Propositional Logic Programming and

Knowledge Representation with Horn Clauses

Logic Programming

- Important Formalism for Knowledge Representation and Reasoning (KRR)
- Has its roots in Theorem Proving, based on the Resolution Calculus (Robinson)
- Express knowledge in terms of facts and rules
- **Algorithm = Logic + Control** (Kowalski)
- PROLOG emerged a general purpose programming language
- Modern languages: enhanced with features such as constraint solving, non-monotonic negation for KRR
- Here: Consider core language, plus some extensions

Positive Propositional Logic Programs

$$
\begin{aligned}
shut\_down &:- overheat \\
shut\_down &:- leak \\
leak &:- valve\_closed, pressure\_loss \\
valve\_closed &:- signal\_1 \\
pressure\_loss &:- signal\_2 \\
overheat &:- signal\_3 \\
signal\_1 &:- \\
signal\_2 &:-
\end{aligned}
$$

- This program captures (simplified) knowledge about a steam engine equipped with three signal gauges.
- Informally, the rules tell that the system has to be shut down if it is in a dangerous state.
- Such states are connected to causes and signals by respective rules.

Logic Program Syntax

- A *Horn clause* is a rule of the form

$$A_0 \leftarrow A_1, \ldots, A_m \qquad (m \geq 0),$$

where each A_i is a propositional atom.

- The parts on the left and on the right of "$\leftarrow$" are called the *head* and the *body* of the rule, respectively.

- A rule r of the form $A_0 \leftarrow$, i.e., whose body is empty, is called a *fact*.

- A *logic program* is a finite set of Horn clauses.

Logic Program Semantics

- An atom A *is true* w.r.t. program P (denoted $P \models A$), if A is a classical consequence of P.

$$
\begin{aligned}
\mathit{shut\_down} &:- \mathit{overheat} \\
\mathit{shut\_down} &:- \mathit{leak} \\
\mathit{leak} &:- \mathit{valve\_closed}, \mathit{pressure\_loss} \\
\mathit{valve\_closed} &:- \mathit{signal\_1} \\
\mathit{pressure\_loss} &:- \mathit{signal\_2} \\
\mathit{overheat} &:- \mathit{signal\_3} \\
\mathit{signal\_1} &:- \\
\mathit{signal\_2} &:-
\end{aligned}
$$

- $P \models \mathit{signal\_1}$, $P \models \mathit{signal\_2}$, $P \models \mathit{valve\_closed}$, $\ldots$ $P \models \mathit{leak}$, $\ldots$

Relationship to the SAT Problem

- Each program P can viewed as a classical CNF $\phi(P)$
- Each rule r corresponds to a clause $\phi(r)$:

$$A_0 \leftarrow A_1, \ldots, A_m \qquad \rightleftharpoons \qquad A_0 \vee \neg A_1 \cdots \neg A_m$$

- $\phi(P) = \bigwedge_{r \in P} \phi(r)$

Theorem. $P \models A$ holds if and only if $\phi(P) \wedge \neg A$ is unsatisfiable

Remark: in Logic Programming, a "query" A is often written as $\leftarrow A$.

Logic Programs: Semantics

- The *Herbrand Base B_P* of program P is the set of all atoms occurring in P
- A *Herbrand interpretation* of P is any subset $I \subseteq B_P$

  Intuitively, the atoms in I are true and all others are false.
- A *Herbrand model* of P is any Herbrand interpretation I which satisfies every rule $A_0 \leftarrow A_1, \ldots, A_m$ in P, i.e., $A_0 \in I$ whenever $\{A_1, \ldots, A_m\} \subseteq I$
- The semantics of P is given by the *least Herbrand model* of P, denoted $LM(P)$, i.e., the unique Herbrand model M of P such that each different Herbrand model I of P satisfies $I \not\subseteq M$.

