

CATEGORIES AND NONASSOCIATIVE C^* -ALGEBRAS
IN QUANTUM FIELD THEORY

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OUTLINE

1. Quantum mechanics and C^* -algebras.
2. Gel'fand's Theorem and the Dixmier-Douady obstruction.
3. Twisted compact operators.
4. T-duality.
5. Monoidal categories.
6. Nonassociative C^* -algebras.

The observables generate an algebra of operators on a Hilbert space \mathcal{H} , closed under addition, multiplication, and adjoints.

One can restrict to bounded operators $\mathcal{B}(\mathcal{H})$:

$$\|A\| = \sup\{\|A\psi\| : \|\psi\| = 1\} < \infty$$

A C*-algebra is a norm-closed subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} .

EXAMPLE: THE COMPACT OPERATORS \sim THE CCR ALGEBRA

The compact operators $\mathcal{K}(\mathcal{H})$ are the C^* -algebra generated by the rank-one operators in $\mathcal{B}(\mathcal{H})$, i.e. those of the form

$$|\xi\rangle\langle\eta| : \psi \mapsto \xi\langle\eta, \psi\rangle.$$

If $\mathcal{H} = L^2(X)$, (normalisable wave functions on X , the compact operators K can be represented as integral operators

$$(K\psi)(x) = \int_X K(x, y)\psi(y) dy$$

where the integral kernel $K(x, y)$ is a limit of separable kernels

$$K_N(x, y) = \sum_{j=1}^N \alpha_j(x)\beta_j(y).$$

THE COMPOSITION AND ADJOINT OF COMPACT OPERATORS

The composition is then

$$(K_1 \circ K_2)(x, z) = \int_X K_1(x, y)K_2(y, z) dy$$

The adjoint is

$$K^*(x, y) = \overline{K(y, x)}$$

cf matrices when integral is replaced by a sum.

EXAMPLE: FUNCTION ALGEBRAS

Take E a locally compact Hausdorff topological space with a measure μ ,

let $\mathcal{H} = L^2(E, \mu)$,

multiplication by compactly supported, continuous, complex-valued functions

$f \in C_K(E)$

$$(f \cdot \psi)(x) = f(x)\psi(x)$$

for $x \in E$, $\psi \in L^2(E, \mu)$ gives a subalgebra of $\mathcal{B}(\mathcal{H})$, with

$$(f_1 \circ f_2)(x) = f_1(x)f_2(x), \quad f^*(x) = \overline{f(x)}$$

GEL'FAND'S THEOREM

Every commutative C^* -algebra is $C_K(E)$ for some locally compact Hausdorff space E , and

the category of commutative C^* -algebras is contravariantly equivalent to the category of locally compact Hausdorff spaces, via the functors

$$\begin{array}{ccc} \text{spec}(\mathcal{A}) & \rightarrow & \mathcal{A} \\ E & \leftarrow & C_K(E) \end{array}$$

where the spectrum of \mathcal{A}

$\text{spec}(\mathcal{A}) =$ equivalence classes of irreducible representations \sim maximal ideals.

What if \mathcal{A} is not commutative?

There is a broader class, the continuous trace C^* -algebras which are given by algebra-valued functions over the spectrum.

Continuous trace C^* -algebra $\mathcal{A} \sim$ sections of $\mathcal{K}(\mathcal{H})$ -bundle over spectrum $E = \text{spec}\mathcal{A}$ (equivalence classes of irreducible representations).

The bundle structure is trivial if and only if the Dixmier–Douady obstruction $\delta \in H^2(E, \mathbb{T}) \cong H^3(E, \mathbb{Z})$ is trivial, (Brauer 1927, . . . , Dixmier–Douady 1964)

DIXMIER–DOUADY THEOREM.

For every such E and $\delta \in H^3(E, \mathbb{Z})$ there is a C^* -algebra $\mathcal{A} = CT(E, \delta)$ with spectrum E and Dixmier–Douady obstruction δ , and it is unique up to Morita equivalence.

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Algebras \mathcal{A}_1 and \mathcal{A}_2 are Morita equivalent if there is an additive equivalence between the categories of \mathcal{A}_1 -modules and \mathcal{A}_2 -modules.

Raeburn, Kumjian, Muhly, Renault, Williams: groupoid C^* algebra proof.

