-time.

Categorical Covariance 2.

The substitution of groups with categories as the basic covariance structure of theories should be a key ingredient in the reconstruction of physics from operationally founded principles of information theory (see for example C. Rovelli[92], A. Greenbaum[93],[94]) and, in the context of quantum gravity, also in the formalism of quantum casual histories[95].

---

[92] C. Rovelli, Relational Quantum Mechanics, Int. J. Theor. Phys., 35, 1637 (1996). quant-ph/9609002.

[93] A. Grinbaum, Elements of Information-Theoretic Derivation of the Formalism of Quantum Theory, International Journal of Quantum Information, **1(3)**, 289-300 (2003), quant-ph/0306079.

[94] A. Grinbaum, The Significance of Information in Quantum Theory, Ph.D. Thesis, Ecole Polytechnique, Paris (2004), quant-ph/0410071.

[95] F. Markopoulou, New Directions in Background Independent Quantum Gravity, gr-qc/0703097.

Categorical Covariance 3.

In the direction of the idea of categorical covariance, we mention several new works by R. Brunetti-K. Fredenhagen-R. Verch[96], and R. Brunetti-M. Porrmann-G. Ruzzi[97] that following and idea of J. Dimock[98] aim at a generalization of H. Araki-R. Haag-D. Kastler algebraic quantum field theory axiomatic, that is suitable for an Einstein covariant background.

---

[96] R. Brunetti, K. Fredenhagen, R. Verch, The Generally Covariant Locality Principle - A New Paradigm for Local Quantum Physics, Commun. Math. Phys., **237**, 31-68 (2003), `math-ph/0112041`.

[97] R. Brunetti, M. Porrmann, G. Ruzzi, General Covariance in Algebraic Quantum Field Theory, `math-ph/0512059`.

[98] J. Dimock, Algebras of Local Observables on a Manifold, Commun. Math. Phys., **77**, 219 (1980); Dirac Quantum Fields on a Manifold, Trans. Amer. Math. Soc., **269**, 133 (1982).

Categorical Covariance 4.

Similar ideas are also used in the non-commutative versions of the axioms recently proposed by M. Paschke and R. Verch[99].

In the framework of topos theoretic foundations for physics, C. Heunen-N. Landsman-B. Spitters have proposed a principle of general tovariance, [100] in which covariance is implemented via the category of geometric morphisms between topoi.

[99] M. Paschke, R. Verch, Local Covariant Quantum Field Theory over Spectral Geometries, gr-qc/0405057.

[100] C. Heunen, N. Landsman, B. Spitters, The Principle of General Tovariance, http://philsci-archive.pitt.edu/archive/00003931.

Non-commutative Space-Time 1.

There are 3 main reasons for the introduction of non-commutative space-time in physics:

1) Quantum effects (Heisenberg uncertainty principle), coupled to the general relativistic effect of the stress-energy tensor on the curvature of space-time (Einstein equation), entail that at very small scales the space-time manifold structure might be "unphysical". (B. Riemann, A. Einstein, S. Doplicher-K. Fredenhagen-J. Roberts[101]).

---

[101] S. Doplicher, K. Fredenhagen, J. Roberts, The Structure of Spacetime at the Planck Scale and Quantum Fields, Commun. Math. Phys., **172**, 187 (1995), `hep-th/0303037`.

Non-commutative Space-Time 2.

2) Modification to the short scale structure of space-time might help to resolve the problems of "ultraviolet divergences" in QFT (W. Heisenberg, H. Snyder[102] and many others) and of "singularities" in General Relativity.
3) A. Connes' view of the standard model in particle physics as a "classical" non-commutative geometry of space-time (with spectral triples).[103]

[102] H. Snyder, Quantized Spacetime, Phys. Rev., **71**, 38-41 (1947).

[103] A. Connes, Essay on Physics and Noncommutative Geometry, in: The Interface of Mathematics and Particle Physics, ed. D. Quillen, Clarendon Press, (1990).

Spectral Space-Time 1.

