The periodic table of n-categories

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7 January 2009

Plan

1. Degeneracy

Plan

1. Degeneracy
2. The Eckmann-Hilton argument

Plan

1. Degeneracy
2. The Eckmann-Hilton argument
3. Inconvenient elements

Plan

1. Degeneracy
2. The Eckmann-Hilton argument
3. Inconvenient elements
4. Higher maps

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1. Degeneracy
2. The Eckmann-Hilton argument
3. Inconvenient elements
4. Higher maps
5. Stabilisation

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1. Degeneracy
2. The Eckmann-Hilton argument
3. Inconvenient elements
4. Higher maps
5. Stabilisation
6. Other reasons to care

References

- J. Baez and J. Dolan. Higher-dimensional algebra and topological quantum field theory.

References

- J. Baez and J. Dolan. Higher-dimensional algebra and topological quantum field theory. *Journ. Math. Phys.*, 36:6073–6105, 1995. E-print `q-alg/9503002v2` .

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References

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Definition

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A k-degenerate n-category is an n-category with:

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A k-degenerate n-category is an n-category with:

- only one 0-cell

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A k-degenerate n-category is an n-category with:

- only one 0-cell
- only one 1-cell

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A k-degenerate n-category is an n-category with:

- only one 0-cell
- only one 1-cell
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A k-degenerate n-category is an n-category with:

- only one 0-cell
- only one 1-cell
- only one 2-cell

$$\vdots$$

- only one $(k-1)$-cell

Definition

A k-degenerate n-category is an n-category with:

- only one 0-cell
- only one 1-cell
- only one 2-cell

$$\vdots$$

- only one $(k-1)$-cell

So the first non-trivial dimension is k.

Dimension-shift for k-degenerate n-categories

Dimension-shift for k-degenerate n-categories

"old"		**"new"**
n**-category**	$\longrightarrow$	$(n-k)$**-category**

Dimension-shift for k-degenerate n-categories

"old" n**-category**	$\longrightarrow$	**"new"** $(n-k)$**-category**
0-cells		

Dimension-shift for k-degenerate n-categories

"old" n**-category**	$\longrightarrow$	**"new"** $(n-k)$**-category**
0-cells		
1-cells		

Dimension-shift for k-degenerate n-categories

"old" n**-category**	$\longrightarrow$	**"new"** $(n-k)$**-category**
0-cells		
1-cells		
$\vdots$		
$(k-1)$-cells		

Dimension-shift for k-degenerate n-categories

$$\begin{array}{ccc} \textbf{“old”} & & \textbf{“new”} \\ n\textbf{-category} & \longrightarrow & (n-k)\textbf{-category} \end{array}$$

$$\left.\begin{array}{c} \text{0-cells} \\ \text{1-cells} \\ \vdots \\ (k-1)\text{-cells} \end{array}\right\} \text{trivial}$$

Dimension-shift for k-degenerate n-categories

$$
\begin{array}{ccc}
\textbf{``old''} & & \textbf{``new''} \\
n\textbf{-category} & \longrightarrow & (n-k)\textbf{-category} \\
 & & \\
\left.\begin{array}{c} \text{0-cells} \\ \text{1-cells} \\ \vdots \\ (k-1)\text{-cells} \end{array}\right\} \text{trivial} & & \\
 & & \\
k\text{-cells} & \longrightarrow & \text{0-cells}
\end{array}
$$

Dimension-shift for k-degenerate n-categories

$$\begin{array}{ccc}
\textbf{“old”} & & \textbf{“new”} \\
n\textbf{-category} & \longrightarrow & (n-k)\textbf{-category} \\
& & \\
\left.\begin{array}{c} 0\text{-cells} \\ 1\text{-cells} \\ \vdots \\ (k-1)\text{-cells} \end{array}\right\} \text{trivial} & & \\
k\text{-cells} & \longrightarrow & 0\text{-cells} \\
(k+1)\text{-cells} & \longrightarrow & 1\text{-cells}
\end{array}$$

Dimension-shift for k-degenerate n-categories

$$
\begin{array}{ccc}
\textbf{“old”} & & \textbf{“new”} \\
n\textbf{-category} & \longrightarrow & (n-k)\textbf{-category} \\
& & \\
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k\text{-cells} & \longrightarrow & \text{0-cells} \\
(k+1)\text{-cells} & \longrightarrow & \text{1-cells} \\
\vdots & \vdots & \vdots
\end{array}
$$

Dimension-shift for k-degenerate n-categories

"old" n**-category**	$\longrightarrow$	**"new"** $(n-k)$**-category**
$\left.\begin{array}{c}\text{0-cells}\\ \text{1-cells}\\ \vdots\\ (k-1)\text{-cells}\end{array}\right\}$ trivial		
k-cells	$\longrightarrow$	0-cells
$(k+1)$-cells	$\longrightarrow$	1-cells
$\vdots$	$\vdots$	$\vdots$
n-cells	$\longrightarrow$	$(n-k)$-cells

Degenerate categories

Degenerate categories

A category with only one object is a monoid.

Degenerate categories

A category with only one object is a monoid.

“old” category **“new” monoid**

Degenerate categories

A category with only one object is a monoid.

"old" category	**"new" monoid**
objects – trivial	

Degenerate categories

A category with only one object is a monoid.

"old" category		**"new" monoid**
objects – trivial		
morphisms	$\longrightarrow$	objects

Degenerate categories

A category with only one object is a monoid.

“old” category		**“new” monoid**
objects – trivial		
morphisms	$\longrightarrow$	objects
composition	$\longrightarrow$	multiplication

Degenerate categories

A category with only one object is a monoid.

“old” category		**“new” monoid**
objects – trivial		
morphisms	$\longrightarrow$	objects
composition	$\longrightarrow$	multiplication
identity	$\longrightarrow$	unit

Degenerate bicategories

Degenerate bicategories

A bicategory with only one object is a monoidal category.

"old"
bicategory $\longrightarrow$ **"new"**
monoidal category

"old" bicategory	$\longrightarrow$	**"new" monoidal category**

0-cells – trivial

“old” **bicategory**	$\longrightarrow$	**“new”** **monoidal category**
0-cells – trivial		
1-cells	$\longrightarrow$	objects

"old" **bicategory**	$\longrightarrow$	**"new"** **monoidal category**
0-cells – trivial		
1-cells	$\longrightarrow$	objects
2-cells	$\longrightarrow$	morphisms

“old” bicategory	$\longrightarrow$	**“new” monoidal category**
0-cells – trivial		
1-cells	$\longrightarrow$	objects
2-cells	$\longrightarrow$	morphisms
composition		

"old" bicategory		"new" monoidal category
bicategory	$\longrightarrow$	**monoidal category**
0-cells – trivial		
1-cells	$\longrightarrow$	objects
2-cells	$\longrightarrow$	morphisms
composition of 1-cells	$\longrightarrow$	$\otimes$ of objects

1. Degeneracy

“old” bicategory		**“new” monoidal category**
0-cells – trivial		
1-cells	$\longrightarrow\!\!\!\!\triangleright$	objects
2-cells	$\longrightarrow\!\!\!\!\triangleright$	morphisms
composition		
of 1-cells	$\longrightarrow\!\!\!\!\triangleright$	$\otimes$ of objects
of 2-cells $\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd}$	$\longrightarrow\!\!\!\!\triangleright$	$\otimes$ of morphisms

“old” **bicategory**	$\longrightarrow$	**“new”** **monoidal category**
0-cells – trivial		
1-cells	$\longrightarrow$	objects
2-cells	$\longrightarrow$	morphisms
composition		
of 1-cells	$\longrightarrow$	$\otimes$ of objects
of 2-cells $\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd}$	$\longrightarrow$	$\otimes$ of morphisms
of 2-cells $\begin{tikzcd} \cdot \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow"] & \cdot \end{tikzcd}$	$\longrightarrow$	composition of morphisms

In a k-degenerate n-category:

In a k-degenerate n-category:

composition of k-cells $\longrightarrow$ $\otimes$

In a k-degenerate n-category:

composition of k-cells $\longrightarrow$ $\otimes$

k-different types

In a k-degenerate n-category:

composition of k-cells	$\longrightarrow$	$\otimes$
k-different types		k-different types

Example: 2-degenerate

Example: 2-degenerate

“old” $\longrightarrow$ **“new”**

Example: 2-degenerate

$$\textbf{“old”} \qquad \longrightarrow \qquad \textbf{“new”}$$

$$\left.\begin{array}{l}\text{0-cells}\\ \text{1-cells}\end{array}\right\}\ \text{trivial}$$

Example: 2-degenerate

“old” $\longrightarrow$ **“new”**

$\left.\begin{array}{l}\text{0-cells}\\ \text{1-cells}\end{array}\right\}$ trivial

2-cells $\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd}$ $\longrightarrow$ 0-cells

Example: 2-degenerate

"old" $\longrightarrow$ **"new"**

$$\left.\begin{array}{l}\text{0-cells}\\ \text{1-cells}\end{array}\right\}\ \text{trivial}$$

2-cells
$$\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd}$$
$\longrightarrow$ 0-cells

composition

Example: 2-degenerate

"old"	$\longrightarrow$	**"new"**
$\left.\begin{array}{l}\text{0-cells}\\ \text{1-cells}\end{array}\right\}$ trivial		
2-cells $\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd}$	$\longrightarrow$	0-cells
composition		
$\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd}$	$\longrightarrow$	$\otimes_0$

Example: 2-degenerate

“old”	$\longrightarrow$	**“new”**
$\left.\begin{array}{l}\text{0-cells}\\ \text{1-cells}\end{array}\right\}$ trivial		
2-cells $\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd}$	$\longrightarrow$	0-cells
composition		
$\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd}$	$\longrightarrow$	$\otimes_0$
$\begin{tikzcd} \cdot \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2, "\Downarrow"] \arrow[r, phantom, shift right=2, "\Downarrow"] & \cdot \end{tikzcd}$	$\longrightarrow$	$\otimes_1$

Example: 2-degenerate

"old" $\longrightarrow$ **"new"**

$\left.\begin{array}{l}\text{0-cells}\\ \text{1-cells}\end{array}\right\}$ trivial

2-cells $\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd}$ $\longrightarrow$ 0-cells

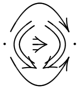
composition

$\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd}$ $\longrightarrow$ $\otimes_0$

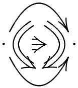
$\begin{tikzcd} \cdot \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2, "\Downarrow"] \arrow[r, phantom, shift right=2, "\Downarrow"] & \cdot \end{tikzcd}$ $\longrightarrow$ $\otimes_1$

?

