TOPOS-THEORETIC MODELS OF THE CONTINUUM

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Constructive approaches

First try: one-sided Dedekind cuts

$$R_\ell^+ = \{ L : PQ \mid (\top \vdash (\exists x)(x \in L)),$$
$$(((x<y) \wedge (y \in L)) \vdash (x \in L)),$$
$$((x \in L) \vdash (\exists y)((x<y) \wedge (y \in L))) \}$$

Call L proper if we also have $(\top \vdash (\exists x) \neg (x \in L))$.

$\mathcal{F}_\ell(R^+)$ = frame generated by symbols $(p, \infty]$, $p \in \mathbb{Q}$, subject to

$$(p,\infty] \wedge (q,\infty] = (\max\{p,q\}, \infty]$$
$$\top = \bigvee_{p \in \mathbb{Q}} (p,\infty]$$
$$(p,\infty] = \bigvee_{q>p} (q,\infty]$$

$\mathcal{F}_\ell(R^+) \cong \mathcal{O}(R_\ell^+)$ is spatial (constructively)

$Sh(R_\ell^+)$ is a classifying topos for one-sided cuts:

in particular one-sided cuts in $Sh(X)$ correspond to continuous maps $X \longrightarrow R_\ell^+$,

i.e. to lower semicontinuous functions $X \longrightarrow \mathbb{R} \cup \{\infty\}$.

Better: use two-sided Dedekind cuts

$$R_d = \{ \langle L, U \rangle : PQ \times PQ \mid (\top \vdash (\exists x)(x \in L)), (\top \vdash (\exists x)(x \in U)),$$
$$(((x<y) \wedge (y \in L)) \vdash (x \in L)),$$
$$((x \in L) \vdash (\exists y)((x<y) \wedge (y \in L))),$$
$$(((x<y) \wedge (x \in U)) \vdash (y \in U)),$$
$$((x \in U) \vdash (\exists y)((y<x) \wedge (y \in U))),$$
$$(((x \in L) \wedge (x \in U)) \vdash \bot),$$
$$((x<y) \vdash ((x \in L) \vee (y \in U))) \}$$

Formal reals $\mathcal{F}(R)$ = frame generated by symbols (p,q), $p,q \in \mathbb{Q}$, subject to

$$(p,q) \wedge (r,s) = (\max\{p,r\}, \min\{q,s\})$$

$$\top = \bigvee_{p,q} (p,q)$$

$$(p,q) = \bigvee_{p<r<s<q} (r,s)$$

(in particular $(p,q) = \bot$ if $p \geqslant q$)

$$(p,q) = (p,r) \vee (q,s) \quad \text{provided } p \leq q < r \leq s$$

Write R_f for the locale whose frame of opens is $\mathcal{F}(R)$; then $Sh(R_f)$ is a classifying topos for two-sided Dedekind cuts: in particular two-sided cuts in $Sh(X)$ correspond to continuous maps $X \longrightarrow R_f$.

(If X is spatial they correspond to cts. maps $X \longrightarrow R_d$.)

Is R_f spatial?

R_d is the set of points of R_f;
in particular it's a sober space when equipped with the topology induced by $\mathcal{F}(R)$.

The proof that $\mathcal{O}(R_d) \cong \mathcal{F}(R)$ uses Heine-Borel (for R_d). Hence the isomorphism holds in any Boolean topos (and in any Grothendieck topos with enough points).

But Heine-Borel is essential, since $\mathcal{F}(R)$ is constructively locally compact (i.e. the closed sublocale $[0,1]_f$ complementary to $\bigvee \{ (p,q) \mid (p>1) \vee (q<0) \}$ is compact).

And there are Grothendieck toposes where Heine-Borel fails (Fourman-Hyland, Joyal); hence also it fails in the free topos.

In fact for a topos $\mathcal{E}$, the following are equivalent:

(1) Heine-Borel holds in $\mathcal{E}$.

(2) R_f is spatial

(3) $(R_d, +)$ is a localic group.

(N.B.: $(R_f, +)$ is always a localic group, and $(R_d, +)$ is a spatial group.)

Why not use Cauchy sequences?

Can form the objects

$$C = \{ f : Q^N \mid ((m<n) \vdash (|f(m) - f(n)| < 1/n)) \}$$

$$E = \{ \langle f, g \rangle : C \times C \mid ((n > 0) \vdash (|f(n) - g(n)| < \tfrac{3}{n})) \}$$

E is an equivalence relation on C, and we can consider $R_c = C/E$.

We have $Q \rightarrowtail R_c \rightarrowtail R_d$;

the second inclusion is an isomorphism if countable choice holds in $\mathcal{E}$ (so, for example, in the effective topos) but not in general.

R_c behaves 'unpredictably' in the absence of countable choice: e.g. in $Sh(\mathbb{R})$, R_c is the sheaf of locally constant real-valued fns.

but in $Sh(\mathbb{Q})$ it's the sheaf of continuous real-valued functions.

Also, R_c isn't sober unless it coincides with R_d.

[**Why not use continued fractions?** Even worse: can form an object R_{cf}, but Q is complemented in it (in fact it's the union of $Q \rightarrowtail R_c$ and its Heyting negation).

And we can't even define addition on R_{cf}.]

Order & Algebraic Properties

Can define $<$ and $\leq$ on R_d by

$$\langle L_1, U_1\rangle < \langle L_2, U_2\rangle \quad \text{iff} \quad (\exists x)((x \in U_1) \wedge (x \in L_2))$$

$$\langle L_1, U_1\rangle \leq \langle L_2, U_2\rangle \quad \text{iff} \quad L_1 \subseteq L_2 \quad (\text{iff } U_2 \subseteq U_1).$$

(Can also define them for R_c)

We have

$$(x \leq y) \quad \text{iff} \quad \neg(y < x)$$

$$(x \text{ invertible}) \quad \text{iff} \quad ((x > 0) \vee (x < 0))$$

Hence

$$\neg(x \text{ invertible}) \quad \text{iff} \quad (x = 0)$$

i.e. R_d is a residue field.

