Factorised Relational Databases

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Factorised Representations of Relations

Cust

ckey	name
1	Joe
2	Dan
3	Li
4	Mo

Ord

ckey	okey	date
1	1	1995
1	2	1996
2	3	1994
2	4	1993
3	5	1995
3	6	1996

Item

okey	disc
1	0.1
1	0.2
3	0.4
3	0.1
4	0.4
5	0.1

Consider a query joining the three relations above:

Cust $\bowtie_{ckey}$ Ord $\bowtie_{okey}$ Item

ckey	name	okey	date	disc
1	Joe	1	1995	0.1
1	Joe	1	1995	0.2
2	Dan	3	1994	0.4
2	Dan	3	1994	0.1
2	Dan	4	1993	0.4
3	Li	5	1995	0.1

Factorised Representations of Relations

Cust $\bowtie_{ckey}$ Ord $\bowtie_{okey}$ Item				
ckey	name	okey	date	disc
1	Joe	1	1995	0.1
1	Joe	1	1995	0.2
2	Dan	3	1994	0.4
2	Dan	3	1994	0.1
2	Dan	4	1993	0.4
3	Li	5	1995	0.1

A *flat* relational algebra expression of the query result is:

$$
\begin{array}{ccccccccccc}
\langle 1\rangle & \times & \langle Joe\rangle & \times & \langle 1\rangle & \times & \langle 1995\rangle & \times & \langle 0.1\rangle & \cup \\
\langle 1\rangle & \times & \langle Joe\rangle & \times & \langle 1\rangle & \times & \langle 1995\rangle & \times & \langle 0.2\rangle & \cup \\
\langle 2\rangle & \times & \langle Dan\rangle & \times & \langle 3\rangle & \times & \langle 1994\rangle & \times & \langle 0.4\rangle & \cup \\
\langle 2\rangle & \times & \langle Dan\rangle & \times & \langle 3\rangle & \times & \langle 1994\rangle & \times & \langle 0.1\rangle & \cup \\
\langle 2\rangle & \times & \langle Dan\rangle & \times & \langle 4\rangle & \times & \langle 1993\rangle & \times & \langle 0.4\rangle & \cup \\
\langle 3\rangle & \times & \langle Li\rangle & \times & \langle 5\rangle & \times & \langle 1995\rangle & \times & \langle 0.1\rangle &
\end{array}
$$

It uses relational product ($\times$), union ($\cup$), and unary relations (e.g., $\langle 1\rangle$).

Factorised Representations of Relations

$\langle 1 \rangle$	$\times$	$\langle Joe \rangle$	$\times$	$\langle 1 \rangle$	$\times$	$\langle 1995 \rangle$	$\times$	$\langle 0.1 \rangle$	$\cup$
$\langle 1 \rangle$	$\times$	$\langle Joe \rangle$	$\times$	$\langle 1 \rangle$	$\times$	$\langle 1995 \rangle$	$\times$	$\langle 0.2 \rangle$	$\cup$
$\langle 2 \rangle$	$\times$	$\langle Dan \rangle$	$\times$	$\langle 3 \rangle$	$\times$	$\langle 1994 \rangle$	$\times$	$\langle 0.4 \rangle$	$\cup$
$\langle 2 \rangle$	$\times$	$\langle Dan \rangle$	$\times$	$\langle 3 \rangle$	$\times$	$\langle 1994 \rangle$	$\times$	$\langle 0.1 \rangle$	$\cup$
$\langle 2 \rangle$	$\times$	$\langle Dan \rangle$	$\times$	$\langle 4 \rangle$	$\times$	$\langle 1993 \rangle$	$\times$	$\langle 0.4 \rangle$	$\cup$
$\langle 3 \rangle$	$\times$	$\langle Li \rangle$	$\times$	$\langle 5 \rangle$	$\times$	$\langle 1995 \rangle$	$\times$	$\langle 0.1 \rangle$	

A *factorised* representation (or f-representation) of the query result is:

$$\langle 1 \rangle \times \langle Joe \rangle \times \langle 1 \rangle \times \langle 1995 \rangle \times (\langle 0.1 \rangle \cup \langle 0.2 \rangle) \cup$$
$$\langle 2 \rangle \times \langle Dan \rangle \times (\langle 3 \rangle \times \langle 1994 \rangle \times (\langle 0.4 \rangle \cup \langle 0.1 \rangle) \cup \langle 4 \rangle \times \langle 1993 \rangle \times \langle 0.4 \rangle) \cup$$
$$\langle 3 \rangle \times \langle Li \rangle \times \langle 5 \rangle \times \langle 1995 \rangle \times \langle 0.1 \rangle$$

There are several *algebraically equivalent* factorised representations defined by distributivity of product over union and commutativity of product and union.