By "spectral space-time" we mean the idea that space-time (commutative or not) has to be "reconstructed a posteriori", in a spectral way, from other operationally defined degrees of freedom (geometrical or not). The origin of this "pregeometrical philosophy" is not clear:

- Space-time as a "relational" a posteriori entity originate from G.W. Leibnitz, G. Berkeley, E. Mach.

Spectral Space-Time 2.

- Pregeometrical speculations date as back as Pythagoras, but in their modern form, they start with J.A. Wheeler's "pregeometry"[104],[105] and "it from bit"[106] proposals.
- R. Geroch [107] has been the first to suggest a "shift" from space-time to algebras of functions over it, in order to address the problems of singularities in general relativity.

[104] J.A. Wheeler, Pregeometry: Motivations and Prospects, in: Quantum Theory and Gravitation, ed. A. Marlov, Academic Press (1980).

[105] D. Meschini, M. Lehto, J. Piilonen, Geometry, Pregeometry and Beyond, Stud. Hist. Philos. Mod. Phys., **36**, 435-464 (2005), gr-qc/0411053.

[106] J.A. Wheeler, It from Bit, Sakharov Memorial Lectures on Physics, vol. 2, Nova Science (1992).

[107] R. Geroch, Einstein Algebras, Commun. Math. Phys., **26**, 271-275 (1972).

Spectral Space-Time 3.

- R. Feynman-F. Dyson proof of Maxwell equations, from non-relativistic QM of a free particle, indicates that essential information about the underlying space-time is already contained in the algebra of observables of the system[108].
- The "reconstruction" of (classical Minkowski) space-time from suitable states over the observable algebra in algebraic quantum field theory has been considered by S. Doplicher[109], A. Ocneanu[110], U. Bannier[111].

[108] An argument recently revised and extended to non-commutative configuration spaces by
T. Kopf-M. Paschke `arXiv:math-ph/0301040,0708.0388`

[109] S. Doplicher, private conversation, Rome, April 1995.

[110] As reported in: A. Jadczyk, Algebras Symmetries, Spaces, in: Quantum Groups, H. D. Doebner, J. D. Hennig ed., Springer (1990).

[111] U. Bannier, Intrinsic Algebraic Characterization of Space-Time Structure, Int. J. Theor. Phys., **33**, 1797-1809 (1994).

Spectral Space-Time 4.

- Extremely important rigorous results on the "reconstruction of classical Minkowski space-time" from the vacuum state in algebraic quantum field theory, via Tomita-Takesaki modular theory, have been obtained in the "geometric modular action" program by D. Buchholz-S. Summers[112],[113],[114].

[112] D. Buchholz, O. Dreyer, M. Florig, S. Summers, Geometric Modular Action and Spacetime Symmetry Groups, Rev. Math. Phys., 12, 475-560 (2000), math-ph/9805026.

[113] S. Summers, Yet More Ado About Nothing: The Remarkable Relativistic Vacuum State, arXiv:0802.1854v1

[114] S. J. Summers, R. K. White, On Deriving Space-Time from Quantum Observables and States, hep-th/0304179.

Spectral Space-Time 5.

- Tomita-Takesaki modular theory is also used in the "modular localization program" by R. Brunetti-D. Guido-R. Longo[115]. In this context a reconstruction of space-time has been conjectured by N. Pinamonti[116].

[115] R. Brunetti, D. Guido, R. Longo, Modular Localization and Wigner Particles, arXiv:math-ph/0203021.

[116] N. Pinamonti, On Localization of Position Operators in Möbius covariant Theories, math-ph/0610070.

Spectral Space-Time 6.

- That non-commutative geometry provides a suitable environment for the implementation of spectral reconstruction of space-time from states and observables in quantum physics has been my research motivation since 1990 and it is still an open work in progress[117].
- The idea that space-time might be spectrally reconstructed, via non-commutative geometry, from Tomita-Takesaki modular theory applied to the algebra of physical observables was elaborated in 1995 by myself and independently by R. Longo. Since then, this conjecture is the main subject and goal of our investigation[118].

[117] P.B., Hypercovariant Theories and Spectral Space-time (2001).
[118] P.B., Modular Spectral Triples in Non-commutative Geometry and Physics, Research Report, Thai Research Fund, (2005).

Quantum Gravity via NCG 1.