Example: 3-degenerate

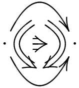
3-cells  $\longrightarrow$ 0-cells

Example: 3-degenerate

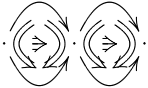
3-cells  $\longrightarrow$ 0-cells

composition

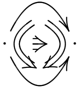
Example: 3-degenerate

3-cells  $\longrightarrow$ 0-cells

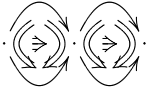
composition

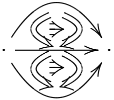
 $\longrightarrow$ $\otimes_0$

Example: 3-degenerate

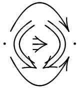
3-cells  $\longrightarrow$ 0-cells

composition

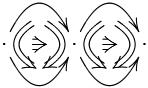
 $\longrightarrow$ $\otimes_0$

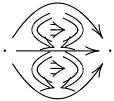
 $\longrightarrow$ $\otimes_1$

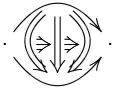
Example: 3-degenerate

3-cells  $\longrightarrow$ 0-cells

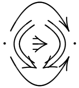
composition

 $\longrightarrow$ $\bigotimes_0$

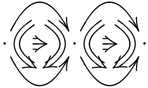
 $\longrightarrow$ $\bigotimes_1$

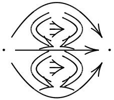
 $\longrightarrow$ $\bigotimes_2$

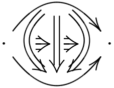
Example: 3-degenerate

3-cells  $\longrightarrow$ 0-cells

composition

 $\longrightarrow$ $\otimes_0$

 $\longrightarrow$ $\otimes_1$

 $\longrightarrow$ $\otimes_2$

?

We have coherence cells for interchange:

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$$\begin{tikzcd}[row sep=small]
\cdot \arrow[r, bend left=50] \arrow[r] \arrow[r, phantom, shift left=2ex, "\Downarrow"] & \cdot \arrow[r, bend left=50] \arrow[r] \arrow[r, phantom, shift left=2ex, "\Downarrow"] & \cdot \\
\cdot \arrow[r, bend right=50] \arrow[r] \arrow[r, phantom, shift right=2ex, "\Downarrow"] & \cdot \arrow[r, bend right=50] \arrow[r] \arrow[r, phantom, shift right=2ex, "\Downarrow"] & \cdot
\end{tikzcd}
\cong
\begin{tikzcd}
\cdot \arrow[r, bend left=50] \arrow[r] \arrow[r, bend right=50] \arrow[r, phantom, shift left=2ex, "\Downarrow"] \arrow[r, phantom, shift right=2ex, "\Downarrow"] & \cdot & \cdot \arrow[r, bend left=50] \arrow[r] \arrow[r, bend right=50] \arrow[r, phantom, shift left=2ex, "\Downarrow"] \arrow[r, phantom, shift right=2ex, "\Downarrow"] & \cdot
\end{tikzcd}$$

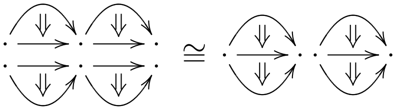
We have coherence cells for interchange:

$$
\begin{tikzcd}[column sep=large]
\cdot \arrow[r, bend left=60] \arrow[r] \arrow[r, phantom, shift left=3ex, "\Downarrow"] & \cdot \arrow[r, bend left=60] \arrow[r] \arrow[r, phantom, shift left=3ex, "\Downarrow"] & \cdot \\
\cdot \arrow[r, bend right=60] \arrow[r] \arrow[r, phantom, shift right=3ex, "\Downarrow"] & \cdot \arrow[r, bend right=60] \arrow[r] \arrow[r, phantom, shift right=3ex, "\Downarrow"] & \cdot
\end{tikzcd}
\quad \cong \quad
\begin{tikzcd}[column sep=large]
\cdot \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3ex, "\Downarrow"] \arrow[r, phantom, shift right=3ex, "\Downarrow"] & \cdot
\end{tikzcd}
\;
\begin{tikzcd}[column sep=large]
\cdot \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3ex, "\Downarrow"] \arrow[r, phantom, shift right=3ex, "\Downarrow"] & \cdot
\end{tikzcd}
$$

becomes

1. Degeneracy

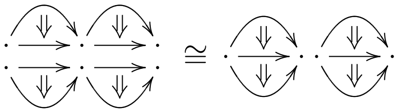
We have coherence cells for interchange:



becomes

$$(a \otimes_0 b) \otimes_1 (c \otimes_0 d) \cong (a \otimes_1 c) \otimes_0 (b \otimes_1 d).$$

We have coherence cells for interchange:



becomes

$$(a \otimes_0 b) \otimes_1 (c \otimes_0 d) \cong (a \otimes_1 c) \otimes_0 (b \otimes_1 d).$$

We have analogous coherence cells for all dimensions, with axioms.

1. Degeneracy

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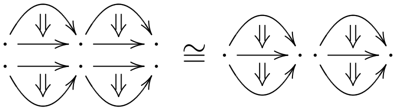

becomes

$$(a \otimes_0 b) \otimes_1 (c \otimes_0 d) \cong (a \otimes_1 c) \otimes_0 (b \otimes_1 d).$$

We have analogous coherence cells for all dimensions, with axioms.

This is called k-**tuply monoidal**.

Slogan:

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k-degenerate n-categories

"are"

k-tuply monoidal $(n-k)$-categories.

Slogan:

k-degenerate n-categories

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k-tuply monoidal $(n-k)$-categories.

—but this is only part of the point.

1. Degeneracy

1. Degeneracy

only one object	symmetric mon.	symmetric mon.	symmetric mon.	...
set				

1. Degeneracy

set	category			

1. Degeneracy

set	category			

1. Degeneracy

set	category			
monoid ≡ category with only one object				

1. Degeneracy

set	category	2-category		
monoid $\equiv$ category with only one object				

1. Degeneracy

set	category	2-category		
monoid $\equiv$ category with only one object				

1. Degeneracy

set	category	2-category		
monoid ≡ category with only one object	monoidal cat. ≡ 2-category with only one object			

1. Degeneracy

set	category	2-category		
monoid ≡ category with only one object	monoidal cat. ≡ 2-category with only one object			

1. Degeneracy

set	category	2-category
monoid ≡ category with only one object	monoidal cat. ≡ 2-category with only one object	
commutative monoid ≡ 2-cat. with only one 1-cell		

1. Degeneracy

set	category	2-category	3-category	
monoid $\equiv$ category with only one object	monoidal cat. $\equiv$ 2-category with only one object			
commutative monoid $\equiv$ 2-cat. with only one 1-cell				

1. Degeneracy

set	category	2-category	3-category	
monoid ≡ category with only one object	monoidal cat. ≡ 2-category with only one object			
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set	category	2-category	3-category	
monoid ≡ category with only one object	monoidal cat. ≡ 2-category with only one object	monoidal 2-cat. ≡ 3-category with only one object		
commutative monoid ≡ 2-cat. with only one 1-cell				

1. Degeneracy

set	category	2-category	3-category	
monoid ≡ category with only one object	monoidal cat. ≡ 2-category with only one object	monoidal 2-cat. ≡ 3-category with only one object		
commutative monoid ≡ 2-cat. with only one 1-cell				

1. Degeneracy

set	category	2-category	3-category	
monoid ≡ category with only one object	monoidal cat. ≡ 2-category with only one object	monoidal 2-cat. ≡ 3-category with only one object		
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" $\equiv$ 3-cat. with only one 2-cell				

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〃 ≡ 3-cat. with only one 2-cell				
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" $\vdots$				

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" ⋮				

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commutative monoid $\equiv$ 2-cat. with only one 1-cell	braided mon. category $\equiv$ 3-cat. with only one 1-cell	braided mon. 2-category $\equiv$ 4-cat. with only one 1-cell		
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" $\equiv$ 4-cat. with only one 3-cell	" $\equiv$ 5-cat. with only one 3-cell			
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〃 ≡ 4-cat. with only one 3-cell	〃 ≡ 5-cat. with only one 3-cell	**symmetric mon. 2-category** ≡ 6-cat. with only one 3-cell	*** mon. 3-category ≡ 7-cat. with only one 3-cell	
〃 ⋮	〃 ⋮	〃 ⋮		

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″ ≡ 3-cat. with only one 2-cell	**symmetric mon. category** ≡ 4-cat. with only one 2-cell	sylleptic mon. 2-category ≡ 5-cat. with only one 2-cell	sylleptic mon. 3-category ≡ 6-cat. with only one 2-cell	⋯
″ ≡ 4-cat. with only one 3-cell	″ ≡ 5-cat. with only one 3-cell	**symmetric mon. 2-category** ≡ 6-cat. with only one 3-cell	*** mon. 3-category ≡ 7-cat. with only one 3-cell	⋯
″ ⋮	″ ⋮	″ ⋮	**symmetric mon. 3-category** ⋮	⋯

2. The Eckmann-Hilton argument

Let A be a set with two binary operations $*$ and $\circ$ such that

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$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d).$$

Then $*$ and $\circ$ are in fact equal and this operation is commutative.

