Tutorial on Conceptual Issues of Quantum Theory

Categories, Logic and Foundations of Physics (CLAP)
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"When one man speaks to another man who doesn't understand him, and when the man who's speaking no longer understands, it's metaphysics."

(Voltaire, *Candide*, 1759)

"The issue remains, when will we ever stop burdening the taxpayer with conferences devoted to the quantum foundations?"

(Chris Fuchs)

A few general remarks

We will be concerned with *conceptual* aspects of quantum theory. These include metaphysical questions as well as interpretational issues on the relation between the mathematical formalism and the physical theory as ‘referring to the world’.

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We will present a number of mathematical results (well-known in physics). Some of the mathematical results are interesting *as mathematics*, others are quite straightforward. Our main interest lies in the *physical meaning and consequences* of the mathematical results.

We will typically choose the simplest mathematical setting in which a result can be obtained.

Some sources and references

- The Stanford Encyclopedia of Philosophy (plato.stanford.edu)
- C.J. Isham, *Lectures on Quantum Theory: Mathematical and Structural Foundations*, Imperial College Press (1995)
- J.A. Wheeler, W.H. Zurek, *Quantum Theory and Measurement*, Princeton (1983)
- R. Penrose, *The Road to Reality*, Jonathan Cape (2004)

The Copenhagen Interpretation

The Copenhagen interpretation of quantum mechanics goes back to Bohr, Heisenberg, Born and others. It is the orthodox interpretation and still prevalent among working physicists.

Bohr was the leading intellectual figure in the development of the CI, but he is hard to read.

"Never express yourself more clearly than you are able to think."
(Niels Bohr)

What we call the CI today mostly goes back to Heisenberg.

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- the eigenvector-eigenvalue link: if, for some $\widehat{A} \in \mathcal{A}(S)$, we have $\widehat{A}(\psi) = a\psi$, i.e., if ψ is an eigenvector (also called an eigenstate) of $\widehat{A}$, then the physical quantity A represented by $\widehat{A}$ has the value a.

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- upon measurement of A, the state collapses into an eigenstate of $\widehat{A}$ corresponding to the measured (eigen)value $a \in \mathrm{sp}(\widehat{A})$. The Born rule allows to calculate probabilities of outcomes of measurements and expectation values.

The physical interpretation

The state ψ of a quantum system has quite a different role from the state s of a classical system. ψ can be seen as a pure calculational device, encoding the observer's knowledge about the quantum system (Heisenberg's positivist approach), or as a symbolic representation of the quantum world (Bohr), in contrast to classical physics, where the state gives a pictorial or literal representation.

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In particular, the quantum state does *not* assign values to all physical quantities at once. The formalism of QM fundamentally restricts the possibility of such value assignments, as we will see from the uncertainty principle.

Measurements hence do not simply reveal preexisting values.

The physical interpretation - Bohr, 1

The following is taken from the article "Copenhagen Interpretation of Quantum Mechanics" from the SEP (plato.stanford.edu). It summarises Bohr's mature view on quantum theory after EPR. A few comments are added.

1. The interpretation of a physical theory has to rely on an experimental practice.
2. The experimental practice presupposes a certain pre-scientific practice of description, which establishes the norm for experimental measurement apparatus, and consequently what counts as scientific experience.
3. Our pre-scientific practice of understanding our environment is an adaptation to the sense experience of separation, orientation, identification and reidentification over time of physical objects.
4. This pre-scientific experience is grasped in terms of common categories like [a] thing's position and change of position, duration and change of duration, and the relation of cause and effect, terms and principles that are now parts of our common language.
5. These common categories yield the preconditions for objective knowledge, and any description of nature has to use these concepts to be objective.

These points express a neo-Kantian point of view.

The physical interpretation - Bohr, 2

6. The concepts of classical physics are merely exact specifications of the above categories.
7. The classical concepts—and not classical physics itself—are therefore necessary in any description of physical experience in order to understand what we are doing and to be able to communicate our results to others, in particular in the description of quantum phenomena as they present themselves in experiments;
8. Planck's empirical discovery of the quantization of action [*i.e., the quantum of action* $\hbar$] requires a revision of the foundation for the use of classical concepts, because they are not all applicable at the same time. Their use is well defined only if they apply to experimental interactions in which the quantization of action can be regarded as negligible.
9. In experimental cases where the quantization of action plays a significant role, the application of a classical concept does not refer to independent properties of the object; rather the ascription of either kinematic or dynamic properties to the object as it exists independently of a specific experimental interaction is ill-defined.
10. The quantization of action demands a limitation of the use of classical concepts so that these concepts apply only to a phenomenon, which Bohr understood as the macroscopic manifestation of a measurement on the object, i.e. the uncontrollable interaction between the object and the apparatus.