Applications of Factorised Representations

- Succinct representation of large intermediate/final results in query evaluation
  - Equality joins induce regularity in the query result and make it factorisable.
- Provenance databases and probabilistic databases
  - Compact encoding of large provenance (10MB/record in GeneOntology DB)
  - Factorisation of provenance polynomials is used for efficient query evaluation.
- Incompleteness and non-determinism (choice) in design specifications
  - Whenever we need to deal with a large space of possibilities or choices.
- Compiled relational databases
  - Compile data into compact factorised representation to speed up processing of many subsequent queries.
- Configuration problems
  - Represent the space of feasible solutions (valid combinations of components)

Properties of Factorised Representations of Relations

Factorised Representations

- Are relational algebra expressions.
- Can be exponentially more succinct than the relations they encode.
- Allow for fast (constant-delay) enumeration of tuples
- Reduce data redundancy and boost query performance using a mixture of
  - vertical data partitioning (product) and
  - horizontal data partitioning (union).

Key Challenges and Talk Overview

1. Characterise conjunctive queries based on succinctness of their factorised results.

2. Build a relational DBMS that uses f-representations at the physical layer.

Overview of the Rest of the Talk:

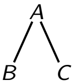
- Factorisations whose nesting structures are inferred from the query
- Tight bounds on size and readability of factorised query results
- FDB: Query engine for factorised databases

Factorisation Trees

A *factorisation tree* (f-tree) $\mathcal{T}$ over relational schema $\mathcal{S}$ is a rooted forest with nodes labelled by attributes from $\mathcal{S}$.

$\mathcal{T}$ defines a nesting structure for f-representations of relations over $\mathcal{S}$.

Example f-trees and corresponding factorisations over $\mathcal{S} = \{A, B, C\}$:



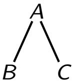
$$\longleftrightarrow \quad \bigcup_{a \in A} (\langle a \rangle \times (\bigcup_{b \in B} \langle b \rangle) \times (\bigcup_{c \in C} \langle c \rangle)).$$

$$\longleftrightarrow \quad \bigcup_{a \in A} (\langle a \rangle \times (\bigcup_{b \in B} \langle b \rangle \times (\bigcup_{c \in C} \langle c \rangle))).$$

Factorisation Trees for Relations

However, not all f-trees work for all relations.

The f-tree



cannot factorise the relation R

R

A	B	C
1	1	1
1	2	2

because

- For $A = 1$, the values of B and C are *dependent*, i.e.,
- Relation $\pi_{B,C}\sigma_{A=1}(R)$ cannot be factorised as $(\bigcup_{b \in B} \langle b \rangle) \times (\bigcup_{c \in C} \langle c \rangle)$:

$$[(\langle 1 \rangle \cup \langle 2 \rangle) \times (\langle 1 \rangle \cup \langle 2 \rangle)] \neq [(\langle 1 \rangle \times \langle 1 \rangle) \cup (\langle 2 \rangle \times \langle 2 \rangle)]$$

Factorisation Trees for Query Results

We statically infer from queries which f-trees *always* work for their results.

For a query Q (without projections) and f-tree $\mathcal{T}$

the result $Q(\mathbf{D})$ can be factorised according to $\mathcal{T}$ for *any* database

iff

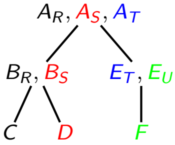
for all relations of Q, all attributes are on a single root-to-leaf path.

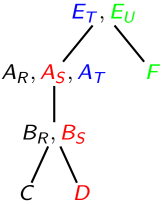
Similar but more involved condition holds for arbitrary conjunctive queries.

Factorisation Trees for Query Results

Consider query $Q = \sigma_\phi(R \times S \times T \times U)$, with

- schemas $R(A_R, B_R, C)$, $S(A_S, B_S, D)$, $T(A_T, E_T)$, and $U(E_U, F)$,
- condition $\phi = (A_R = A_S = A_T, B_R = B_S, E_T = E_U)$.