In a bicategory we have the interchange laws:

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$$
\begin{tikzcd}
\cdot \arrow[r, bend left=60] \arrow[r] \arrow[r, phantom, shift left=2ex, "\Downarrow"] & \cdot \arrow[r, bend left=60] \arrow[r] \arrow[r, phantom, shift left=2ex, "\Downarrow"] & \cdot \\[-3ex]
\cdot \arrow[r, bend right=60] \arrow[r] \arrow[r, phantom, shift right=2ex, "\Downarrow"] & \cdot \arrow[r, bend right=60] \arrow[r] \arrow[r, phantom, shift right=2ex, "\Downarrow"] & \cdot
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\begin{tikzcd}[row sep=tiny]
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\cdot \arrow[r, bend right=60] \arrow[r, phantom, shift right=2ex, "\Downarrow"] \arrow[r] & \cdot \arrow[r, bend right=60] \arrow[r, phantom, shift right=2ex, "\Downarrow"] \arrow[r] & \cdot
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\end{tikzcd}
$$

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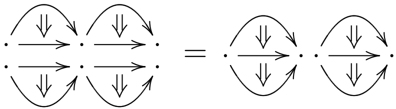

becomes

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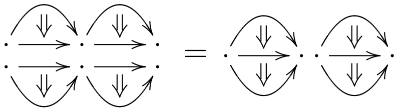

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This is exactly the condition needed for the Eckmann-Hilton argument

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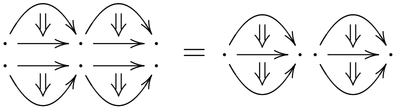

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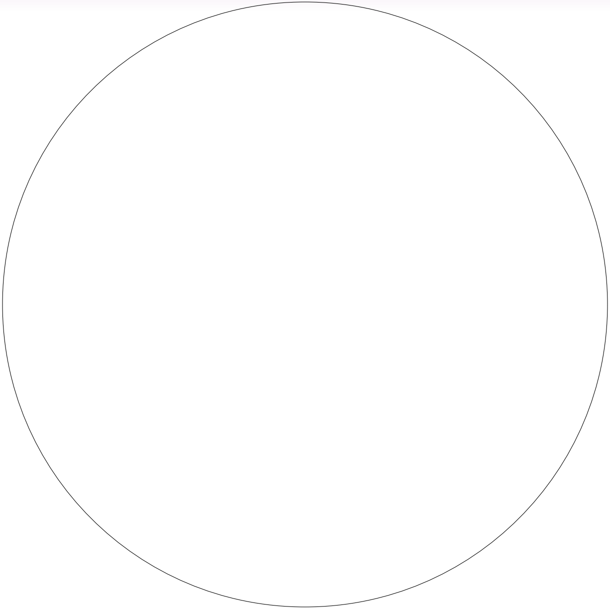
2. The Eckmann-Hilton argument



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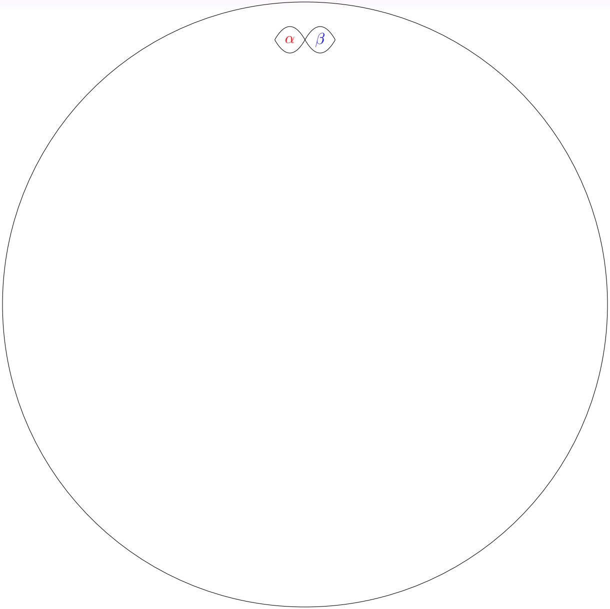

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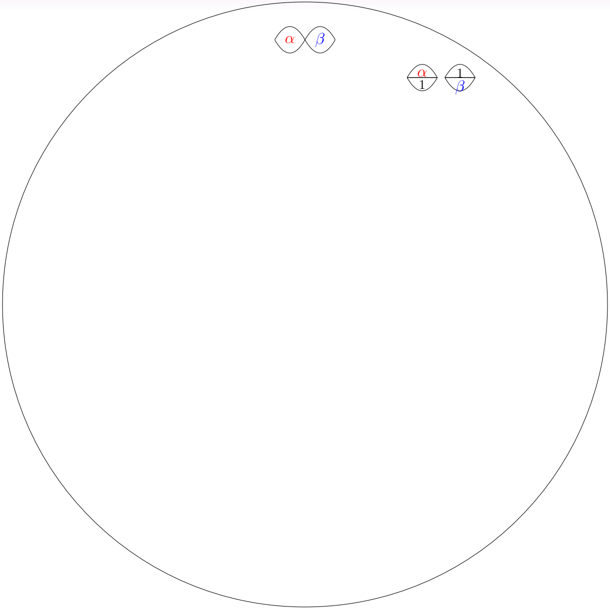

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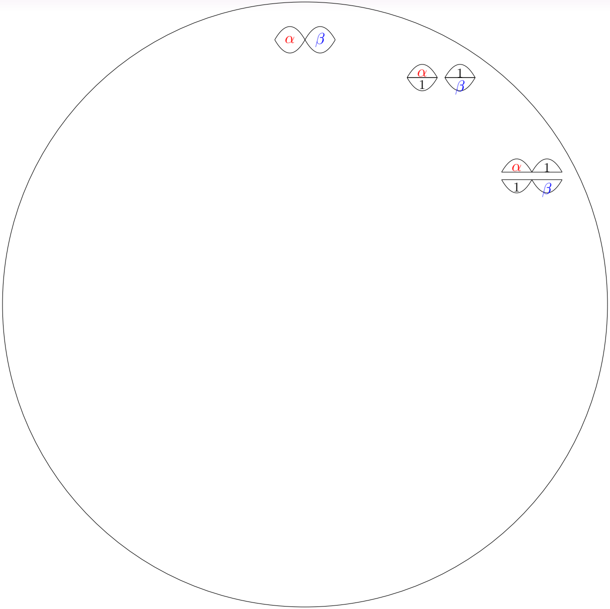

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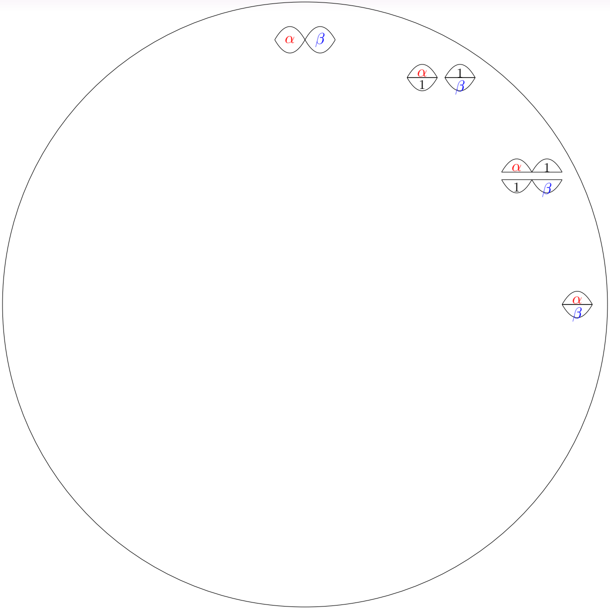

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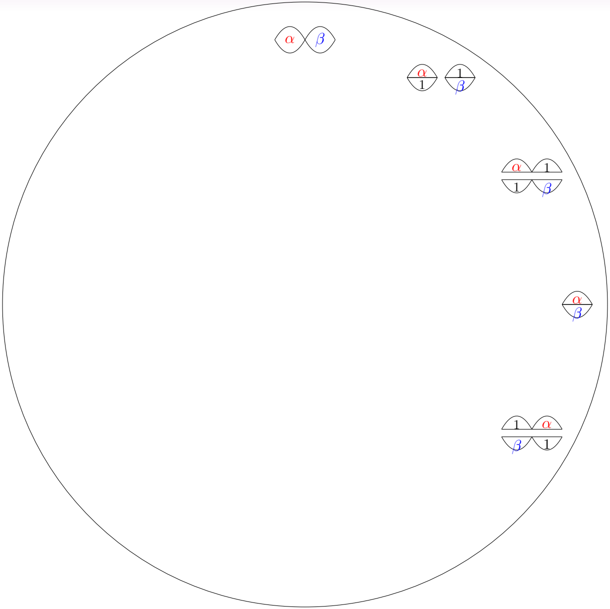