In particular, R_d is reduced (i.e. $((x^2 = 0) \vdash (x = 0))$).

Also, R_d is a local ring (i.e.

$$(\top \vdash ((x \text{ invertible}) \vee (1-x \text{ invertible}))) \ )$$

and satisfies $((x > 0) \vdash (\exists y)((y > 0) \wedge (y^2 = x)))$ etc.

In fact R_d is a separably real-closed local ring.

R_d has binary max and min operations

(hence in particular it has $|x| = \max\{x, -x\}$)

but it isn't (conditionally) order-complete.

(Can remedy this by modifying the last axiom for a 2-sided cut, but the resulting object R_m loses the local ring property.)

The Continuity Principle

$(\forall f : R_d^{R_d})\,(f \text{ is continuous})$

Holds in $\mathrm{Sh}(\mathbb{R})$ (Hyland)

and (better) in the 'gros topos' $\mathrm{Sh}(\mathcal{C})$

where $\mathcal{C}$ is a suitable full subcategory of **Top** (or **Loc**)

containing $\mathbb{R}$ and the function-space $[\mathbb{R}, \mathbb{R}]$, equipped with the Grothendieck topology generated by open covers.

In this topos, R_d is the sheaf $\mathcal{C}(-, \mathbb{R})$

and $R_d^{R_d}$ is $\mathcal{C}(-, [\mathbb{R}, \mathbb{R}])$.

Hence (Joyal) the continuity principle also holds in the free topos.

Note: the Continuity Principle implies that R_d is indecomposable, at least in the sense that

$$\big(((A \cup B = R_d) \wedge (A \cap B = \emptyset) \wedge (A \text{ inhabited}) \wedge (B \text{ inhabited})) \vdash \bot\big).$$

Axiomatic Approaches

Suppose we want every function $R \to R$ to be (not just continuous but) differentiable (or even smooth).

Impossible with R_d, since we have the function $x \mapsto |x|$.

Realizing it **seems** to require nilpotents; but R_d doesn't have those.

Solution (Lawvere 1967): axiomatize properties of a 'ring of smooth reals', then look for models in suitable toposes.

Key axiom: set $D = \{d : R \mid d^2 = 0\}$. Then

$$(\forall f : R^D)(\exists!\, a, b : R)(\forall d : D)(f(d) = a + bd).$$

Then, for any $f : R^R$, can define f' by setting $f'(a)$ to be the unique elt. of R such that $(\forall d : D)(f(a+d) = f(a) + f'(a)d)$.

Models for this axiom are **surprisingly** easy to find:

e.g. the generic local ring (better, the generic local $\mathbb{R}$-algebra).

Algebraic and order axioms: we still want R to be a local ring (indeed, separably real-closed local — or better, a local C^∞-ring).

It can't be a residue field, but it can satisfy the dual field of fractions axiom $\quad \neg(x = 0) \dashv\vdash (x \text{ invertible})$

We also want an order relation $<$ satisfying things like

$$(x \text{ invertible}) \dashv\vdash ((x > 0) \vee (x < 0))$$

and we define $(x \leq y)$ to mean $\neg(y < x)$.

(Note that $\leq$ is now only a preorder.)

Remark: these axioms imply that $<$ is $\neg\neg$-stable:

Suppose $\neg\neg(x<y)$. Then $\neg(x=y)$, so $((x<y)\vee(y<x))$.
But $(\neg\neg(x<y)\wedge(y<x)) \vdash \neg\neg((x<y)\wedge(y<x)) \vdash \bot$,
so $(\neg\neg(x<y) \vdash (x<y))$.

The axioms also imply that the 'Penon infinitesimals'

$$\{x:R \mid \neg\neg(x=0)\} = \{x:R \mid (\forall n:N^+)(-\tfrac{1}{n} < x < \tfrac{1}{n})\}$$

are exactly the (Jacobson) radical of R. (The 'Kock-Lawvere infinitesimals', i.e. the nilpotents, are the prime radical of R.)

<u>Theorem</u> (Bell/Lambek) Suppose R is Archimedean (i.e. N is cofinal in R). Then there's a quotient map $R \longrightarrow R_d$ whose kernel is the Penon/Jacobson radical. Moreover, R_d is a field of fractions as well as a residue field.

This can happen in examples; but it's more common to require an enlarged object $\mathcal{N}$ of 'smooth natural numbers' to be cofinal, so that the 'standard' R_d appears as a subring of a quotient of R — and it ceases to be a field of fractions.

Another approach: building the continuum from the infinitesimal?

Bill Lawvere has observed that, in models of SDG, although $D = \{x : R \mid x^2 = 0\}$ is 'tiny', D^D is quite large.

Specifically, D^D is a monoid (under composition) and the multiplicative monoid $(R, \cdot)$ occurs as a retract of it.

Specifically, as a submonoid $(R, \cdot)$ is the centre of D^D; as a quotient, it **may** be the 'abelianization' of D^D, i.e. its quotient by the smallest congruence containing all pairs $\langle fg, gf \rangle$ $(f, g \in D^D)$.

Although this recovers only the multiplicative structure of R, there are (at least in theory) ways of recovering the additive structure too.

So: could one begin by axiomatizing D, and then extract all the required properties of R by considering it as a suitable retract of D^D?