MORITA EQUIVALENCE

THEOREM (MORITA–RIEFFEL) For each additive equivalence from \mathcal{A}_1 -modules to \mathcal{A}_2 -modules there exists a left \mathcal{A}_2 -right \mathcal{A}_1 -bimodule E such that the equivalence is given by $E \otimes_{\mathcal{A}_1} \cdot$.

The compact operators $\mathcal{K}(\mathcal{H})$ are Morita equivalent to \mathbb{C} via the left $\mathcal{K}(\mathcal{H})$ -right \mathbb{C} -bimodule \mathcal{H} .

(Uniqueness of the Canonical Commutation Relations)

SUMMARY

loc. cpt Hausdorff space	\longleftrightarrow	comm. C^* -algebra
cpt Hausdorff space	\longleftrightarrow	comm. C^* -algebra with 1
E	\longrightarrow	$C_0(E)$
spectrum $\text{spec}(\mathcal{A})$	\longleftarrow	algebra \mathcal{A}
noncommutative geometry	\longleftrightarrow	continuous trace C^* -algebra
flux $H \in H^3(E, \mathbb{Z})$	\longleftrightarrow	DD class $\delta \in H^2(E, \mathbb{T})$

Take the same integral operators $\mathcal{K}(L^2(X))$, but with a composition

$$(K_1 * K_2)(x, z) = \int_X \frac{K_1(x, y)K_2(y, z)}{\phi(x, y, z)} dy$$

for a scalar function $\phi : X \times X \times X \rightarrow U(1) = \{z \in \mathbb{C} : |z| = 1\}$.

Problem this is not generally associative:

$$(K_1 * K_2) * K_3 \neq K_1 * (K_2 * K_3)$$

unless $\phi(x, y, z)\phi(x, z, w) = \phi(x, y, w)\phi(y, z, w)$

EINSTEIN'S PRINCIPLE OF GENERAL COVARIANCE:

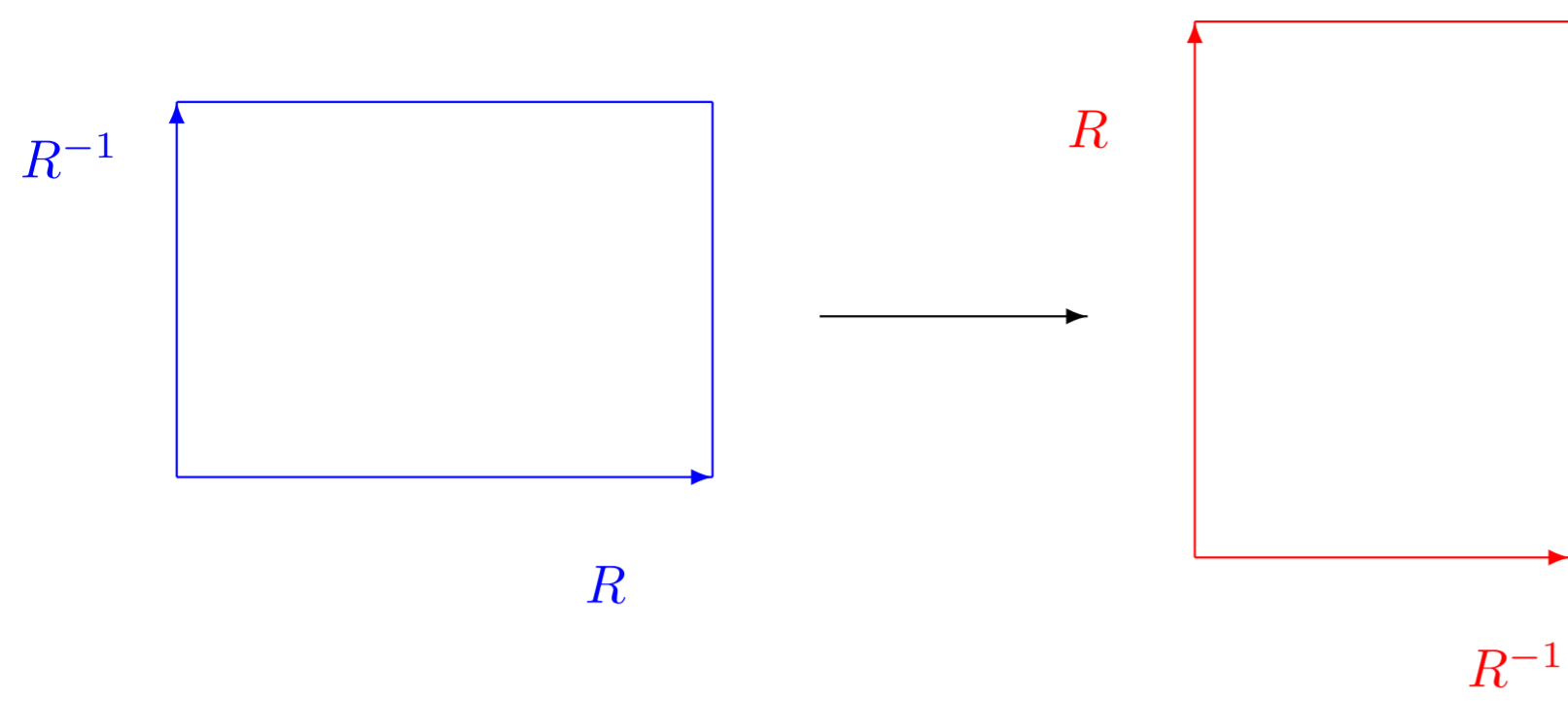
Physical theories should be completely invariant under coordinate transformations.