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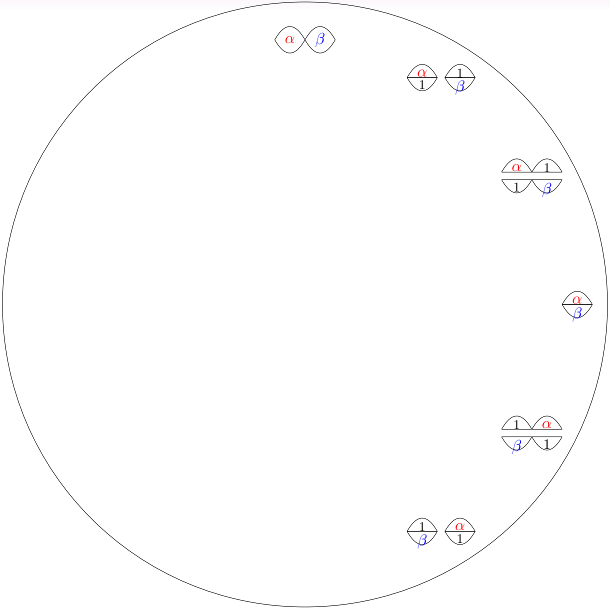

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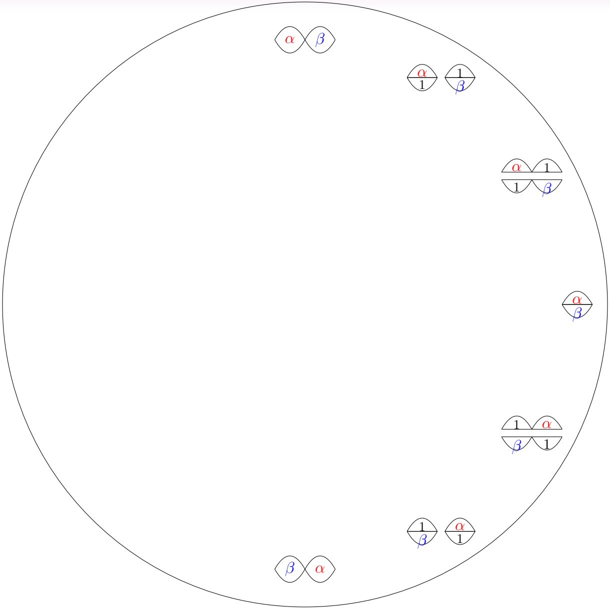

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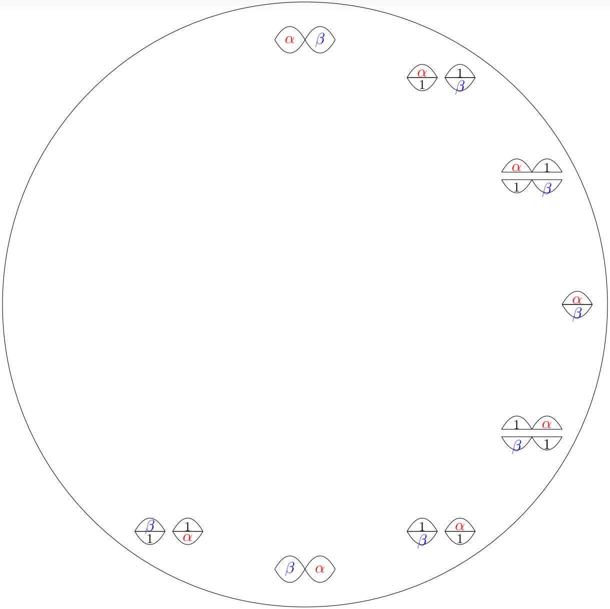

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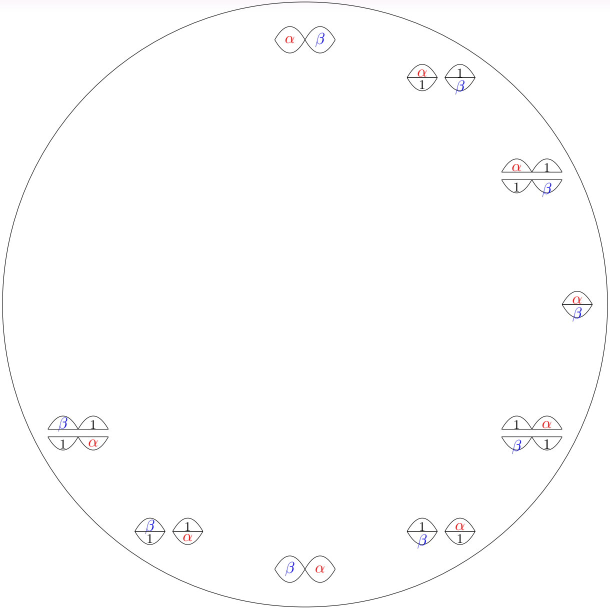

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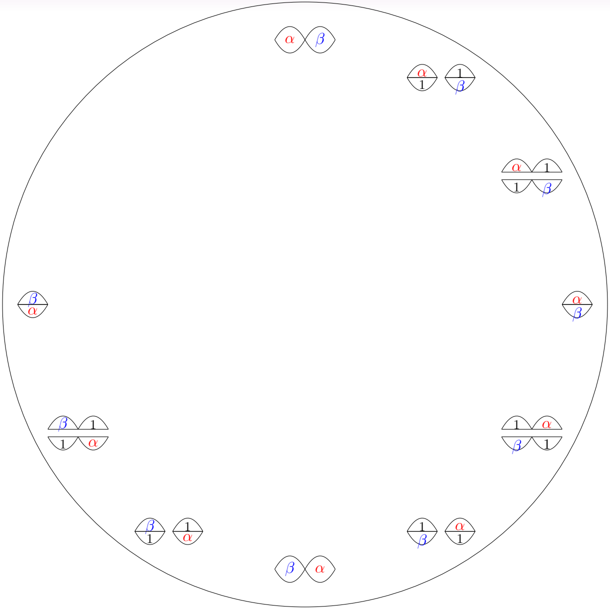

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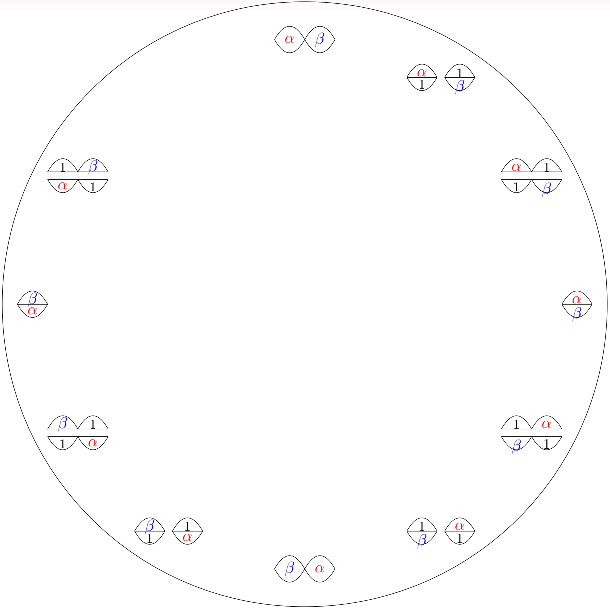

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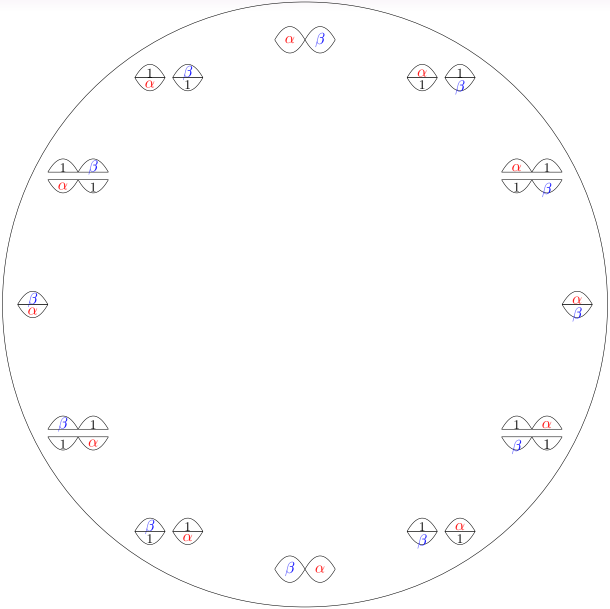

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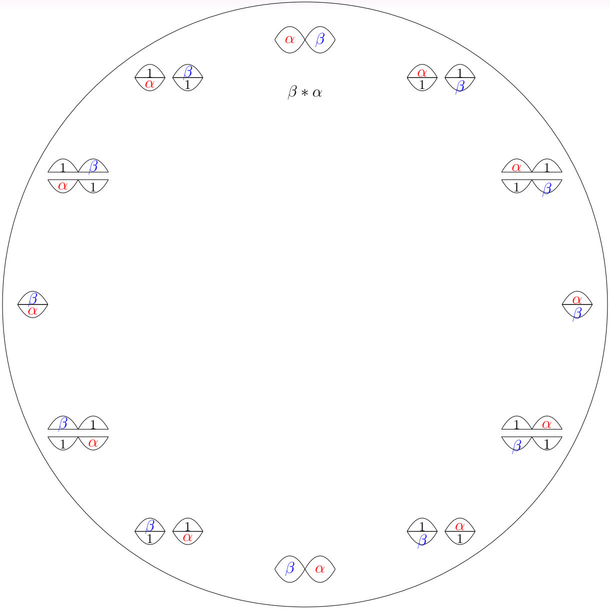