This can be summarised as environmental contextuality.

The physical interpretation - Bohr, 3

11. The quantum mechanical description of the object differs from the classical description of the measuring apparatus, and this requires that the object and the measuring device should be separated in the description, but the line of separation is not the one between macroscopic instruments and microscopic objects. It has been argued in detail (Howard 1994) that Bohr pointed out that parts of the measuring device may sometimes be treated as parts of the object in the quantum mechanical description.
12. The quantum mechanical formalism does not provide physicists with a pictorial representation: the ψ-function does not, as Schrödinger had hoped, represent a new kind of reality. Instead, as Born suggested, the square of the absolute value of the ψ-function expresses a probability amplitude for the outcome of a measurement. Due to the fact that the wave equation involves an imaginary quantity this equation can have only a symbolic character, but the formalism may be used to predict the outcome of a measurement that establishes the conditions under which concepts like position, momentum, time and energy apply to the phenomena.
13. The ascription of these classical concepts to the phenomena of measurements rely on the experimental context of the phenomena, so that the entire setup provides us with the defining conditions for the application of kinematic and dynamic concepts in the domain of quantum physics.
14. Such phenomena are complementary in the sense that their manifestations depend on mutually exclusive measurements, but that the information gained through these various experiments exhausts all possible objective knowledge of the object.

Keywords: complementarity and doctrine of classical concepts.

Heisenberg's uncertainty principle

Let ψ be a state. The standard deviation of $\widehat{p}$ in the state ψ is

$$\Delta_\psi(\widehat{p}) = \sqrt{(\langle \widehat{p}^2 \rangle_\psi) - (\langle \widehat{p} \rangle_\psi)^2}.$$

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$$\Delta_\psi(\widehat{A})^2\Delta_\psi(\widehat{B})^2 \geq \frac{1}{4}|\langle[\widehat{A},\widehat{B}]\rangle_\psi|^2 + \frac{1}{4}|\langle\{\widehat{A} - \langle\widehat{A}\rangle_\psi, \widehat{B} - \langle\widehat{B}\rangle_\psi\}\rangle_\psi|^2.$$

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An important question is if the UP gives an *epistemological constraint* only, due to unavoidable disturbances during measurement, or if it is an *ontological principle*, constraining the nature of reality itself, independent of measurements.

Bohr and Heisenberg - degrees of instrumentalism

Bohr:

- An entity realist (possibly), but not a theory realist. QM gives a symbolic, but not a pictorial or literal description of the quantum world. In particular, the use of imaginary numbers (states are vectors in *complex* Hilbert spaces) points to this.
- Complementarity similar in spirit to relativity. Both create a kind of contextuality for observations: relativity due to the existence of a maximal speed, QT due to the existence of a minimal action.

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Heisenberg:

- Positivist and instrumentalist, QM as giving calculational devices.
- It is meaningless to speak of the value of a physical quantity before a measurement is made.

Heisenberg's view was and still is very influential. He coined the expression "Copenhagen interpretation" in 1955.

The measurement problem, 1

"The dynamics and the postulate of collapse are flatly in contradiction with one another ... the postulate of collapse seems to be right about what happens when we make measurements, and the dynamics seems to be bizarrely wrong about what happens when we make measurements, and yet the dynamics seems to be right about what happens whenever we aren't making measurements."

D. Albert, *Quantum Mechanics and Experience*, Harvard University Press, Cambridge (1992)

The measurement problem, 2

The dynamics of QM is linear, given by the Schrödinger equation:

$$\frac{\partial\psi(\mathbf{r},t)}{\partial t} = -\frac{i}{\hbar}\widehat{H}(\psi(\mathbf{r},t)).$$

Here, $\widehat{H}$ is the Hamiltonian, which is the operator representing the physical quantity energy.