Example /2

shut_down :– *overheat*

shut_down :– *leak*

leak :– *valve_closed*, *pressure_loss*

valve_closed :– *signal_1*

pressure_loss :– *signal_2*

overheat :– *signal_3*

signal_1 :–

signal_2 :–

- M_1 = { <all atoms> } = B_P
- M_2 = { *signal_1*, *signal_2*, *valve_closed*, *pressure_loss*, *leak shut_down*, *leak*, *overheat* }
- M_3 = { *signal_1*, *signal_2*, *valve_closed*, *pressure_loss*, *shut_down*, *leak* }

Operational Characterization

- We can compute $lm(P)$ by fixpoint iteration of the *immediate consequence* operator

$$T_P : 2^{B_P} \longrightarrow 2^{B_P}$$

defined by

$$T_P(I) = \{A_0 \in B_P \mid P \text{ contains a rule } A_0 \text{ :- } A_1, \ldots, A_m \text{ such that } \{A_1, \ldots, A_m\} \subseteq I \text{ } \textit{holds} \}.$$

- Intuition: all facts provable by rules in P from I in one step.
- Notice: The operator T_P is *monotone*, i.e., $I \subseteq J$ implies $T_P(I) \subseteq T_P(J)$

Fixpoint Results

Well-known results in Logic Programming:

- **Theorem.** T_P has a *least fixpoint* T_P^∞, which is the limit of the sequence $\langle T_P^i \rangle_{i \geq 0}$ defined by

$$\begin{aligned} T_P^0 &= \emptyset, \\ T_P^{i+1} &= T_P(T_P^i), i \geq 0. \end{aligned}$$

- **Theorem.** $T_P^\infty = \{A \in B_P \mid P \models A\}$

- **Theorem.** $T_P^\infty = LM(P)$

Example (continued)

$$
\begin{aligned}
\textit{shut\_down} &:- \textit{overheat}\\
\textit{shut\_down} &:- \textit{leak}\\
\textit{leak} &:- \textit{valve\_closed}, \textit{pressure\_loss}\\
\textit{valve\_closed} &:- \textit{signal\_1}\\
\textit{pressure\_loss} &:- \textit{signal\_2}\\
\textit{overheat} &:- \textit{signal\_3}\\
\textit{signal\_1} &:- \quad \textit{signal\_2} :- .
\end{aligned}
$$

$$
\begin{aligned}
T_P^0 &= \emptyset,\\
T_P^1 &= \{\textit{signal\_1}, \textit{signal\_2}\},\\
T_P^2 &= T_P^1 \cup \{\textit{valve\_closed}, \textit{pressure\_loss}\},\\
T_P^3 &= T_P^2 \cup \{\textit{leak}\},\\
T_P^4 &= T_P^\infty = T_P^3 \cup \{\textit{shutdown}\}.
\end{aligned}
$$

Thus, the least fixpoint is reached in four steps

Logic Programs: Inference if negative Information

- Conclude negative information under *Negation as Failure*:

- **Definition.** $T \models \neg A$ if $T \not\models A$

Example: $P \models \neg$*overheat*, because $P \not\models$ *overheat*

- Evaluation inference: For each atom A,

  - $P \models A \Leftrightarrow A \in LM(P)$
  - $P \models \neg A \Leftrightarrow A \notin LM(P)$

- This constructively implements the *Closed World Assumption*

Complexity of Propositional Logic Programs

- Existence of $lm(P)$ is trivial (it always exists)
- Reasoning: Given a program P and an atom A, decide whether $A \in lm(P)$

Theorem. Deciding whether $A \in lm(P)$ is **P**-complete.

Proof.

- **Membership:** Computing T_P^∞ is feasible in polynomial time, and then we only need to check whether $A \in T_P^\infty$.
- **Hardness:** Encoding of a deterministic Turing Machine (DTM).