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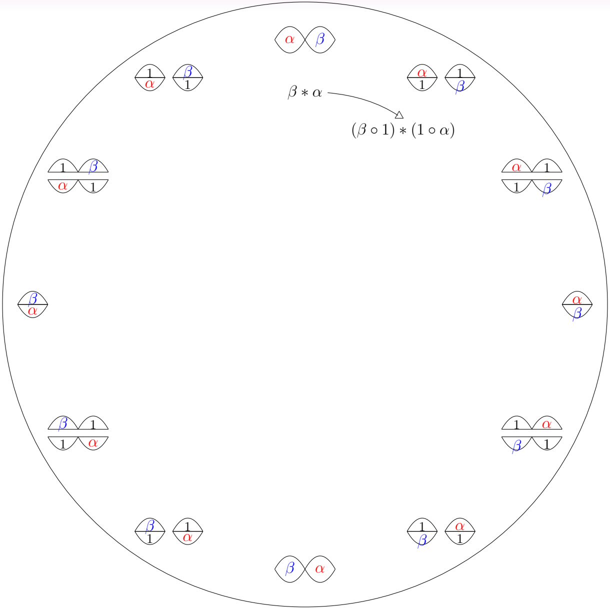

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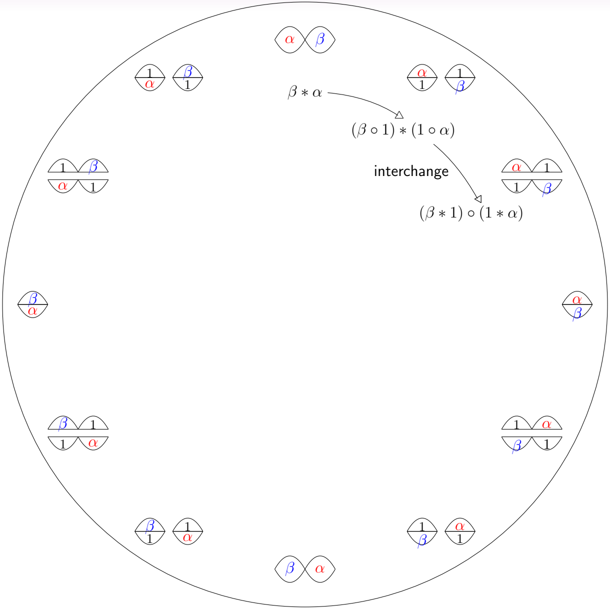

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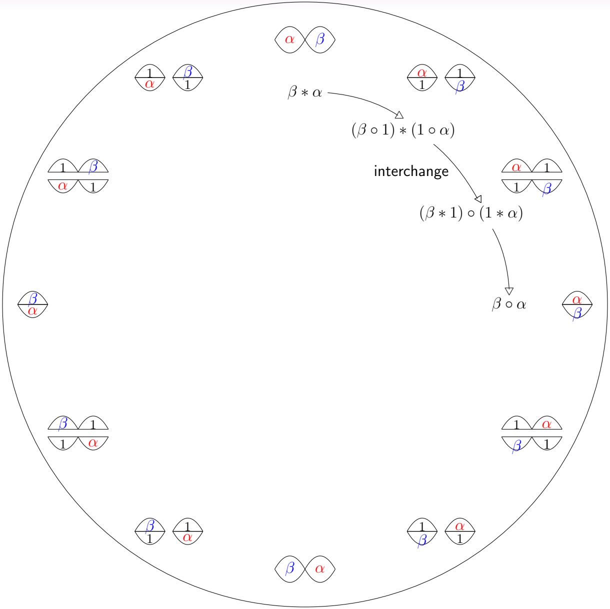

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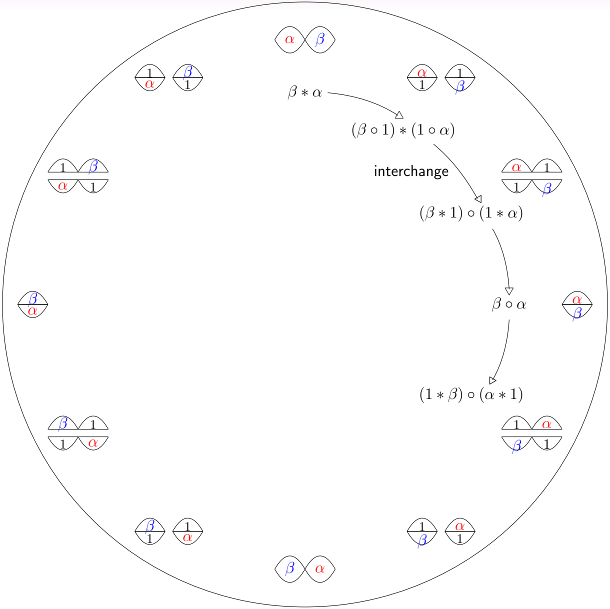

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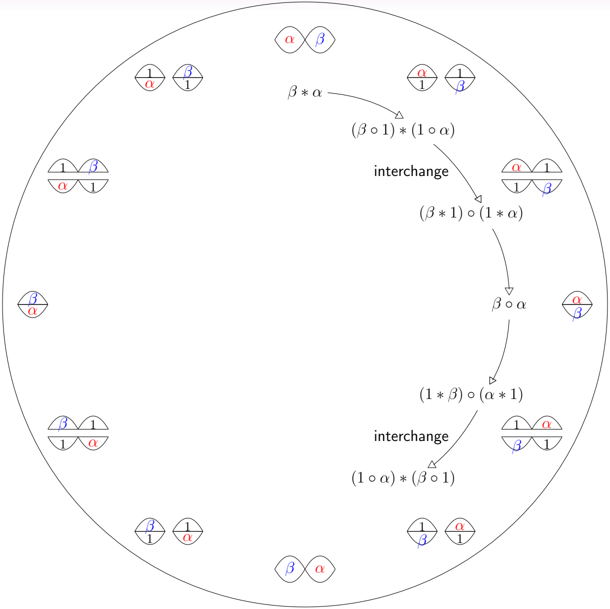

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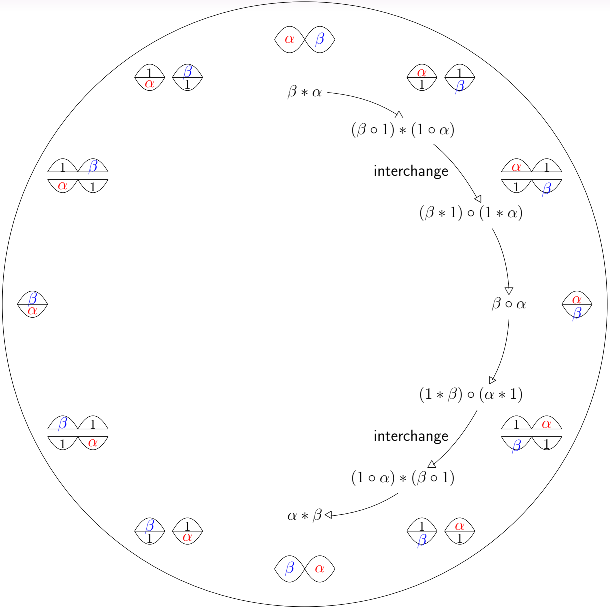

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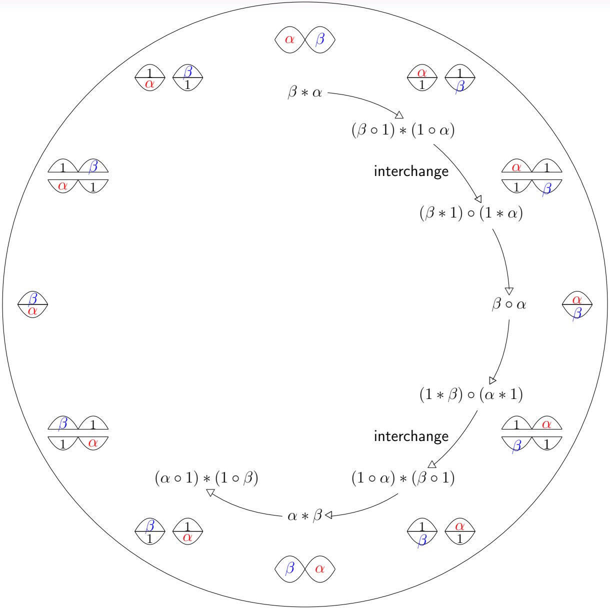

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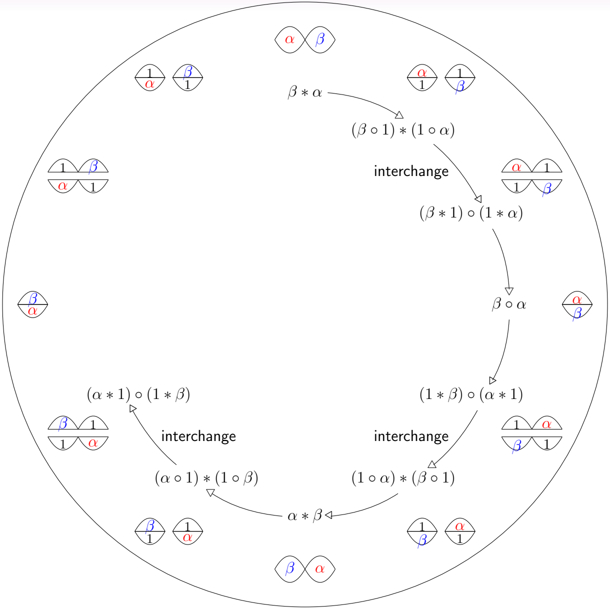