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Assume that we have some physical quantity A such that the eigenstates of $\widehat{A}$ form a basis $(\phi_i^{\widehat{A}})_{i\in I}$ of Hilbert space. Moreover, assume that for $t = 0$, the state $\psi(\mathbf{r}, 0)$ is an eigenstate of $\widehat{A}$. After some time t, the state will have evolved to

$$\psi(\mathbf{r},t) = \sum_{i\in I} c_i\phi_i^{\widehat{A}},$$

which is not an eigenstate of $\widehat{A}$ anymore in general. Rather, $\psi(\mathbf{r},t)$ is a superposition of eigenstates of $\widehat{A}$.

The measurement problem, 3

This kind of linear evolution very successfully describes microphysics when no measurement is made. Yet, measurement itself is *not* described by this linear evolution, but by the so-called projection postulate, which goes back to von Neumann: if the physical quantity A is measured and the result $a_i \in \mathrm{sp}(\widehat{A})$ is obtained (which happens with probability c_i^2 according to the Born rule), then the state of the quantum system changes to

$$\psi(\mathbf{r}, t) = \phi_i^{\widehat{A}},$$

the eigenstate of $\widehat{A}$ corresponding to the eigenvalue a_i. (For simplicity, we assume here that $\widehat{A}$ is non-degenerate and has discrete spectrum). Clearly, this change is *discontinuous* in general.

The measurement problem, 4

We do not observe superpositions of macroscopic objects, which needs to be explained as well. Many people see the measurement problem as the question

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- What is a measurement? Which physical interactions are measurements and which are not?

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It has become clear already that the status of the measurement problem depends on the status one ascribes to the state ψ—as a state of knowledge, as a symbolic device, or as describing reality. More generally, potential solutions to the measurement problem (and even the question what could possibly count as a solution) depend on one's philosphical and metaphysical position with respect to QM and to physical theories in general.

EPR - completeness of QM and local realism

Einstein, Podolsky, Rosen, *"Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?"*, Phys. Rev. **47**, 777–780 (1935)

This famous paper is a point of culmination in the Bohr-Einstein debate. It discusses the completeness of QM. The *"condition of completeness"* for a physical theory is: *"[E]very element of the physical reality must have a counterpart in the physical theory."*

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EPR suggest the following sufficient, but not necessary criterion for identifying physical reality: *"If, without in any way disturbing the system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."*

EPR argument, 1

Acoording to the eigenvector-eigenvalue link, a physical quantity A has a definite value a in a given state ψ if $\widehat{A}(\psi) = a\psi$. Hence, there is an element of physical reality corresponding to the physical quantity A.

If, on the other hand, we consider another physical quantity B such that ψ is not an eigenstate of B, then B has no definite value in the state ψ. EPR conclude that such a physical quantity has no physical reality in the state ψ.

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NB: The latter statement is somewhat imprecise, since two non-commuting operators can have common eigenvectors. Yet, there are pairs of operators (like spin-x and spin-z) which have no common eigenvectors. We concentrate on these.

EPR argument, 2

EPR then consider an entangled state (without using the word) of a composite system $S = S_1 \diamond S_2$ whose components S_1, S_2 do not interact anymore. (One may think of a spatially separated pair of electrons.) Let A, B denote two physical quantities pertaining to the first component.

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The state of the composite system can be written in two ways: let $(u_i(S_1))_{i=1,\dots,n}$ denote the eigenbasis of $\widehat{A}$, with eigenvalues a_i, and let $(v_j(S_1))_{j=1,\dots,n}$ be the eigenbasis of $\widehat{B}$, with eigenvalues b_j. Then

$$\psi(S) = \sum_{i=1}^{n} u_i(S_1) \otimes \psi_i(S_2) = \sum_{j=1}^{n} v_j(S_1) \otimes \phi_j(S_2).$$

EPR argument, 3

Now perform a measurement of A on S_1. Let us assume we obtain the value a_k as measurement result. The state of the composite system changes from $\sum_{i=1}^{n} u_i(S_1) \otimes \psi_i(S_2)$ to

$$u_k(S_1) \otimes \psi_k(S_2).$$

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The second component, S_2, hence is left in the state $\psi_k(S_2)$ after measurement. On the other hand, if we measure B on S_1 and obtain the value b_l, then the state changes to

$$v_l(S_1) \otimes \phi_l(S_2),$$

so S_2 is in the state $\phi_l(S_2)$ after measurement.