F-representations modelled on the left f-tree have the structure:

$$\bigcup_{a \in A_R, A_S, A_T} \left[\langle a \rangle \times \bigcup_{b \in B_R, B_S} \left(\langle b \rangle \times \left(\bigcup_{c \in C} \langle c \rangle\right) \times \left(\bigcup_{d \in D} \langle d \rangle\right)\right) \times \bigcup_{e \in E_T, E_U} \left(\langle e \rangle \times \left(\bigcup_{f \in F} \langle f \rangle\right)\right)\right]$$

Size of Factorised Representations

The *size* of an f-representation is the number of its singleton data elements.

$$|(\langle 1\rangle \cup \langle 2\rangle \cup \langle 3\rangle)(\langle 1\rangle \cup \langle 2\rangle)| = 5,$$
$$|(\langle 1\rangle\langle 1\rangle \cup \langle 1\rangle\langle 2\rangle \cup \langle 2\rangle\langle 1\rangle \cup \langle 2\rangle\langle 2\rangle \cup \langle 3\rangle\langle 1\rangle \cup \langle 3\rangle\langle 2\rangle)| = 12.$$

The two sizes above differ, although

$$(\langle 1\rangle \cup \langle 2\rangle \cup \langle 3\rangle)(\langle 1\rangle \cup \langle 2\rangle) = (\langle 1\rangle\langle 1\rangle \cup \langle 1\rangle\langle 2\rangle \cup \langle 2\rangle\langle 1\rangle \cup \langle 2\rangle\langle 2\rangle \cup \langle 3\rangle\langle 1\rangle \cup \langle 3\rangle\langle 2\rangle)$$

How much space do we save by factorisation?

Tight Bounds on the Size of Factorised Representations

Given a query Q, for any f-tree $\mathcal{T}$ of Q there is a rational number $s(\mathcal{T})$ such that:

- For any database $\mathbf{D}$, the factorisation of $Q(\mathbf{D})$ over $\mathcal{T}$ has size $O(|\mathbf{D}|^{s(\mathcal{T})})$.
- There exist arbitrarily large databases $\mathbf{D}$ for which the factorisation of $Q(\mathbf{D})$ over $\mathcal{T}$ has size $\Theta(|\mathbf{D}|^{s(\mathcal{T})})$.

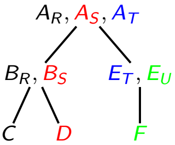
The parameter $s(\mathcal{T})$ is

- a feasible solution to a linear program,
- the *fractional edge cover number of a sub-query of* Q.
  - this sub-query depends on the shape of $\mathcal{T}$.
  - $1 \leq s(\mathcal{T}) \leq |Q|$.

Example of Computing $s(\mathcal{T})$

Consider the following f-tree $\mathcal{T}$.

- Attributes with the same colour belong to the same input relation.



Number of relations covering the path from root to any node X:

- For each node X except for F, this number is 1.

- For node F, this number is 2.

$s(\mathcal{T})$ is the maximum of the number of covering relations for each node.

- Thus, $s(\mathcal{T}) = 2$.

Tight Bounds on the Size of Factorised Representations

Size bounds for f-trees can be lifted to queries by finding an optimal f-tree:

$$s(Q) = \min_{\mathcal{T}} s(\mathcal{T}).$$

$s(Q)$ characterises queries by factorisability of their results.

- For any database $\mathbf{D}$, there is a factorisation of $Q(\mathbf{D})$ with size $O(|\mathbf{D}|^{s(Q)})$.
- For f-trees derived from Q, this bound is best possible.

Readability of Factorised Representations

- Assume we annotate tuples by distinct variables (= provenance, keys).
- Factorised representations can be seen as polynomials over such variables
  - $\cup$ becomes sum $(+)$ and $\times$ becomes product $(\cdot)$.