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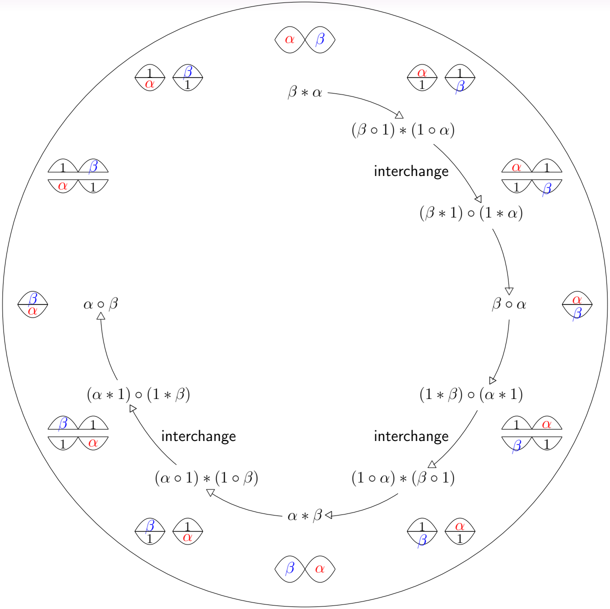

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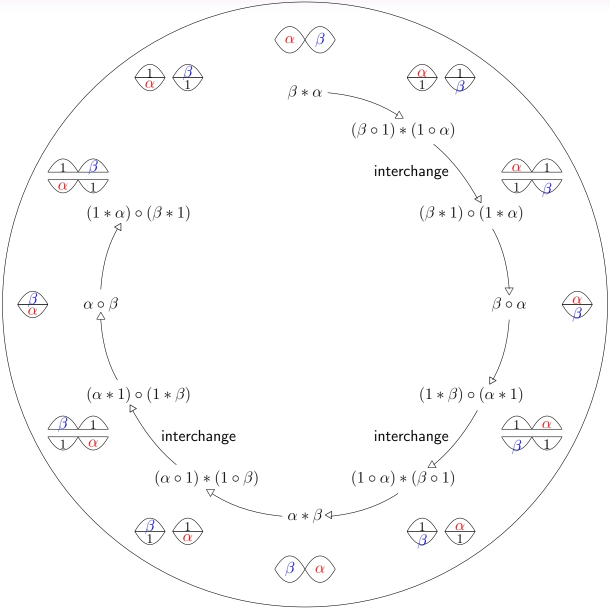

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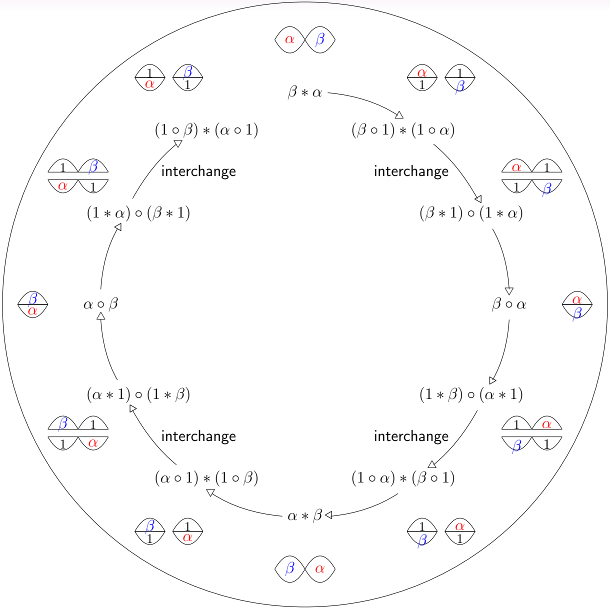

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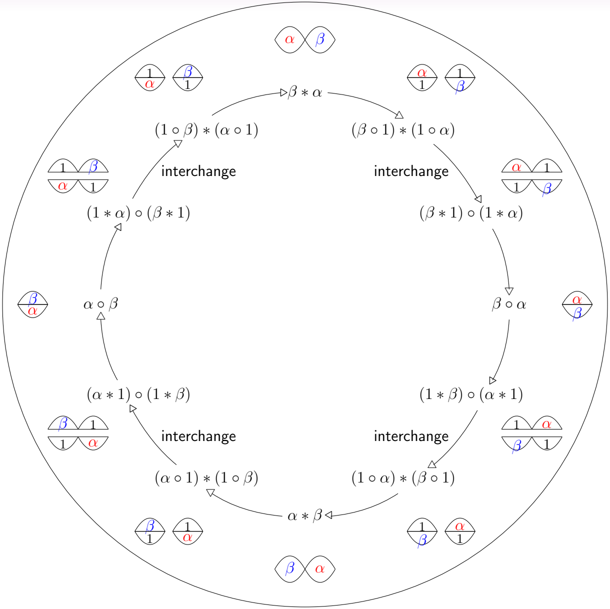

Increasing dimensions: tricategories

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—a braiding.

Slight problem

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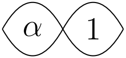
Horizontal composition of 2-cells is not strictly unital in a bicategory.



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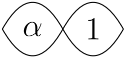

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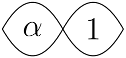

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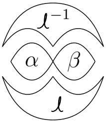

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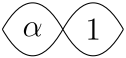

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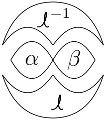

This issue gets worse as we increase dimensions.

Doubly degenerate tricategories

Doubly degenerate tricategories

$$\left.\begin{array}{l}\text{0-cells}\\ \text{1-cells}\end{array}\right\}\text{trivial}$$

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2-cells $\longrightarrow$ objects

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composition of 2-cells

1-composition	$\longrightarrow$	$\otimes$

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Doubly degenerate tricategories

$$\left.\begin{array}{l}\text{0-cells}\\ \text{1-cells}\end{array}\right\}\ \text{trivial}$$

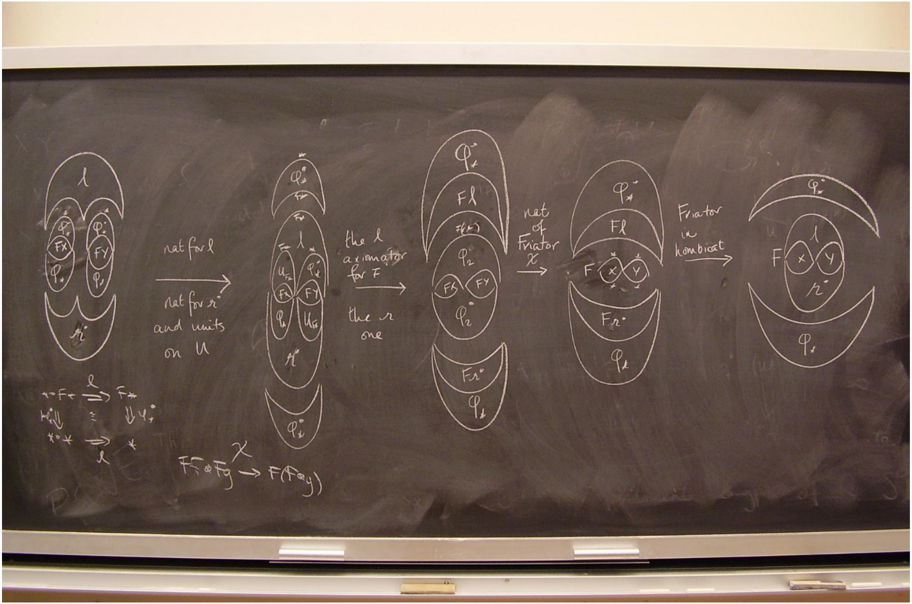
2-cells	$\longrightarrow\!\!\triangleright$	objects
3-cells	$\longrightarrow\!\!\triangleright$	morphisms

composition of 2-cells

1-composition	$\longrightarrow\!\!\triangleright$	$\otimes$
0-composition	$\longrightarrow\!\!\triangleright$	not quite a $\otimes$

Nevertheless, a lengthy calculation shows that a doubly degenerate tricategory is indeed a braided monoidal category.

2. The Eckmann-Hilton argument



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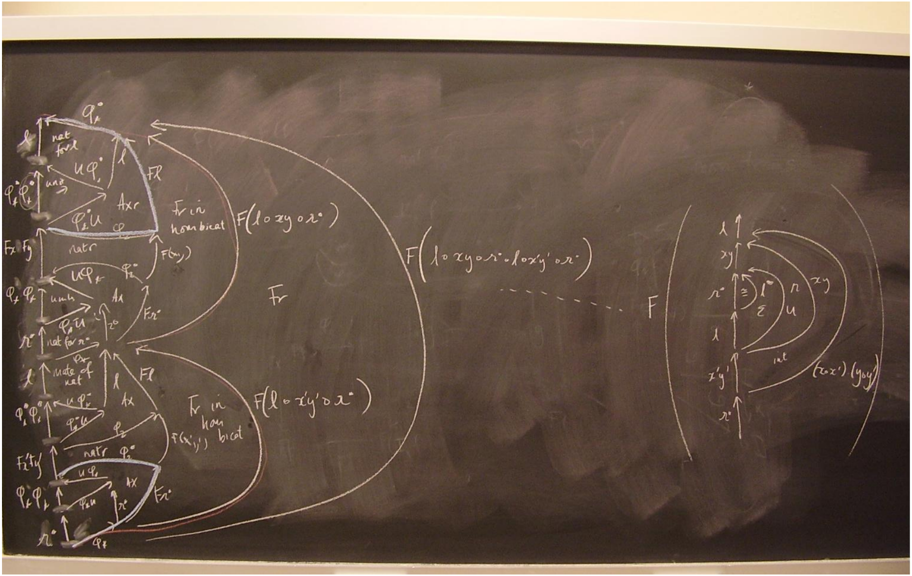