EPR argument, 4

EPR argue that, despite the fact that S_2 is left in two different states after the measurements of A resp. B on S_1, there is no real change taking place in the second system, since S_1 and S_2 do not interact anymore. They continue: *"Thus, it is possible to assign two different wave functions ... to the same reality (the second system after the interaction with the first)."*

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It can happen that the two states of system S_2 are eigenstates of physical quantities C, D pertaining to system S_2 such that $\widehat{C}$ and $\widehat{D}$ have no common eigenvectors. EPR show this in an example involving position and momentum of S_2, but it can also be done for spin in x- and spin in z-direction, for example.

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Since eigenstates allow to predict the outcomes of measurements with certainty, one is forced to ascribe reality to both physical quantities C, D, despite the fact that they are represented by non-commuting operators $\widehat{C}, \widehat{D}$ with no common eigenstates.

EPR argument, 5

But, as we saw, QM does not allow to assign reality to two physical quantities represented by non-commuting operators with no common eigenvectors. Hence, EPR conclude, *"the quantum-mechanical description of reality by the wave function is not complete"*. Some remarks:

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- no argument about signals, speed of light, no special relativity
- non-interacting components: mostly interpreted as spatial separation $\rightarrow$ local realism
- Einstein: *"But on one supposition we should, in my opinion, hold absolutely fast: the real factual situation of the system S_2 is inpedendent of what is done with the system S_1, which is spatially separated from the former."*

Bell's theorem, 1

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He considers an EPR situation with two electrons in the singlet state

$$\psi = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle) = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle).$$

The total spin of the composite system is 0. Only spin degrees of freedom are shown in ψ.

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- the result is either 1 or -1 (which holds for *every* spin measurement),
- the result of the measurement of the spin of the second electron (denoted S_2) in some direction $\mathbf{b}$ is correlated to the result of the measurement on S_1: if $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, then the expectation value of the operator $\widehat{\sigma}_{\mathbf{a}} \otimes \widehat{\sigma}_{\mathbf{b}}$ is

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- In particular, for $\mathbf{b} = \mathbf{a}$, we obtain a perfect anticorrelation: the measurement result on S_2 is the negative of the measurement result on S_1.

Bell's theorem, 3

Assume that the result $S_1(\mathbf{a}, \lambda)$ of the spin measurement on S_1 depends on some hidden variables λ (including the quantum state ψ), and likewise does the result $S_2(\mathbf{b}, \lambda)$ of the spin measurement on S_2.

Bell's theorem, 3

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The main assumption of local realism is that $S_1(\mathbf{a}, \lambda)$ does not depend on $\mathbf{b}$ and $S_2(\mathbf{b}, \lambda)$ does not depend on $\mathbf{a}$.

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The main assumption of local realism is that $S_1(\mathbf{a}, \lambda)$ does not depend on $\mathbf{b}$ and $S_2(\mathbf{b}, \lambda)$ does not depend on $\mathbf{a}$. If $\rho(\lambda)$ is the probability distribution of λ, then the expectation value of the product of the outcomes, as given by the local realist theory, is

$$E(\mathbf{a}, \mathbf{b}) = \int d\lambda \rho(\lambda) S_1(\mathbf{a}, \lambda) S_2(\mathbf{b}, \lambda).$$

Here, the integral is over the set of hidden variables λ which reproduce perfect anticorrelation for $\mathbf{a} = -\mathbf{b}$.

Bell shows that this cannot reproduce the QM expectation value $-\mathbf{a} \cdot \mathbf{b}$ for general directions $\mathbf{a}$ and $\mathbf{b}$, not even approximately.

Bell inequality and Aspect experiments

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Any *experimental* violation of this inequality proves that local hidden variable theories cannot reproduce the predictions of QM. In the 60s, no such experiments existed, since the setup of the detectors happened well before the experiment such that communication (mediating correlations) might have been possible.

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In 1982, Alain Aspect conducted experiments that are widely regarded as proving violations of Bell's inequality. Hence, QM is confirmed, while local realism is refuted.