QUANTUM FIELD THEORY AND STRING THEORY:

Symmetries extend to phase space.

$$\text{configuration space } \mathbb{R}^D \longleftrightarrow \text{momentum space } \hat{\mathbb{R}}^D$$

T-DUALITY: $R \leftrightarrow R^{-1}$ PRESERVES THE PHYSICS



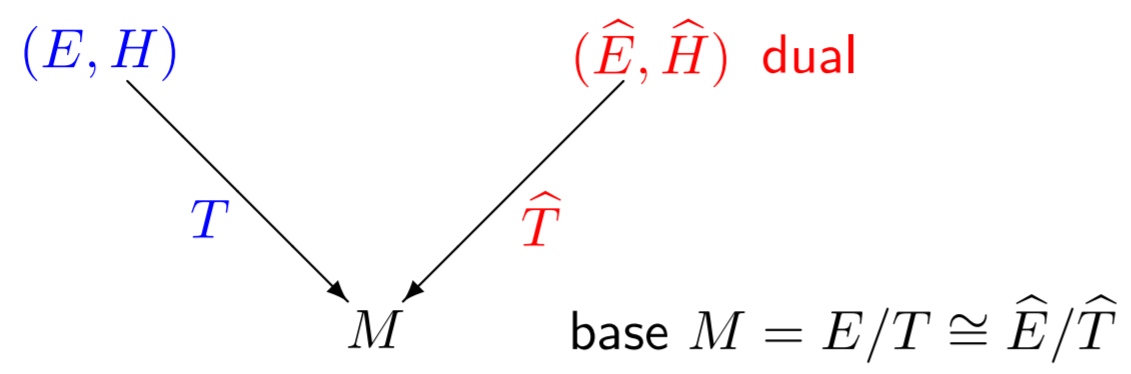
STRING THEORY (HULL AND TOWNSEND)

T-duality: Momentum and winding number interchange

Added ingredient: flux $H \in H^3(E)$

ROUGH PICTURE OF T-DUALITY

T-duality interchanges two principal torus bundles over the same base, and interchanges the curvature of each with the H -flux ($H \in \Omega^3(E)$ or $\widehat{H} \in \Omega^3(\widehat{E})$) of the other.



$$T = \mathbb{T}^k = \mathbb{R}^k / \mathbb{Z}^k \cong \widehat{T}$$

EXAMPLES

It will suffice to consider $E = \mathbb{T}^3$, $T = \mathbb{T}^k$ ($k \leq 3$), and $M = \mathbb{T}^{3-k}$, with H k times the volume 3-form.

$$\begin{array}{c} \mathbb{T}^3 \\ \downarrow \\ \mathbb{T}^k \end{array} \quad \begin{array}{c} \mathbb{T}^3 \\ \downarrow \\ \mathbb{T}^{3-k} \end{array}$$

KNOWN GEOMETRIC DUAL PRINCIPAL TORUS BUNDLES

$$H = H_3 + H_2 + H_1 + H_0 \text{ where } H_p \in \Omega^p(M, \wedge^{3-p}\hat{\mathfrak{t}})$$

with $\hat{\mathfrak{t}}$ the dual of the Lie algebra of T .