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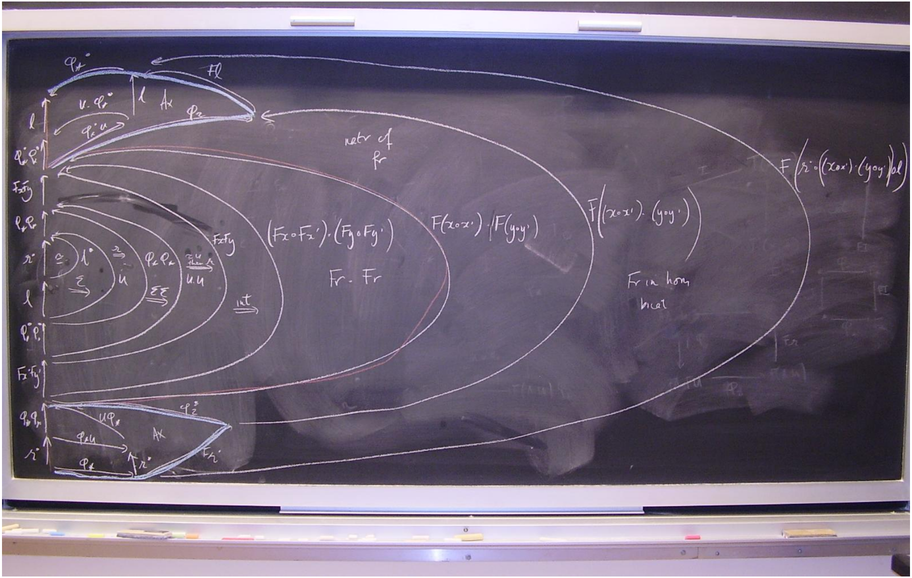

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Our slogan was

k-degenerate n-categories
"are"
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In general

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Moral

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k-degenerate n-categories
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but it does take some effort.

Theorem (Leinster).

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A bicategory with only one 0-cell x and one 1-cell 1_x is precisely a commutative monoid

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0-cells, 1-cells } trivial		
2-cells	$\longrightarrow$	elements
coherence isomorphisms	$\longrightarrow$	invertible elements

Coherence constraints giving distinguished invertible elements

We might expect three such elements: $\mathscr{a}$, $\mathscr{l}$, $\mathscr{r}$.

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This leaves just one distinguished invertible element: $\mathscr{l}$.

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Can we fix this using higher morphisms?

Theorem (Leinster).

A weak functor between doubly degenerate bicategories

$$F : X \longrightarrow Y$$

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The axioms eliminate one of them.

A weak transformation $F \Rightarrow G$

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$$\begin{tikzcd} X \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow"] & Y \end{tikzcd}$$

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A modification "from the assertion $F = G$ to itself"

3. Inconvenient elements

A weak transformation $F \Rightarrow G$

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A modification "from the assertion $F = G$ to itself"

$$\begin{tikzcd} X \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "G"'] \arrow[r, phantom, "\Downarrow \; \Rrightarrow \; \Downarrow"] & Y \end{tikzcd}$$

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is a distinguished element in Y.

Totality of doubly degenerate bicategories

3. Inconvenient elements

Totality of doubly degenerate bicategories $\xrightarrow{\text{structure comparison}}$

3. Inconvenient elements

$$\begin{tikzcd} \text{Totality of doubly degenerate bicategories} \arrow[r, "\text{structure comparison}"] & \text{Totality of commutative monoids} \end{tikzcd}$$

3. Inconvenient elements

$$\text{Totality of doubly degenerate bicategories} \xrightarrow{\text{structure comparison}} \text{Totality of commutative monoids}$$

doubly degenerate
bicategories

3. Inconvenient elements

TOTALITY OF DOUBLY DEGENERATE BICATEGORIES $\xrightarrow{\text{structure comparison}}$ TOTALITY OF COMMUTATIVE MONOIDS

doubly degenerate
bicategories

weak
functors

3. Inconvenient elements

$$\text{TOTALITY OF DOUBLY DEGENERATE BICATEGORIES} \xrightarrow{\textsf{structure comparison}} \text{TOTALITY OF COMMUTATIVE MONOIDS}$$

doubly degenerate bicategories

weak functors

weak transformations

3. Inconvenient elements

Totality of doubly degenerate bicategories $\xrightarrow{\textsf{structure comparison}}$ Totality of commutative monoids

doubly degenerate
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TOTALITY OF DOUBLY DEGENERATE BICATEGORIES $\xrightarrow{\text{structure comparison}}$ TOTALITY OF COMMUTATIVE MONOIDS

doubly degenerate bicategories

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3. Inconvenient elements

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doubly degenerate bicategories

weak functors

weak transformations

modifications

commutative monoids

homomorphisms

3. Inconvenient elements

Totality of doubly degenerate bicategories	$\xrightarrow{\text{structure comparison}}$	Totality of commutative monoids
doubly degenerate bicategories		commutative monoids
weak functors		homomorphisms
weak transformations		identities
modifications		

3. Inconvenient elements

Totality of doubly degenerate bicategories	$\xrightarrow{\text{structure comparison}}$	Totality of commutative monoids
doubly degenerate bicategories		commutative monoids
weak functors		homomorphisms
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3. Inconvenient elements

Totality of doubly degenerate bicategories $\xrightarrow{\text{structure comparison}}$ Totality of commutative monoids

doubly degenerate bicategories	cat.	cat.	commutative monoids
weak functors			homomorphisms
weak transformations			identities
modifications			identities

3. Inconvenient elements

$$\text{TOTALITY OF DOUBLY DEGENERATE BICATEGORIES} \xrightarrow{\text{structure comparison}} \text{TOTALITY OF COMMUTATIVE MONOIDS}$$

not equivalence

$\left.\begin{array}{l}\text{doubly degenerate bicategories}\\ \text{weak functors}\end{array}\right\}$ cat.	$\xrightarrow{\textbf{not equivalence}}$	cat. $\left\{\begin{array}{l}\text{commutative monoids}\\ \text{homomorphisms}\end{array}\right.$
weak transformations		identities
modifications		identities

3. Inconvenient elements

Totality of doubly degenerate bicategories $\xrightarrow{\text{structure comparison}}$ Totality of commutative monoids

not equivalence

doubly degenerate bicategories	cat.	bicat.	bicat.	cat.	commutative monoids
weak functors					homomorphisms
weak transformations					identities
modifications					identities

3. Inconvenient elements

TOTALITY OF DOUBLY DEGENERATE BICATEGORIES $\xrightarrow{\text{structure comparison}}$ TOTALITY OF COMMUTATIVE MONOIDS

not equivalence

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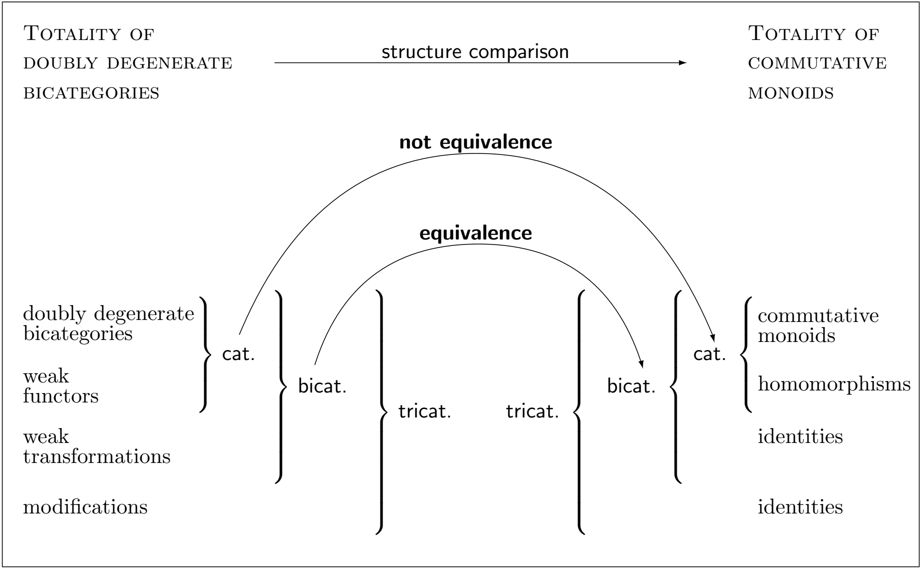
doubly degenerate bicategories	cat.	bicat.	bicat.	cat.	commutative monoids
weak functors	cat.	bicat.	bicat.	cat.	homomorphisms
weak transformations		bicat.	bicat.		identities
modifications					identities

3. Inconvenient elements



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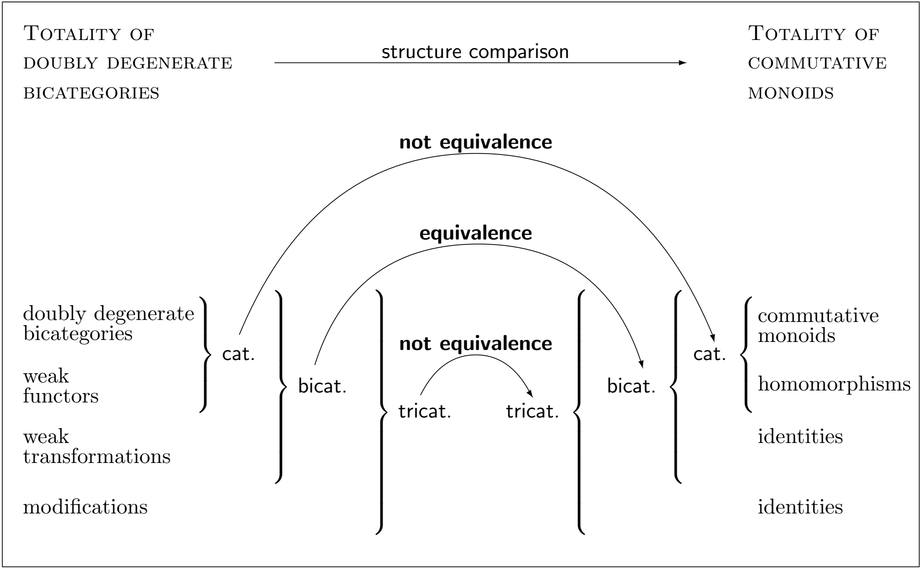