A. Aspect, J. Dalibard, G. Roger, *"Experimental Test of Bell's Inequalities Using Time-Varying Analyzers"*, Phys. Rev. Lett. **49**, 1804–1807 (1982)

CHSH inequality

Bell's original inequality is rarely used. A related inequality is the CHSH inequality:

$$-2 \leq E(\mathbf{a},\mathbf{b}) + E(\mathbf{a},\mathbf{b}') + E(\mathbf{a}',\mathbf{b}) - E(\mathbf{a}',\mathbf{b}') \leq 2,$$

where $\mathbf{a}, \mathbf{a}'$ are spin directions measured on S_1 and $\mathbf{b}, \mathbf{b}'$ are spin directions measured on S_2.

J.F. Clauser, M.A. Horne, A. Shimony and R.A. Holt, *"Proposed experiment to test local hidden-variable theories"*, Phys. Rev. Lett. **23**, 880–884 (1969)

GHZ states and Bell's theorem without inequalities, 1

D.M. Greenberger, M.A. Horne, A. Zeilinger, *"Going beyond Bell's Theorem"*, in Bells Theorem, Quantum Theory and Conceptions of the Universe, ed. M. Kafatos, Kluwer, Dordrecht-Boston-London, 69–72 (1989), also available from arXiv:0712.0921

In this paper, the observation is made that the EPR argument applied to spin measurements uses the case of perfect anticorrelation (spin measurements in the *same* direction on both particles), while Bell needed to consider spin measurements in *different* directions in order for his argument on correlations to work.

GHZ states and Bell's theorem without inequalities, 2

In fact, there exists a classical hidden variables model for the case of perfect anticorrelation. GHZ consider if this is also true beyond the two-particle case considered by Bell: can the case of perfect quantum-theoretic anticorrelation always be modelled by a classical hidden variables model?

GHZ states and Bell's theorem without inequalities, 2

In fact, there exists a classical hidden variables model for the case of perfect anticorrelation. GHZ consider if this is also true beyond the two-particle case considered by Bell: can the case of perfect quantum-theoretic anticorrelation always be modelled by a classical hidden variables model?

They consider a 4-particle state, generated from a Bell-type situation through further decay and show that there cannot be a hidden variables model to reproduce the case of perfect anticorrelation. Soon after, similar arguments were made for 3 entangled particles. The theoretical results have been confirmed experimentally. Multi-partite entanglement is very important in quantum information theory nowadays.

Propositions, projections and probability, 1

A.M. Gleason, *"Measures on the closed subspaces of a Hilbert space"*, Journal of Mathematics and Mechanics **6**, 885–893 (1957)

Via the spectral theorem, a proposition of the form "$A \in \Delta$", that is, "the physical quantity A has a value in the set Δ of real numbers", corresponds to a projection operator $\widehat{P} = \widehat{E}[A \in \Delta]$ on $\mathcal{H}$. If the algebra of physical quantities is a von Neumann algebra $\mathcal{N}$ (that contains the self-adjoint operator $\widehat{A}$ representing the physical quantity A), then the projection is contained in $\mathcal{N}$.

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Taking the probabilistic interpretation of QM seriously, one may wonder

a. What are the minimal requirements for a mapping from the projections (representing propositions) to the unit interval $[0, 1]$ to count as a probability measure?

b. How do these measures relate to quantum states?

These are the questions answered by Gleason in his famous paper.

Propositions, projections and probability, 2

The projections on a Hilbert space $\mathcal{H}$, and, more generally, the projections in any von Neumann algebra $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$, form a complete lattice. The lattice operations are easier to understand for the isomorphic lattice of closed subspaces of Hilbert space.

Let $\widehat{P}, \widehat{Q}$ be two projections, and let $U_{\widehat{P}}, U_{\widehat{Q}}$ be the corresponding closed subspaces. Then $\widehat{P} < \widehat{Q}$ if $U_{\widehat{P}} \subset U_{\widehat{Q}}$, and the meet (minimum) $\widehat{P} \wedge \widehat{Q}$ is the projection corresponding to the intersection $U_{\widehat{P}} \cap U_{\widehat{Q}}$. The join (maximum) $\widehat{P} \vee \widehat{Q}$ corresponds to the closure of the subspace generated by $U_{\widehat{P}} \cup U_{\widehat{Q}}$.

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Two projections $\widehat{P}, \widehat{Q}$ are orthogonal if $U_{\widehat{P}} \cap U_{\widehat{Q}}$ is the null subspace. In this case, $\widehat{P}\widehat{Q} = \widehat{Q}\widehat{P} = \widehat{0}$.