$\dim T$	H		
1	arbitrary	Bouwknegt, Evslin, Mathai	2004
arbitrary	$H_1, H_0 = 0$	Bouwknegt, KCH, Mathai	2004
arbitrary	$H_0 = 0$	Mathai, Rosenberg	2004
arbitrary	arbitrary	Bouwknegt, KCH, Mathai	2005,6

EXAMPLE

\mathbb{T}^3 with volume form: volume is generated by $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ is
 $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$. for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$.

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- as a \mathbb{T}^2 -bundle over \mathbb{T} : $H = H_1$: noncommutative dual (Mathai–Rosenberg 2004)

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- as a \mathbb{T} -bundle over \mathbb{T}^2 : $H = H_2$ geometric dual;
- as a \mathbb{T}^2 -bundle over \mathbb{T} : $H = H_1$: noncommutative dual (Mathai–Rosenberg 2004)
- as a \mathbb{T}^3 bundle over a point: $H = H_0$: nonassociative dual (Bouwknegt, KCH, Mathai 2005,6).

H-FLUX

Gel'fand's theorem allows one to replace E by a C^* -algebra.

The exact sequence of groups

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{T} \rightarrow 0$$

gives an isomorphism $H^3(E, \mathbb{Z}) \cong H^2(E, \mathbb{T})$:

Identify H with the Dixmier–Douady class δ and replace (E, H) by $CT(E, \delta)$.

AUTOMORPHISM GROUPS

Now consider a principal $T = G/N$ -bundle E over M .

Does G act as automorphisms of $CT(E, \delta)$?

Suppose $\alpha : G \rightarrow \text{Aut}(\mathcal{A})$, and the same subgroup N stabilises each irreducible.

Then $E = \text{spec}(\mathcal{A})$ is a $T = G/N$ -bundle over $M = \text{spec}(\mathcal{A})/G$.

If $G = \mathbb{R}$ every principal G/N -bundle arises in this way, but for general groups G this is not always true.

CROSSED PRODUCTS

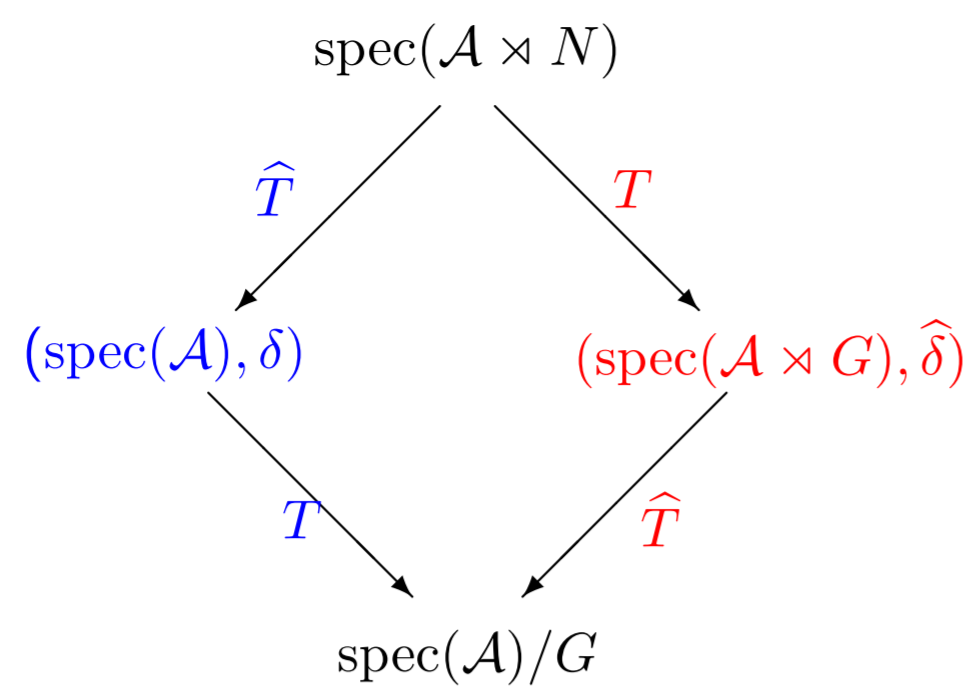
Crossed product $\mathcal{A} \rtimes G = C_0(G, \mathcal{A})$

$$(f * g)(x) = \int_G f(y) \alpha_y[g(y^{-1}x)] dy, \quad f^*(x) = \alpha_x[f(x^{-1})]^*$$

FACTS.