3. Inconvenient elements

set	category	2-category	3-category	···
monoid ≡ category with only one object	monoidal cat. ≡ 2-category with only one object	monoidal 2-cat. ≡ 3-category with only one object	monoidal 3-cat. ≡ 4-category with only one object	···
commutative monoid ≡ 2-cat. with only one 1-cell	braided mon. category ≡ 3-cat. with only one 1-cell	braided mon. 2-category ≡ 4-cat. with only one 1-cell	braided mon. 3-category ≡ 5-cat. with only one 1-cell	···
〃 ≡ 3-cat. with only one 2-cell	**symmetric mon. category** ≡ 4-cat. with only one 2-cell	sylleptic mon. 2-category ≡ 5-cat. with only one 2-cell	sylleptic mon. 3-category ≡ 6-cat. with only one 2-cell	···
〃 ≡ 4-cat. with only one 3-cell	〃 ≡ 5-cat. with only one 3-cell	**symmetric mon. 2-category** ≡ 6-cat. with only one 3-cell	*** mon. 3-category ≡ 7-cat. with only one 3-cell	···
〃 ⋮	〃 ⋮	〃 ⋮	**symmetric mon. 3-category** ⋮	···

3. Inconvenient elements

Totality of triply degenerate tricategories

3. Inconvenient elements

Totality of triply degenerate tricategories $\xrightarrow{\text{structure comparison}}$

3. Inconvenient elements

$$\begin{array}{ccc} \text{Totality of} & & \text{Totality of} \\ \text{triply degenerate} & \xrightarrow{\text{structure comparison}} & \text{commutative} \\ \text{tricategories} & & \text{monoids} \end{array}$$

3. Inconvenient elements

TOTALITY OF TRIPLY DEGENERATE TRICATEGORIES $\xrightarrow{\text{structure comparison}}$ TOTALITY OF COMMUTATIVE MONOIDS

triply degenerate
tricategories

3. Inconvenient elements

TOTALITY OF TRIPLY DEGENERATE TRICATEGORIES $\xrightarrow{\text{structure comparison}}$ TOTALITY OF COMMUTATIVE MONOIDS

triply degenerate
tricategories

weak
functors

3. Inconvenient elements

$$\begin{array}{ccc}
\textsc{Totality of triply degenerate tricategories} & \xrightarrow{\textsf{structure comparison}} & \textsc{Totality of commutative monoids}
\end{array}$$

triply degenerate
tricategories

weak
functors

tritransformations

3. Inconvenient elements

$$\begin{array}{ccc} \textsc{Totality of triply degenerate tricategories} & \xrightarrow{\textsf{structure comparison}} & \textsc{Totality of commutative monoids} \end{array}$$

triply degenerate
tricategories

weak
functors

tritransformations

trimodifications

3. Inconvenient elements

Totality of triply degenerate tricategories $\xrightarrow{\text{structure comparison}}$ Totality of commutative monoids

triply degenerate tricategories

weak functors

tritransformations

trimodifications

perturbations

3. Inconvenient elements

Totality of triply degenerate tricategories	$\xrightarrow{\text{structure comparison}}$	Totality of commutative monoids
triply degenerate tricategories		commutative monoids
weak functors		homomorphisms
tritransformations		identities
trimodifications		identities
perturbations		identities

3. Inconvenient elements

Totality of triply degenerate tricategories	$\xrightarrow{\text{structure comparison}}$	Totality of commutative monoids
triply degenerate tricategories	not a category	commutative monoids
weak functors		homomorphisms
tritransformations		identities
trimodifications		identities
perturbations		identities

3. Inconvenient elements

TOTALITY OF TRIPLY DEGENERATE TRICATEGORIES $\xrightarrow{\text{structure comparison}}$ TOTALITY OF COMMUTATIVE MONOIDS

triply degenerate tricategories		commutative monoids
weak functors	not a bicategory	homomorphisms
tritransformations		identities
trimodifications		identities
perturbations		identities

3. Inconvenient elements

$$\text{Totality of triply degenerate tricategories} \xrightarrow{\textsf{structure comparison}} \text{Totality of commutative monoids}$$

$$\left.\begin{array}{l}\text{triply degenerate tricategories}\\ \text{weak functors}\\ \text{tritransformations}\\ \text{trimodifications}\end{array}\right\}\ \textsf{tricategory}$$

triply degenerate tricategories	commutative monoids
weak functors	homomorphisms
tritransformations	identities
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triply degenerate tricategories	commutative monoids
weak functors	homomorphisms
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$\left.\begin{array}{l}\text{triply degenerate tricategories}\\ \text{weak functors}\\ \text{tritransformations}\\ \text{trimodifications}\end{array}\right\}$ tricategory

tricategory $\left\{\begin{array}{l}\text{commutative monoids}\\ \text{homomorphisms}\\ \text{identities}\\ \text{identities}\end{array}\right.$

3. Inconvenient elements

Totality of triply degenerate tricategories $\xrightarrow{\text{structure comparison}}$ Totality of commutative monoids

equivalence

triply degenerate tricategories	tricategory	tricategory	commutative monoids
weak functors			homomorphisms
tritransformations			identities
trimodifications			identities
perturbations			identities

3. Inconvenient elements

Totality of triply degenerate tricategories $\xrightarrow{\text{structure comparison}}$ Totality of commutative monoids

equivalence

triply degenerate tricategories	tricategory	tetracategory		tricategory	commutative monoids
weak functors					homomorphisms
tritransformations					identities
trimodifications					identities
perturbations					identities

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Totality of triply degenerate tricategories $\xrightarrow{\text{structure comparison}}$ Totality of commutative monoids

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tricategory

tetracategory

tetracategory

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equivalence

not equivalence

triply degenerate tricategories					commutative monoids
weak functors	tricategory			tricategory	homomorphisms
tritransformations		tetracategory	tetracategory		identities
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In general

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However there are still other problems.

4. Higher morphisms

Degenerate bicategories

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- A bicategory with only one 0-cell is a monoidal category.
- A weak functor between such is a monoidal functor.
- A weak transformation between such is quite different from a monoidal transformation.

A weak transformation of degenerate bicategories

$$\begin{tikzcd} X \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd}$$

A weak transformation of degenerate bicategories

$$\begin{tikzcd} X \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\alpha"] & Y \end{tikzcd}$$

is an object $\alpha \in Y$ together with

A weak transformation of degenerate bicategories

$$
\begin{tikzcd}
X \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y
\end{tikzcd}
$$

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for all $A \in Y$ a morphism

$$\alpha_A : GA \otimes \alpha \longrightarrow \alpha \otimes FA$$

A weak transformation of degenerate bicategories

$$
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\end{tikzcd}
$$

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satisfying axioms.

A weak transformation of degenerate bicategories

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is an object $\alpha \in Y$ together with

for all $A \in Y$ a morphism

$$\alpha_A : GA \otimes \alpha \longrightarrow \alpha \otimes FA$$

satisfying axioms.

This is very different from a monoidal transformation,

A weak transformation of degenerate bicategories

$$\begin{tikzcd} X \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & Y \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

is an object $\alpha \in Y$ together with

for all $A \in Y$ a morphism

$$\alpha_A : GA \otimes \alpha \longrightarrow \alpha \otimes FA$$

satisfying axioms.

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  —this works, but it isn't a restriction of **Bicat**.

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Moral so far:

The second step is more precise than the first.

5. Stabilisation

set	category	2-category	3-category	···
monoid $\equiv$ category with only one object	monoidal cat. $\equiv$ 2-category with only one object	monoidal 2-cat. $\equiv$ 3-category with only one object	monoidal 3-cat. $\equiv$ 4-category with only one object	···
commutative monoid $\equiv$ 2-cat. with only one 1-cell	braided mon. category $\equiv$ 3-cat. with only one 1-cell	braided mon. 2-category $\equiv$ 4-cat. with only one 1-cell	braided mon. 3-category $\equiv$ 5-cat. with only one 1-cell	···
" $\equiv$ 3-cat. with only one 2-cell	**symmetric mon. category** $\equiv$ 4-cat. with only one 2-cell	sylleptic mon. 2-category $\equiv$ 5-cat. with only one 2-cell	sylleptic mon. 3-category $\equiv$ 6-cat. with only one 2-cell	···
" $\equiv$ 4-cat. with only one 3-cell	" $\equiv$ 5-cat. with only one 3-cell	**symmetric mon. 2-category** $\equiv$ 6-cat. with only one 3-cell	*** mon. 3-category $\equiv$ 7-cat. with only one 3-cell	···
"	"	"	**symmetric mon. 3-category**	···
⋮	⋮	⋮	⋮	

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—we get symmetric monoidal n-categories.

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Extended TQFT Hypothesis (Baez-Dolan)

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The n-category of which n-dimensional extended TQFTs are representations is the free stable weak n-category with duals on one object.

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There are many more connections with topology.

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The obstruction is braidings.

All the difficulties come from having non-trivial morphisms between identity cells.

Slogan

The Periodic Table measures the difference between weak and strict n-categories.