Readability:

- A representation Φ is read-k if the maximum number of occurrences of any variable in Φ is k.
- The readability of Φ is the smallest number k such that there is a read-k representation equivalent to Φ.
- Readability has been proposed in the context of factorisation of Boolean functions [Golumbic et al.'06].
- Example: ψ_1 is read-3 and ψ_2 is read-1. They are equivalent and have readability one.

$$\psi_1 = c_1 o_1 i_1 + c_1 o_1 i_2 + c_2 o_3 i_3 + c_2 o_3 i_4 + c_2 o_4 i_5 + c_3 o_5 i_6.$$
$$\psi_2 = c_1 o_1 (i_1 + i_2) + c_2 (o_3 (i_3 + i_4) + o_4 i_5) + c_3 o_5 i_6.$$

Two Readability Dichotomies

1. Let Q be a query.
   - If Q is *hierarchical*, the readability of $Q(\mathbf{D})$ for any database $\mathbf{D}$ is bounded by a constant.
   - If Q is non-hierarchical, for any f-tree $\mathcal{T}$ of Q there exist arbitrarily large databases $\mathbf{D}$ such that $\mathcal{T}(\mathbf{D})$ is read-$\Omega(|\mathbf{D}|)$.

2. Let Q be a query without repeating relation symbols.
   - If Q is hierarchical, the readability of $Q(\mathbf{D})$ is 1 for any database $\mathbf{D}$.
   - If Q is non-hierarchical, there exist arbitrarily large databases $\mathbf{D}$ such that the readability of $Q(\mathbf{D})$ is $\Omega(\sqrt{|\mathbf{D}|})$.

What are these hierarchical queries?

Hierarchical query Q:

- For any two equivalence classes of attributes in Q, either their sets of relation symbols are disjoint, or one is included in the other.

This is a key property for query characterisation in many applications:

- In probabilistic databases, any tractable non-repeating conjunctive query is hierarchical; non-hierarchical queries are intractable [Suciu&Dalvi'07].
- In the finite cursor machine model of computation [Grohe et al'07], any query that can be evaluated in one pass is hierarchical; non-hierarchical queries need more passes.
  - Assumption: we are allowed to first sort the input relations.
- In the Massively Parallel computation model, any query that can be evaluated with one synchronisation step is hierarchical. [Suciu et al'11]

Readability Width of a Query

There is a rational number $r(Q)$ with properties similar to those of $s(Q)$:

- For any database $\mathbf{D}$, the readability of the query result $Q(\mathbf{D})$ is at most $M \cdot |\mathbf{D}|^{r(Q)}$, where M is the max number of repeating relation symbols in Q.
- For any f-tree $\mathcal{T}$ of Q there exist arbitrarily large databases $\mathbf{D}$ such that the f-representation $\mathcal{T}(\mathbf{D})$ is at least read-$(|\mathbf{D}|/|Q|)^{r(Q)}$.
- $r(Q) = 0$ for hierarchical queries Q only and $r(Q) > 0$ for all others.
- $r(Q)$ defines the *readability width* of Q.

FDB: A Query Engine for Factorised Databases

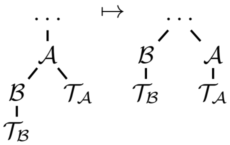
- Uses f-representations to encode relational data
- Query evaluation
  - Relational operators: selection, projection, product
  - New operators for restructuring factorisations
  - Any query can be evaluated by a sequence of operators
- Query optimisation
  - Find the best query **and** factorisation plan
- Implementation of an in-memory engine in C++
  - flat/factorised data $\rightarrow$ flat/factorised data
- Experimental evaluation with FDB and relational engines
  - Factorised query results up to 6 orders of magnitude smaller than equivalent relations.
  - FDB up to 5 orders of magnitude faster than PostgreSQL/SQLite/our in-memory relational engine.

Thanks!

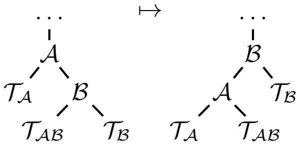
Query Operators

Restructuring operators

- **Normalisation** factors out expressions common to all terms of a union.
  Example: f-tree nodes $\mathcal{A}$ and $\mathcal{B}$ do not have dependent attributes.