  Given a DTM T, an input string I and a number of steps N (where N is a polynomial in $|I|$), construct in logspace a program $P = P(T, I, N)$ and an atom A such that $P \models A$ iff T accepts input I within N steps

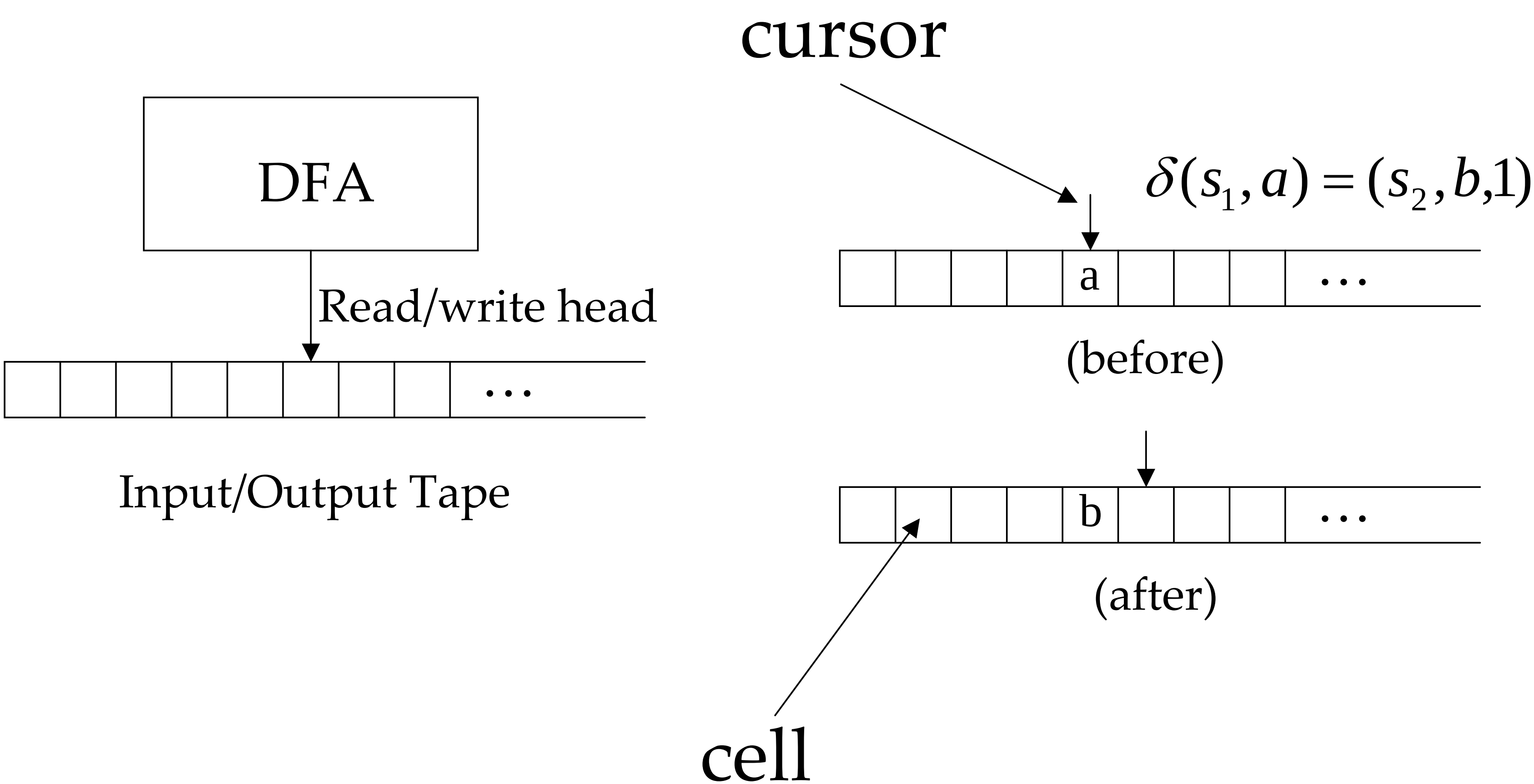
P-hardness:

Deterministic Turing Machine (DTM) $T = (S, \Sigma, \delta, s_0)$

where: $\sqcup \in \Sigma$, s_0, accept $\in S$, $\delta: S \times \Sigma \to (S \times \Sigma \times \{-1,0,1\})$

- T divided into *cells*, *cursor* move along the tape
- An input string I is written on the tape: the first $|I|$ cells $c_0,\ldots,c_{|I|-1}$ of the tape, all other cells contain $\sqcup$
- T takes successive *steps* of computation according to δ.