It is often claimed that NCG provides the right mathematics (a kind of quantum version of Riemannian geometry) for a mathematically sound theory of quantum gravity, [119],[120].

Among the available approaches to quantum gravity via NCG:

- J. Madore's "derivation based approach"[121];
- S. Majid's "quantum group approach"[122].

[119] L. Smolin, Three Roads to Quantum Gravity, Weidenfeld & Nicolson (2000).

[120] P. Martinetti, What Kind of Noncommutative Geometry for Quantum Gravity?, Mod. Phys. Lett., **A20**, 1315 (2005), `gr-qc/0501022`.

[121] J. Madore, An Introduction to Non-commutative Geometry and its Physical Applications, Cambridge University Press (1999).

[122] S. Majid, Hopf Algebras for Physics at the Planck Scale, J. Classical and Quantum Gravity, **5**, 1587-1606 (1988). S. Majid, Algebraic Approach to Quantum Gravity I,II,III, `arXiv:hep-th/0604130`, `arXiv:hep-th/0604182`, `http://philsci-archive.pitt.edu/archive/00003345`.

Quantum Gravity via NCG 2.

Current applications of NCG to quantum gravity have been limited to some example or to attempts to make use of its mathematical framework "inside" some already established theories such as "strings" or "loops". Among these, we mention:

- the interesting examples studied by C. Rovelli[123] and F. Besard[124];
- the applications to string theory in the work by A. Connes-M. Douglas-A. Schwarz[125],

---

[123] C. Rovelli, Spectral Noncommutative Geometry and Quantization: a Simple Example, Phys. Rev. Lett., **83**, 1079-1083 (1999), `gr-qc/9904029`.

[124] F. Besnard, Canonical Quantization and Spectral Action, a Nice Example, `gr-qc/0702049`.

[125] A. Connes, M. Douglas, A. Schwarz, Noncommutative Geometry and Matrix Theory: Compactification on Tori, `hep-th/9711162`.

Quantum Gravity via NCG 3.

- the links between loop quantum gravity (spin networks), quantum information and NCG described by F. Girelli-E. Livine[126].
- the intriguing interrelations with loop quantum gravity in the recent works by J. Aastrup-J. Grimstrum-R. Nest[127],[128],[129].

---

[126] F. Girelli, E. Livine, Reconstructing Quantum Geometry from Quantum Information: Spin Networks as Harmonic Oscillators, Class. Quant. Grav., **22**, 3295-3314 (2005), `gr-qc/0501075`.

[127] J. Aastrup, J. Grimstrup, Spectral Triples of Holonomy Loops, `arXiv:hep-th/0503246`.

[128] J. Aastrup, J. Grimstrup, Intersecting Connes Noncommutative Geometry with Quantum Gravity, `hep-th/0601127`.

[129] J. Aastrup, J. Grimstrup, R. Nest, On Spectral Triples in Quantum Gravity I-II, `arXiv:0802.1783v1`, `arXiv:0802.1784v1`.

Quantum Gravity via NCG 4.

Unfortunately, with the only notable exception of two programs partially outlined in

- M. Paschke, An Essay on the Spectral Action and its Relation to Quantum Gravity, in: Quantum Gravity, Mathematical Models and Experimental Bounds, Birkäuser (2007),
- A. Connes, M. Marcolli, Noncommutative Geometry Quantum Fields and Motives, July 2007,

a foundational approach to quantum physics based on A. Connes' NCG has never been proposed.
The obstacles are both technical and conceptual.

Modular Algebraic Quantum Gravity 1.

Our ongoing research project[130] is developing a new approach to the foundations of quantum physics technically based on algebraic quantum theory (operator algebras - AQFT) and A. Connes' NCG whose main objective is a "spectral" reconstruction of non-commutative space-time from Tomita-Takesaki modular theory:

[130] P.B., R. Conti, W. Lewkeeratiyutkul, Algebraic Quantum Gravity, work in progress.

Modular Algebraic Quantum Gravity 2.