1. Under suitable assumptions $\widehat{\mathcal{A}} = \mathcal{A} \rtimes G$ is also a continuous trace algebra with an action of the dual group \widehat{G} ;
2. (TAKAI-TAKESAKI DUALITY) $\widehat{\mathcal{A}} \rtimes \widehat{G} \cong \mathcal{A} \otimes \mathcal{K}(L^2(G)) \sim_M \mathcal{A}$.

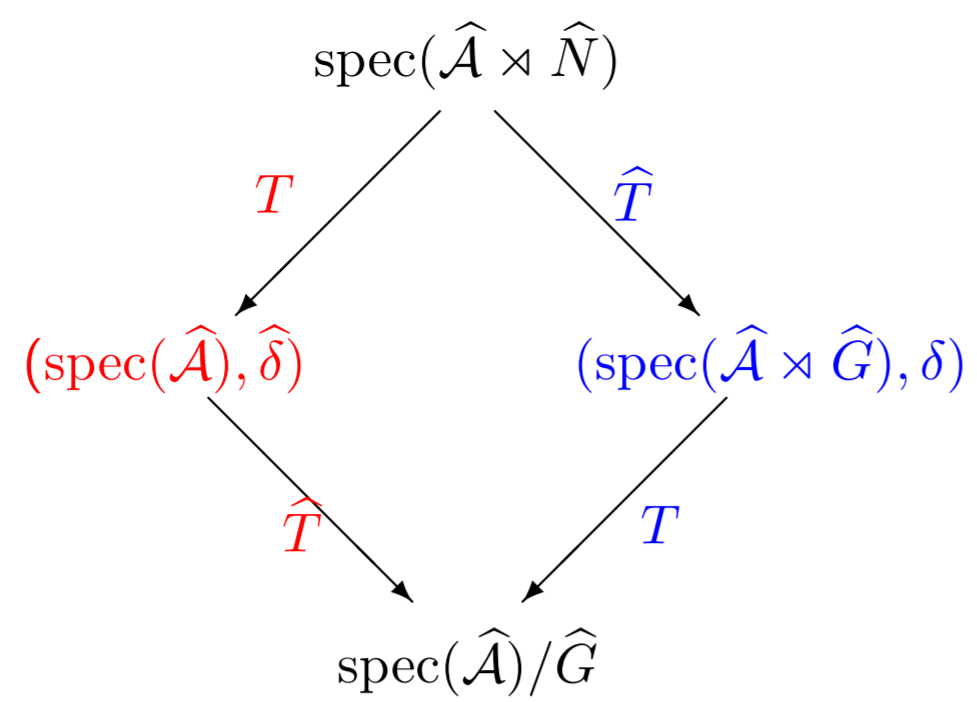
DUALITY



\hat{T} is isomorphic to the group-theoretic dual of N .

Connes' Thom isomorphism theorem: $K_*(\mathcal{A} \rtimes \mathbb{R}^D) \cong K_{*+D}(\mathcal{A})$.

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REMAINING CASE

PUZZLE:

$H_0 \neq 0$ never seems to show up in C^* -algebra literature.

There are spaces with any H , so problem must lie with group action.