$$\delta(s,\sigma) = (s', \sigma', d) \ (d = -1 \text{ or } 0 \text{ or } 1)$$



DFA
Read/write head
…
Input/Output Tape
cursor
$\delta(s_1, a) = (s_2, b, 1)$
a
…
(before)
b
…
(after)
cell

T: DTM

Transition function $t=\langle s,\sigma, s',\sigma',d\rangle$ expresses the following if-then-rule:

If at some time instant τ the DTM is in state s, the cursor points to cell number π, and this cell contains symbol σ

Then at instant $\tau +1$ the DTM is in state s', cell number π contains symbol σ', and the cursor points to cell number $\pi+d$

- Possible to describe the computation of a DTM ***T*** on input string ***I*** from its initial configuration at time instance 0 to the configuration at instant ***N*** by a (horn) propositional logic program ***L(T,I,N)***

- The goal: encode the PTIME Turing computation of ***T*** on input ***I*** with a horn logic program ***L*** and an atom ***G***, s.t. $L \models G$ iff ***T*** accepts ***I*** in at most ***N*** steps

Propositional atoms

(there are many, but only polynomially many…)

- ***symbol***$_\alpha[\tau,\pi]$ for $0 \leq \tau \leq N$, $0 \leq \pi \leq N$ and $\alpha \in \Sigma$. Intuitive meaning: at instant τ of the computation, cell number π contains symbol α

- ***cursor*** $[\tau,\pi]$ for $0 \leq \tau \leq N$, $0 \leq \pi \leq N$. Intuitive meaning: at instant τ of the cursor points to cell number π

- ***state***$_s[\tau]$ for $0 \leq \tau \leq N$ and $s \in S$. Intuitive meaning: at instant τ the DTM T is in state s

- ***accept*** Intuitive meaning: T has accepted.

- Initialization facts:

$symbol_{\sigma}[0,\pi] \leftarrow$ for $0 \leq \pi \leq |I|$, where $I_{\pi} = \sigma$

$symbol_{\sqcup}[0,\pi] \leftarrow$ for $|I| \leq \pi \leq N$

$cursor[0,0] \leftarrow$

$state_{s_0}[0] \leftarrow$

- Transition rules: $t=\langle s,\sigma, s',\sigma',d\rangle$ $0 \le \tau,\pi \le N$

$symbol_{\sigma'}[\tau+1,\pi] \leftarrow state_s[\tau], symbol_{\sigma}[\tau,\pi], cursor[\tau,\pi]$

$cursor\,[\tau+1,\pi+d] \leftarrow state_s[\tau], symbol_{\sigma}[\tau,\pi], cursor[\tau,\pi]$

$state_{s'}[0] \leftarrow state_s[\tau], symbol_{\sigma}[\tau,\pi], cursor[\tau,\pi]$

- Inertia rules: $0 \le \pi \neq \pi' \le N$

$symbol_{\sigma}[\tau+1,\pi] \leftarrow symbol_{\sigma}[\tau,\pi], cursor[\tau,\pi']$

- Accept rules: $0 \le \tau \le N$

$accept \leftarrow state_{\mathrm{accept}}[\tau]$

Our encoding precisely simulates the behaviour machine T on input I up to N steps.

(This can be formally shown by induction on the time steps.)

Therefore:

L(T,I,N) $\models$ *accept* if and only if the DTM T accepts the input string I within N steps.

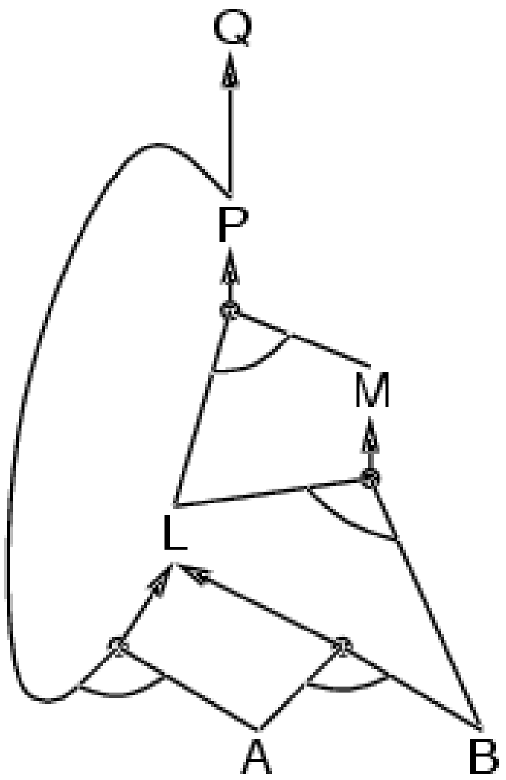
The construction is feasible in Logspace.