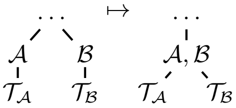
- **Swap** exchanges a node with its parent while preserving normalisation.
  Example: $\mathcal{T}_\mathcal{A}$ depends on $\mathcal{A}$ only, $\mathcal{T}_\mathcal{B}$ depends on $\mathcal{B}$ only, $\mathcal{T}_{\mathcal{A}\mathcal{B}}$ depends on both $\mathcal{A}$ and $\mathcal{B}$



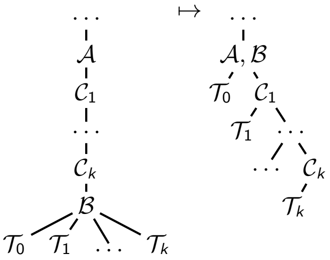
Query Operators

Selection operators $A = B$, where A and B label nodes $\mathcal{A}$ and $\mathcal{B}$ respectively.

- **Merge** siblings $\mathcal{A}$ and $\mathcal{B}$ into a single node



- **Absorb** $\mathcal{B}$ into its ancestor $\mathcal{A}$. Example: $\mathcal{T}_i$ depends on $\mathcal{B}$ and $\mathcal{C}_i$



Select $A\theta c$ does not change the f-tree; it removes from the f-representation all products containing A-singletons $\langle a \rangle$ for which $a \neg\theta c$.

Query Operators

Further query algebra operators

- **Cartesian product** of two f-trees is their forest
- **Projection** on attribute list $\bar{A}$ removes from the f-tree all attributes but those in $\bar{A}$; empty leaf nodes are removed.
  - The projection operation is more involved if the resulting f-tree must not allow f-representations with duplicates.
- Work in progress: **Order-by**, **Group-by**, **Aggregates**.

Query Optimisation

Goal: Find the best f-plan = query **and** factorisation plan

- Optimal f-representation of the query result
- Minimal computation cost, i.e., the sizes of intermediate results
- Cost computation based on $s(Q)$ or cardinality and selectivity estimates

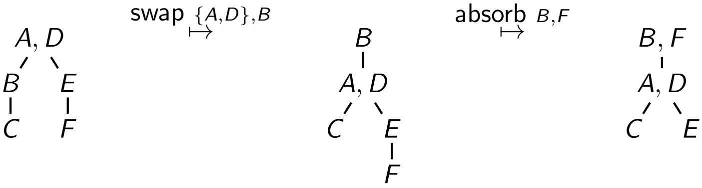
Search space defined by

- selection operators may require several swaps before application,
- choice of selection operators and f-tree transformations for each join,
- choice of order for join conditions,
- projection push-downs.

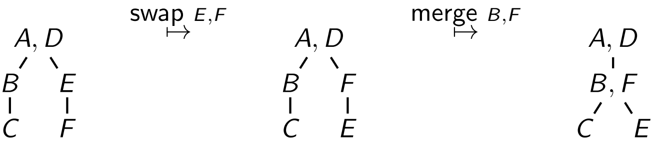
Query Optimisation: Example

Build f-plan for selection $B = F$ on the leftmost f-tree, with dependencies $\{A, B, C\}$ and $\{D, E, F\}$.
Alternative f-plans (cost given by $\max s(\mathcal{T}_i)$ over all $\mathcal{T}_i$'s in the f-plan):

1. Input and output f-trees with cost 1, intermediate with cost 2

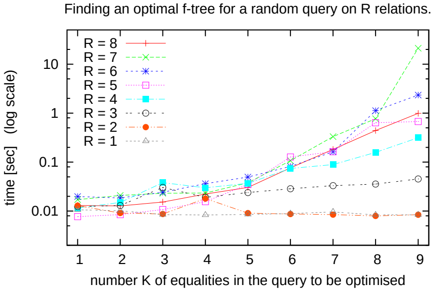


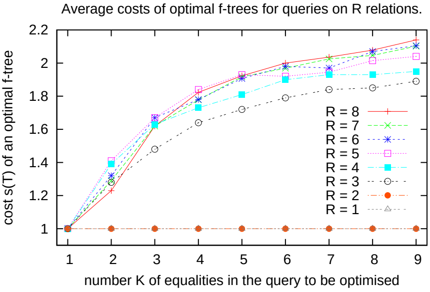
2. All three f-trees have cost 1.



Experimental Evaluation

Query optimisation. K equalities on R relations with $A = 40$ attributes.