NONASSOCIATIVE CASE

Inner automorphisms act trivially on spectrum.

$$G \rightarrow \text{Out}(\mathcal{A}) = \text{Aut}(\mathcal{A})/\text{Inn}(\mathcal{A})$$

Lift to $\alpha : G \rightarrow \text{Aut}(\mathcal{A})$: $\alpha_x \alpha_y = \text{ad}(u(x, y)) \alpha_{xy}$

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$$\text{ad}(u(x, y)) \text{ad}(u(xy, z)) \alpha_{(xy)z} = \text{ad}(\alpha_x[u(y, z)]) \text{ad}(u(x, yz)) \alpha_{x(yz)}$$

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$$\phi(x, y, z) u(x, y) u(xy, z) = \alpha_x[u(y, z)] u(x, yz)$$

PROPERTIES OF ϕ

Central, and satisfies pentagonal cocycle identity

$$\phi(x, y, z)\phi(x, yz, w)\phi(y, z, w) = \phi(xy, z, w)\phi(x, y, zw)$$

ϕ is independent of liftings up to coboundaries

$$\eta(x, y)\eta(xy, z)/\eta(y, z)\eta(x, yz)$$

so only $H^3(G, \mathbb{T})$ class of ϕ matters (but cf. Majid)

$$\phi(\exp(X), \exp(Y), \exp(Z)) = \exp(iH_0(\xi_X, \xi_Y, \xi_Z))$$

where ξ_X is vector field generated by X .

For \mathbb{T}^3 with $k \times \text{vol}$: $\phi(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \exp(2\pi i k[\mathbf{a}, \mathbf{b}, \mathbf{c}])$

†-Category \mathcal{C}_G of \widehat{G} -modules $\sim C_0(G)$ -modules with G -morphisms, and

- module tensor product ($f \in C_0(G)$ acting via comultiplication $(\Delta f)(x, y) = f(xy)$), † action multiplies by $f^*(x) = \overline{f(x)}$;

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- identity object: trivial module \mathbb{C} (action by the counit $\epsilon(f) = f(1)$);
- associator $\Phi : A \otimes (B \otimes C) \rightarrow (A \otimes B) \otimes C \sim$ the action of

$$\phi \in C(G \times G \times G) = C(G) \otimes C(G) \otimes C(G).$$

THE PENTAGONAL IDENTITY

The pentagonal cocycle identity for ϕ gives

$$\begin{array}{ccc}
 \mathcal{A} \otimes (\mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D})) & \longrightarrow & (\mathcal{A} \otimes \mathcal{B}) \otimes (\mathcal{C} \otimes \mathcal{D}) \\
 \swarrow & & \searrow \\
 \mathcal{A} \otimes ((\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D}) & & ((\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}) \otimes \mathcal{D} \\
 \searrow & & \swarrow \\
 & (\mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})) \otimes \mathcal{D} &
 \end{array}$$

ALGEBRAS

DEF. An *algebra* in \mathcal{C}_G is an object \mathcal{A} with a morphism $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ consistent with Φ :

$$\begin{array}{ccccc} \mathcal{A} \otimes (\mathcal{A} \otimes \mathcal{A}) & \longrightarrow & \mathcal{A} \otimes \mathcal{A} & \longrightarrow & \mathcal{A} \\ \Phi \downarrow & & & & \downarrow \\ (\mathcal{A} \otimes \mathcal{A}) \otimes \mathcal{A} & \longrightarrow & \mathcal{A} \otimes \mathcal{A} & \longrightarrow & \mathcal{A} \end{array}$$

The action of \widehat{G} is automatically by automorphisms.

\mathcal{C}_G is a star/bar/dagger category and so one can also define C^* -algebras and Hilbert spaces in \mathcal{C}_G .

EXAMPLES

- Torus bundle \mathbb{T}^3 over a point, with $H_0 = k\text{vol}$

Associated antisymmetric form on $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathfrak{t} = \mathbb{R}^3$ is then given by

$$\phi(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \exp(-2\pi i f(\mathbf{a}, \mathbf{b}, \mathbf{c})) = \exp(-2k\pi i [\mathbf{a}, \mathbf{b}, \mathbf{c}])$$

EXAMPLES

- Torus bundle \mathbb{T}^3 over a point, with $H_0 = k\text{vol}$

Associated antisymmetric form on $\mathbf{a}, \mathbf{b}, \mathbf{c} \in t = \mathbb{R}^3$ is then given by