➔ Horn clause inference is P-complete

Forward chaining

- Idea: fire any rule whose premises are satisfied in the *KB*
  - add its conclusion to the *KB*, until query is found

$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$

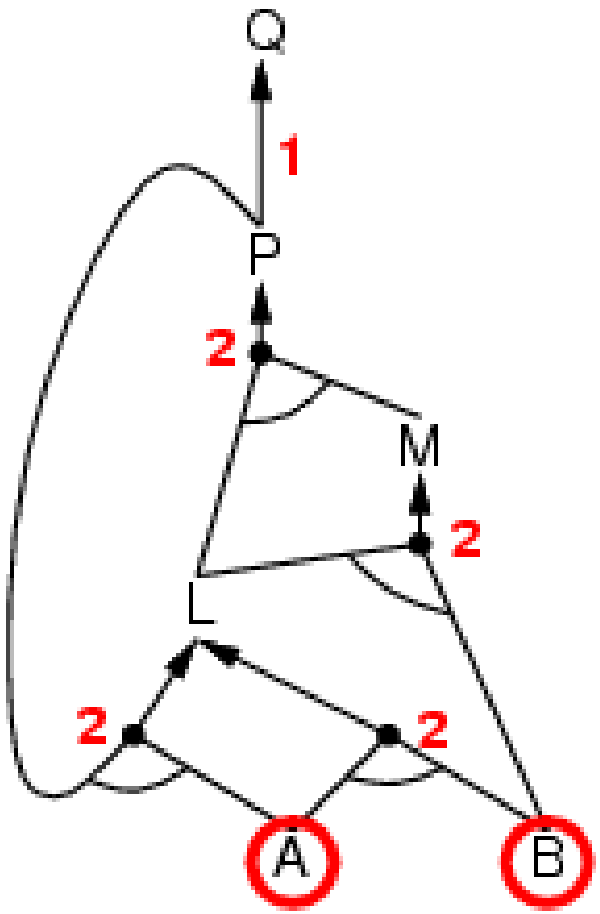


Forward chaining algorithm (Minoux)

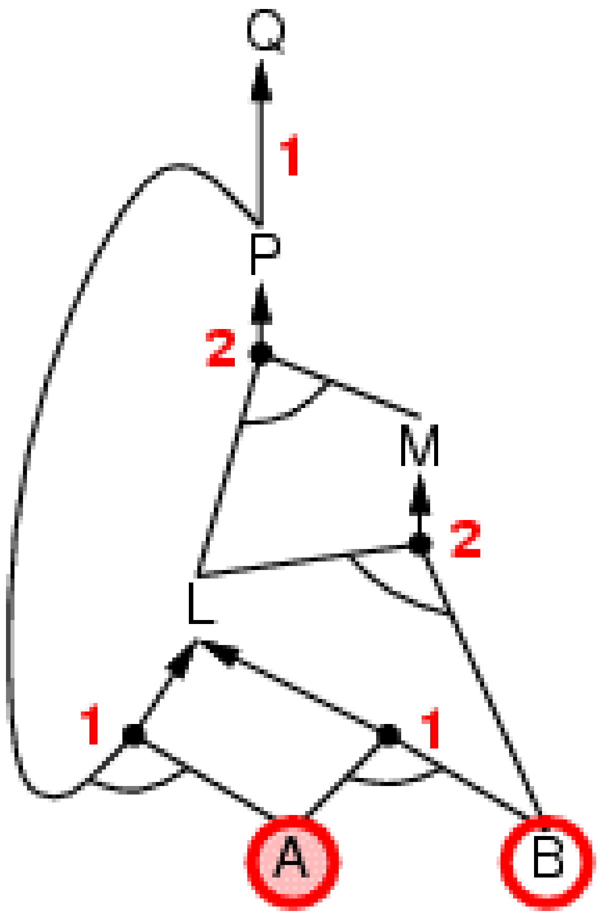
```
function PL-FC-ENTAILS?(KB, q) returns true or false
   local variables: count, a table, indexed by clause, initially the number of premises
                    inferred, a table, indexed by symbol, each entry initially false
                    agenda, a list of symbols, initially the symbols known to be true

   while agenda is not empty do
       p ← POP(agenda)
       unless inferred[p] do
           inferred[p] ← true
           for each Horn clause c in whose premise p appears do
               decrement count[c]
               if count[c] = 0 then do
                   if HEAD[c] = q then return true
                   PUSH(HEAD[c], agenda)
   return false
```