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The lattice $\mathcal{P}(\mathcal{H})$ (or, more generally, $\mathcal{P}(\mathcal{N})$) is ortho-complemented by $\widehat{P} \mapsto \widehat{1} - \widehat{P}$. Clearly, $\widehat{P}\widehat{Q} = \widehat{0}$ if and only if $\widehat{P} \leq \widehat{1} - \widehat{Q}$.

Propositions, projections and probability, 3

The identity operator $\widehat{1}$ represents the trivially true proposition. A probability measure on the projections is a mapping

$$\mu : \mathcal{P}(\mathcal{H}) \longrightarrow [0, 1]$$

such that

- $\mu(\widehat{1}) = 1$,
- if $\widehat{P}\widehat{Q} = 0$, then $\mu(\widehat{P} \vee \widehat{Q}) = \mu(\widehat{P} + \widehat{Q}) = \mu(\widehat{P}) + \mu(\widehat{Q})$.

More specifically, this defines a finitely additive probability measure. If μ behaves additively on countable families of pairwise orthogonal projections, then it is called countably additive.

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More specifically, this defines a finitely additive probability measure. If μ behaves additively on countable families of pairwise orthogonal projections, then it is called countably additive.

Gleason's theorem shows that for $\dim(\mathcal{H}) \geq 3$, every countably additive probability measure on projections is of the form

$$\forall \widehat{P} \in \mathcal{P}(\mathcal{H}) : \mu(\widehat{P}) = \mathrm{tr}(\rho\widehat{P}),$$

where ρ is a density operator.

Density matrices and normal states

A density operator ρ is a positive trace class operator of trace 1. It induces a linear functional on $\mathcal{B}(\mathcal{H})$ (or some von Neumann algebra $\mathcal{N}$) of norm 1, i.e. a state (in the mathematical sense) by

$$\begin{aligned} \mathrm{tr}(\rho\_) : \mathcal{B}(\mathcal{H}) &\longrightarrow \mathbb{C} \\ \widehat{A} &\longmapsto \mathrm{tr}(\rho\widehat{A}). \end{aligned}$$

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States of this form a normal, i.e., for every countable family $(\widehat{P}_i)_{i\in I}$ of pairwise orthogonal projections, one has

$$\mathrm{tr}(\rho \bigvee_{i\in I} \widehat{P}_i) = \sum_{i\in I} \mathrm{tr}(\rho\widehat{P}_i).$$

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Normal states (or just their density matrices) are usually regarded as physical states, often called mixed states. They generalise the vector states ψ (or rather $\langle\psi, \_\psi\rangle : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$) we encountered before. Normal states can be seen as convex combinations of vector states.

Generalisation of Gleason's theorem

Clearly, for every normal state, $\mathrm{tr}(\rho\_)|_{\mathcal{P}(\mathcal{H})}$ is a countably additive measure.

Gleason's theorem hence shows that there is a bijection between normal states and countably additive measures. This is the justification for calculating expectation values using the trace (which goes back to von Neumann in 1928).

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There are von Neumann algebras, also such of physical significance, which do not possess any normal states. Remarkably, Gleason's theorem can be generalised:

Let $\mathcal{N}$ be a von Neumann algebra with no direct summand of type I_2. Then every finitely additive probability measure on $\mathcal{P}(\mathcal{N})$ can be extended to a state of $\mathcal{N}$.

S. Maeda, *"Probability measures on projections in von Neumann algebras."*, Rev. Math. Phys. **1**, Issue 2/3, 235–290 (1989)

An underlying theory?

S. Kochen, E.P. Specker, *"The problem of hidden variables in quantum mechanics"*, Journal of Mathematics and Mechanics **17**, 59–87 (1967).

While Bell's theorem considers *local* hidden variable theories, Kochen and Specker asked if there can exist theory underlying QM such that a *state space* picture similar to classical physics can be regained. An analogy is (classical) statistical mechanics, which underlies and subsumes thermodynamics.

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In classical physics, physical quantities are represented by real-valued functions on the state space $\mathcal{S}$ of the system. The points of state space are the states, and in any state $s \in \mathcal{S}$, all physical quantities have values, just by evaluation at s.