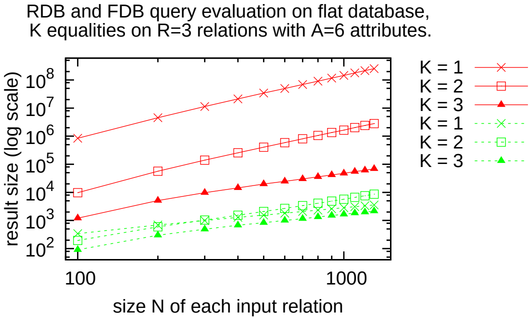


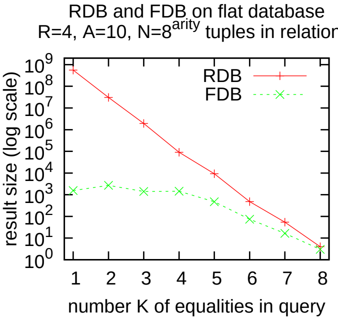


- exhaustive search used above.
- heuristics perform up to 4 orders of magnitude better, the cost differs by at most 0.5.

Experimental Evaluation

Query evaluation on flat data: FDB vs. Relational DB (RDB).

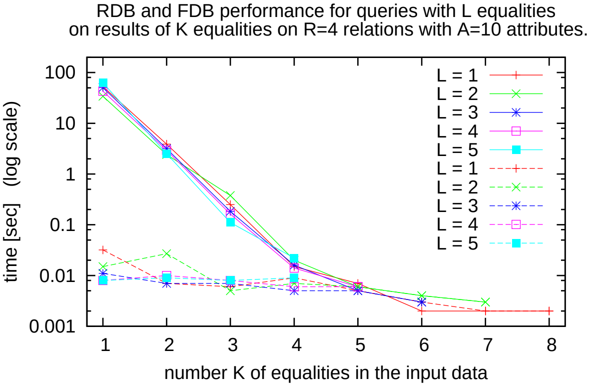




- The trend is the same for time performance.

Experimental Evaluation

Query evaluation on factorised data: FDB (dotted lines) vs. RDB (solid lines).

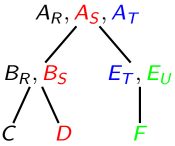


- The trend is the same for size.

In search for the rational number $s(Q)$ - step 1

Disclaimer: Discussion for queries without projection!

For any attribute A in an f-tree $\mathcal{T}$, the number of occurrences of A-values in the factorisation of $Q(\mathbf{D})$ over $\mathcal{T}$ is $|\pi_{\text{path}(A)}(Q(\mathbf{D}))|$.



Example

- $\text{path}(F) = \{A_R, A_S, A_T, E_T, E_U, F\}$.
- The number of occurrences of F-values is then $|\pi_{\text{path}(F)}(Q(\mathbf{D}))|$.

Next step:

- The trouble is that $\pi_{\text{path}(A)}(Q(\mathbf{D}))$ requires to know $Q(\mathbf{D})$.
- We would like to express it as a function of Q and $\mathbf{D}$.

In search for the rational number $s(Q)$ - step 2

Restrict $Q = \pi_{\mathcal{P}}(\sigma_\phi(R_1 \times \cdots \times R_n))$ and **D** to the attributes in $\mathrm{path}(A)$:

- $Q_A = \sigma_{\phi_{\mathrm{path}(A)}}(\pi_{\mathrm{path}(A)} R_1 \times \cdots \times \pi_{\mathrm{path}(A)} R_n)$,
- $\mathbf{D}_A$ obtained by projecting **D** onto $\mathrm{path}(A)$.

Then number of A-values $= |\pi_{\mathrm{path}(A)}(Q(\mathbf{D}))| \leq |Q_A(\mathbf{D}_A)|$.

Rough estimate:

- Cover all attributes of Q_A by $k \leq |Q_A|$ relations.
- Then, $|Q_A(\mathbf{D}_A)| \leq |\mathbf{D}|^k$.
- Best k is the edge cover number of the hypergraph of Q_A.

Better estimate:

- From edge cover number k to fractional edge cover number $\rho^*(Q_A)$.