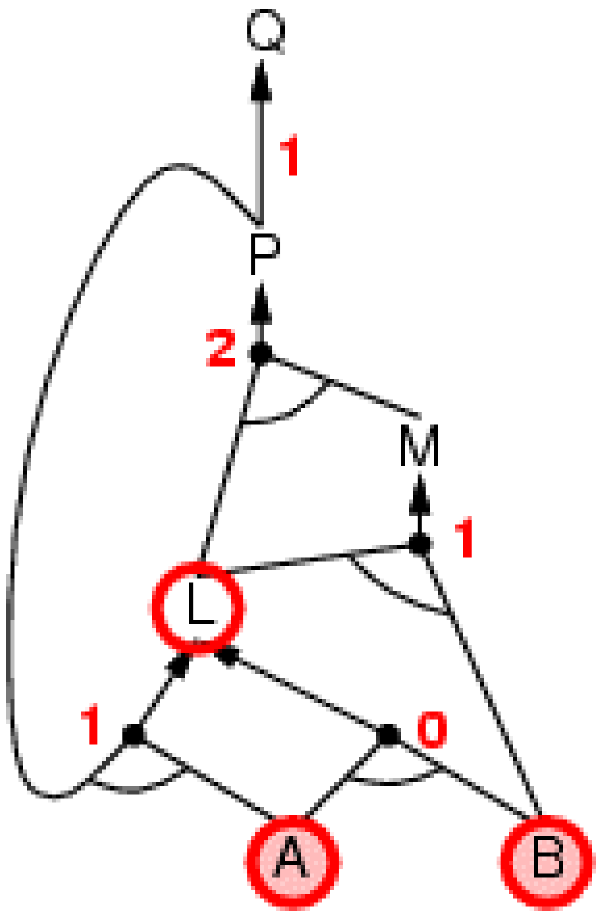
Forward chaining example



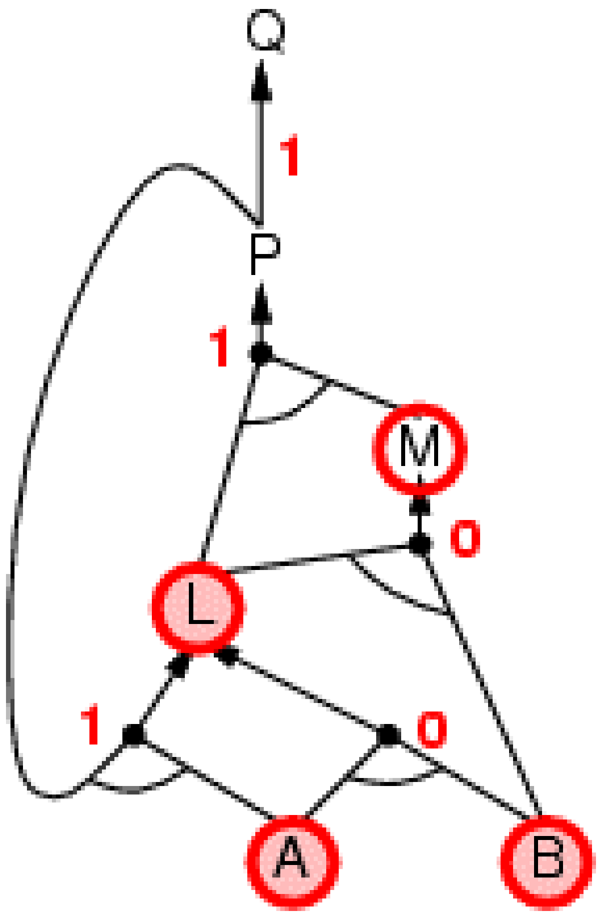
Forward chaining example



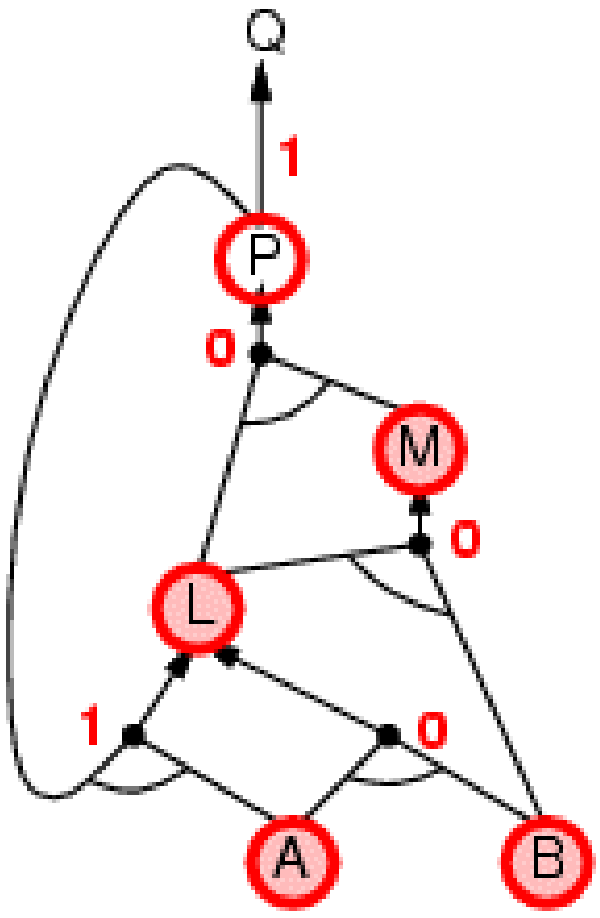