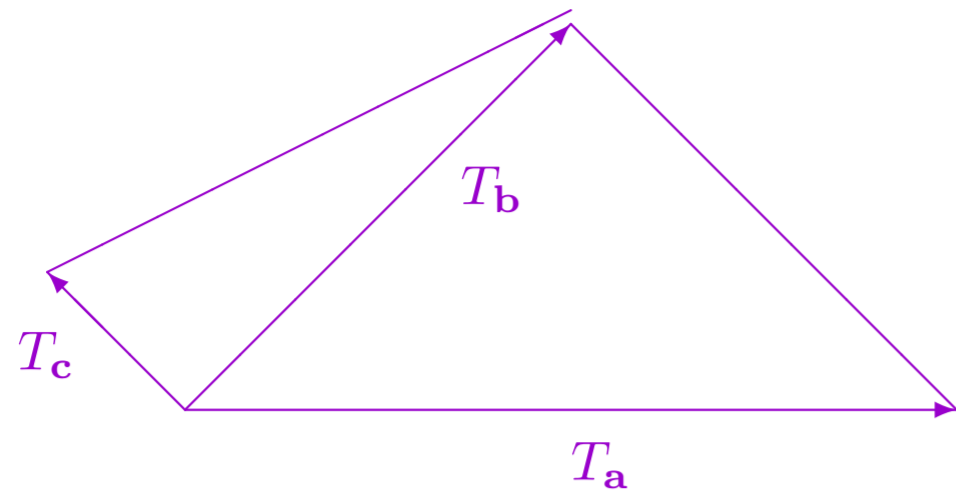
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- $\mathcal{A}_0 = \mathbb{C}$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \sim \{0, 1\}^3 \subseteq \mathbb{R}^3$

$$\phi(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (-1)^{[\mathbf{a}, \mathbf{b}, \mathbf{c}]},$$

suitable u gives the *octonions* (cf Albuquerque and Majid).

Magnetic translations $T_{\mathbf{a}} = \exp(2\pi i \mathbf{a} \cdot \nabla)$



$T_{\mathbf{a}}$ and $T_{\mathbf{b}}$ fail to commute by a factor $\exp(\pi i \Phi)$, where Φ is the flux through face spanned by \mathbf{a} and \mathbf{b} .

$T_{\mathbf{a}}$, $T_{\mathbf{b}}$, and $T_{\mathbf{c}}$ fail to associate by a factor $\exp(\pi i \Phi)$, where Φ is the flux out of the tetrahedron spanned by \mathbf{a} , \mathbf{b} , and \mathbf{c} .

Dirac's monopole argument.

But cf. Carey/Mickelsson

MODULES

DEF. An \mathcal{A} -module in \mathcal{C}_G is an object M with a morphism $\mathcal{A} \otimes M \rightarrow M$ consistent with Φ :

$$\begin{array}{ccccc}
 \mathcal{A} \otimes (\mathcal{A} \otimes M) & \longrightarrow & \mathcal{A} \otimes M & \longrightarrow & M \\
 \Phi \downarrow & & & & \downarrow \\
 (\mathcal{A} \otimes \mathcal{A}) \otimes M & \longrightarrow & \mathcal{A} \otimes M & \longrightarrow & M
 \end{array}$$

The actions of \mathcal{A} and \widehat{G} on M are automatically consistent in that $g[am] = (g[a])(g[m])$, for all $a \in \mathcal{A}$ and $m \in M$, that is one has a covariant representation of $(\widehat{G}, \mathcal{A})$, which is really a representation of $\mathcal{A} \rtimes \widehat{G}$.

THE TWISTED COMPACT OPERATORS

Let \mathcal{H} be a Hilbert space, as right \mathbb{C} -module (with \mathbb{C} the identity object).

Use the Rieffel construction to obtain rank-one operators

$$|\xi\rangle\langle\eta|\zeta = \Phi(\xi\langle\eta, \zeta\rangle)$$

These rank-one operators generate the twisted compact operators having \mathcal{H} as a module.

THE TWISTED COMPACT OPERATORS

When $\mathcal{H} = L^2(X)$ these twisted compact operators can be represented as integral operators with product

$$(K_1 * K_2)(x, z) = \int_X \frac{K_1(x, y)K_2(y, z)}{\phi(x, y, z)} dy$$

THE TWISTED BOUNDED OPERATORS

In the usual case the bounded operators can be characterised as the *adjointable operators*.

Now an adjointable operator A is one for which there exists $A^* : \xi \rightarrow A^*\xi \equiv \xi A$ satisfying

$$\langle A^*\xi, \eta \rangle \equiv \langle \xi A, \eta \rangle = \Phi(\langle \xi, A\eta \rangle)$$