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If a hidden state space theory underlying QT exists, then each self-adjoint operator should correspond to a suitable real-valued function on this state space Ω. The points of this hypothetical state space would *not* be vectors ψ in Hilbert space.

Valuation functions

Let us assume that a self-adjoint operator $\widehat{A} \in \mathcal{B}(\mathcal{H})$ is represented by some function $f_{\widehat{A}} : \Omega \to \mathbb{R}$. It is natural to require that

a. the range of the function $f_{\widehat{A}}$ is the spectrum $\mathrm{sp}(\widehat{A})$,

b. if $\widehat{B} = g(\widehat{A})$ is some self-adjoint operator obtained from $\widehat{A}$ by applying a real-valued (Borel) function, then $f_{\widehat{B}} = f_{g(\widehat{A})} = g \circ f_{\widehat{A}}$.

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Let $\omega \in \Omega$ be a point of the hypothetical state space. Then we can evaluate the functions $f_{\widehat{A}}, \ldots$ at ω. Going back from the functions to the operators, this would give a function

$$v_\omega : \mathcal{B}(\mathcal{H})_{sa} \longrightarrow \mathbb{R}$$

such that

A. for all $\widehat{A} \in \mathcal{B}(\mathcal{H})_{sa}$, we have $v_\omega(\widehat{A}) \in \mathrm{sp}(\widehat{A})$,

B. if $g : \mathbb{R} \to \mathbb{R}$ is a Borel function, then $v_\omega(g(\widehat{A})) = g(v_\omega(\widehat{A}))$.

Such a function is called a valuation function.

The Kochen-Specker construction

The existence of valuation functions is a necessary condition for the existence of state space models of QT, since every point ω of a state space Ω would give a valuation function v_ω.

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For every non-trivial projection $\widehat{P} \in \mathcal{P}(\mathcal{H})$, we have $\mathrm{sp}(\widehat{P}) = \{0, 1\}$. Since every self-adjoint operator is a limit of linear combinations of projection operators via the spectral theorem, it suffices to consider projection operators.

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The question for the existence of valuation functions boils down to the question for the existence of two-valued, finitely additive probability measures on $\mathcal{P}(\mathcal{H})$. Kochen and Specker showed that if $\dim(\mathcal{H}) \geq 3$, then there exist no such measures on $\mathcal{P}(\mathcal{H})$, hence no valuation functions and no state space model of quantum theory.

Interestingly, Kochen and Specker used a *finite* configuration of 117 projections onto one-dimensional subspaces of $\mathbb{C}^3$ in their proof. In 1966, Bell gave the proof of a similar result using Gleason's theorem.

Generalisation of the Kochen-Specker theorem

Using the generalised version of Gleason's theorem, the Kochen-Specker theorem can be generalised to all von Neumann algebras $\mathcal{N}$ without summands of type I_1 and I_2.

A. Döring, *"Kochen-Specker theorem for von Neumann algebras"*, Int. J. Theor. Phys. **44**, 139-160 (2005).

Physically, this means that even if the quantum system has many symmetries and/or superselection rules, there is no state space model.

A logical perspective on the Kochen-Specker theorem

We saw that the KS problem is the question if there are valuation functions

$$v : \mathcal{B}(\mathcal{H})_{sa} \longrightarrow \mathbb{R},$$

but equivalently, the question if there are two-valued measures

$$\mu : \mathcal{P}(\mathcal{H}) \longrightarrow \{0, 1\}$$

on projections. Since every projection represents a proposition (actually, more than one in general), such a measure can be seen as a Boolean truth-value assignment. The KS theorem hence shows that we cannot assign 'true' and 'false' to all propositions of the form "$A \in \Delta$" at once.

Contextuality

Many people interpret the KS theorem as saying that there are no non-contextual hidden states models.

Contextuality has a certain variety of meanings. A context can be an experimental setup, a collection of co-measurable physical quantities, or algebraically, a commutative subalgebra of the non-commutative (C^*- or von Neumann) algebra of all physical quantities of the system under consideration.

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The perspective of KS is non-contextual, since they ask for valuation functions which assign values directly to the self-adjoint operators, without regarding the contexts they lie in.

A contextual theory would allow the value assigned to some operator $\widehat{A}$ to depend on the context considered. In fact, Bohr's view on QM comes close to that.