Forward chaining example



Forward chaining example



Forward chaining example



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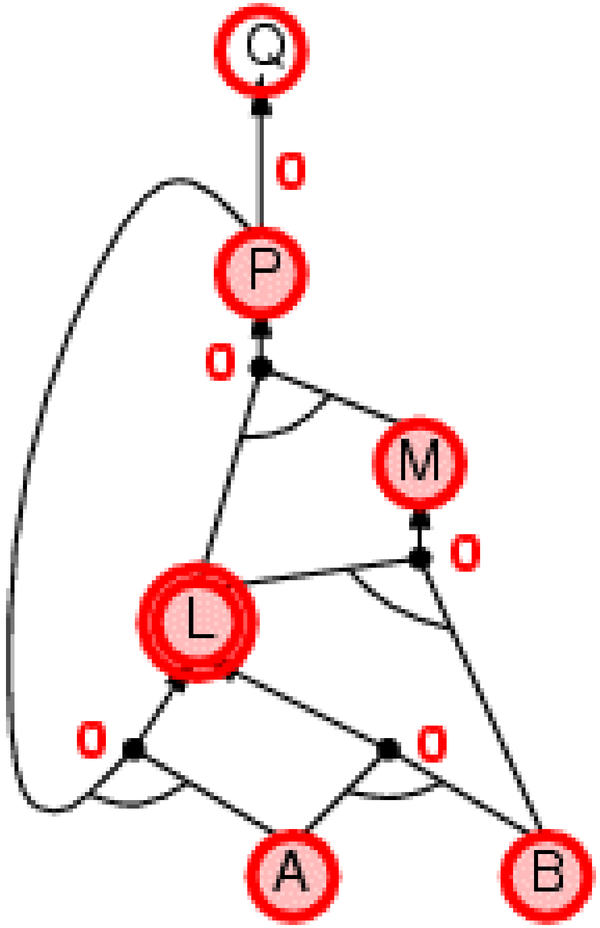