In the specific:

- Building on our previous research on "modular spectral-triples"[131], we make use of Tomita-Takesaki modular theory of operator algebras to associate non-commutative geometrical objects (only formally similar to A. Connes' spectral-triples) to suitable states over involutive normed algebras (in the same direction we stress the important recent work on semifinite spectral triples by A. Carey, J. Phillips, A. Rennie, F. Sukochev as reported in `arXiv:0707.3853`).

[131] P.B., Modular Spectral Triples in Non-commutative Geometry and Physics, Research Report, Thai Research Fund, (2005); P.B., R. Conti, W. Lewkeeratiyutkul, Modular Spectral Triples, in preparation.

Modular Algebraic Quantum Gravity 3.

- We are developing[132] an "event interpretation" of the formalism of states and observables in algebraic quantum physics that is in line with C. Isham's "history projection operator theory" [133] and C. Rovelli's "relational quantum mechanics"[134].

[132] P.B., Algebraic Formalism for Rovelli Quantum Theory, in preparation.

[133] See for example C. Isham, Quantum Logic and the Histories Approach to Quantum Theory, J. Math. Phys., **35**, 2157-2185 (1994), gr-qc/9308006.

[134] C. Rovelli, Relational Quantum Mechanics, arXiv:/quant-ph/9609002.

Modular Algebraic Quantum Gravity 4.

- Making contact with our current research project on "categorical non-commutative geometry" and with other projects in categorical quantum gravity[135],[136], we will generalize the diffeomorphism covariance group of general relativity in a categorical context and use it to "indentify" the degrees of freedom related to the spatio-temporal structure of the physical system.

[135] J. Baez, Higher-Dimensional Algebra and Planck-Scale Physics, arXiv:gr-qc/9902017; J. Baez, Quantum Quandaries: a Category Theoretic Perspective, arXiv:quant-ph/0404040.

[136] L. Crane, Categorical Geometry and the Mathematical Foundations of Quantum Gravity, gr-qc/0602120; L. Crane, What is the Mathematical Structure of Quantum Spacetime, arXiv:0706.4452.

Modular Algebraic Quantum Gravity 5.

- Using techniques from "decoherence/einselection"[137],[138], "emergence/noiseless subsystems"[139],[140], or the "cooling" procedure developed by A. Connes-M. Marcolli[141], we will try to extract from our spectrally defined "modular" non-commutative geometries, a macroscopic space-time for the pair state/system and its classical "residue".

[137] H.D. Zeh, Roots and Fruits of Decoherence, quant-ph/0512078.

[138] W. Zurek, Decoherence and the Transition from Quantum to Classical, Los Alamos Science **27** (2002).

[139] T. Konopka, F. Markopoulou, Constrained Mechanics and Noiseless Subsystems, gr-qc/0601028; F. Markopoulou, Towards Gravity form the Quantum, hep-th/0604120.

[140] O. Dreyer, Emergent Probabilities in Quantum Mechanics, quant-ph/0603202.; O. Dreyer, Emergent General Relativity, gr-qc/0604075.

[141] A. Connes, M. Marcolli, Noncommutative Geometry, Quantum Fields and Motives, July 2007

Algebraic Quantum Gravity 9.

- Possible reproduction of quantum geometries already defined in the context of loop quantum gravity[142] or S. Doplicher-J. Roberts-K. Fredenhagen models[143] will be investigated.
- Important connections of these ideas to "quantum information theory" and "quantum computation" are currently under consideration[144].

---

[142] C. Rovelli, Quantum Gravity, Cambridge University Press (2004).
[143] S. Doplicher, K. Fredenhagen, J. Roberts, The Structure of Spacetime at the Planck Scale and Quantum Fields, Commun. Math. Phys., **172**, 187 (1995), `hep-th/0303037`.
[144] P.B., Hypercovariant Theories and Spectral Space-time, unpublished (2001).

Reference

P.B., R. Conti, W. Lewkeeratiyutkul,
Non-commutative Geometry, Categories and Quantum Physics,
preprint (submitted to East-West Journal of Mathematics,
prooceedings of the "International Conference in Mathematics and
Applications", Mahidol University, 15-17 August 2007),
arXiv:0801.2826v1.

This file has been realized using the **beamer** LaTeX-macro.

Thank You for Your Kind Attention!