In search for the rational number $s(Q)$ - step 3

For a query $Q = \sigma_\phi(R_1 \times \cdots \times R_n)$, the *fractional edge cover number* $\rho^*(Q)$ is the cost of an optimal solution to the linear program with variables $\{x_{R_i}\}_{i=1}^n$:

$$\begin{array}{rll} \text{minimising} & \sum_i x_{R_i} & \\ \text{subject to} & \sum_{i:R_i \in rel(\mathcal{A})} x_{R_i} \geq 1 & \text{for all attribute classes } \mathcal{A}, \\ & x_{R_i} \geq 0 & \text{for all } R_i. \end{array}$$

- x_{R_i} is the weight of relation R_i.
- $rel(\mathcal{A})$ are relations with attributes in $\mathcal{A}$.
- Each node $\mathcal{A}$ has to be covered by relations in $rel(\mathcal{A})$ such that the sum of the weights of these relations is greater than 1.
- The objective is to minimise the sum of the weights of all relations.
- In the non-weighted edge cover, the variables x_{R_i} can only be assigned the values 0 and 1.

Then $|Q(\mathbf{D})| \leq |\mathbf{D}|^{\rho^*(Q)}$ for all databases $\mathbf{D}$. [Atserias, Grohe, Marx; FOCS'08]

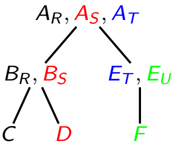
In search for the rational number $s(Q)$ - step 4

1. Number of A-values $= |\pi_{\mathrm{path}(A)}(Q(\mathbf{D}))| \le |Q_A(\mathbf{D}_A)| \le |\mathbf{D}_A|^{\rho^*(Q_A)} \le |\mathbf{D}|^{\rho^*(Q_A)}$.

2. Define $s(\mathcal{T}) = \max_A \rho^*(Q_A)$.
   - $s(\mathcal{T})$ = maximal possible $\rho^*(Q_A)$ over all attributes A from Q.
   - Then, the size of the factorisation of $Q(\mathbf{D})$ over $\mathcal{T}$ is $\le |Q| \cdot |\mathbf{D}|^{s(\mathcal{T})} = O(|\mathbf{D}|^{s(\mathcal{T})})$.

3. Define $s(Q) = \min_{\mathcal{T}} s(\mathcal{T})$.
   - $s(Q)$ = minimum possible $s(\mathcal{T})$ over all f-trees $\mathcal{T}$ for Q.
   - Then, there exists an f-representation of $Q(\mathbf{D})$ with size $O(|\mathbf{D}|^{s(Q)})$.

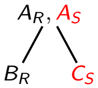
Example of computing $s(Q)$



- For each node X except for F, we have that $\rho^*(Q_X) = 1$ since all attributes in $\mathrm{path}(X)$ are covered by one relation.
- $\mathrm{path}(F)$ is not covered by one relation and $\rho^*(Q_F) = 2$.
- Thus, $s(\mathcal{T}) = 2$.
- $s(Q) = 2$ since $s(\mathcal{T}) = 2$ is the smallest possible value for any f-tree $\mathcal{T}$ of the query Q.

Projections

With $R(A_R, B_R)$,$S(A_S, B_S)$, the query $Q = \sigma_{A_R=A_S}(R \times S)$ has $s(Q) = 1$:



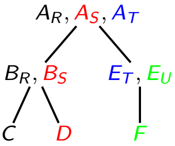
However the query $Q' = \pi_{B_R,C_S}(\sigma_{A_R=A_S}(R \times S))$ has $s(Q') = 2$.

After projecting away A_R, A_S, B_R and C_S are dependent and cannot be siblings.

$\Rightarrow$ Projection may increase the factorisation size.

More Succinct Representations: DAG

Avoid repeating identical expressions: store them once and use pointers.



$$\bigcup_{a \in A_R, A_S, A_T} \left[\langle a \rangle \times \cdots \times \bigcup_{e \in E_T, E_U} \left(\langle e \rangle \times \left(\bigcup_{f \in F} \langle f \rangle \right) \right) \right]$$

- Node $\{F\}$ only depends on $\{E_T, E_U\}$.
- A fixed $\langle e \rangle$ binds with the same $\bigcup_{f \in F} \langle f \rangle$ for each $\langle a \rangle$.

  $\Rightarrow$ store the mapping $\langle e \rangle \mapsto \bigcup_{f \in F} \langle f \rangle$ separately.

$$\bigcup_{a \in A_R, A_S, A_T} \left[\langle a \rangle \times \cdots \times \bigcup_{e \in E_T, E_U} \left(\langle e \rangle \times U_e \right) \right]; \qquad \left\{ U_e = \bigcup_{f \in F} \langle f \rangle \right\}$$