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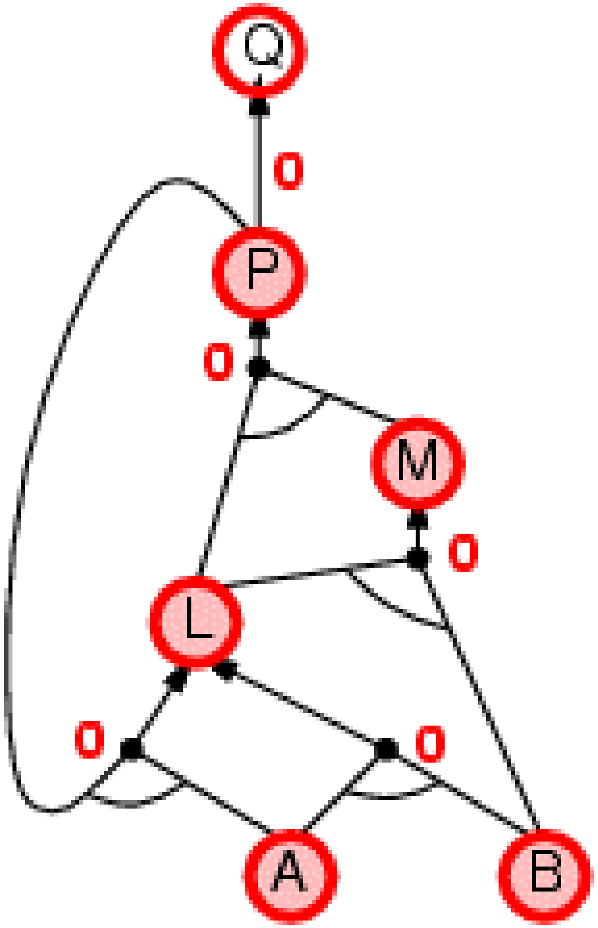
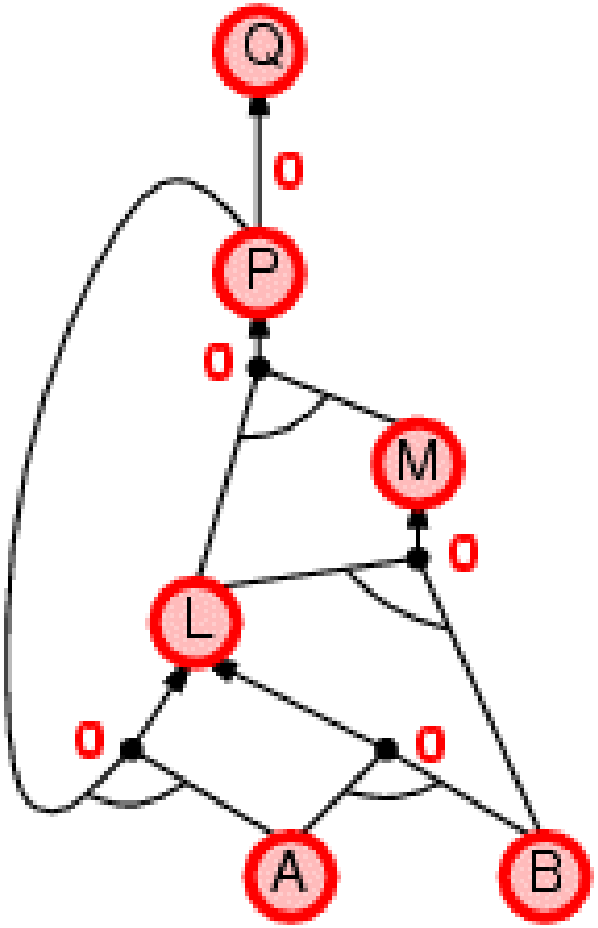


Forward chaining example



This algorithm can be implemented to run in Linear time on a Random Access Machine.

It suffices to use appropriate data structures (arrays)

Read the Minoux Paper

➔ Propositional Horn inference is feasible in Linear Time