# On the Hardness of Robust Classification

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University of Oxford

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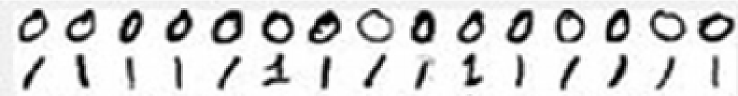

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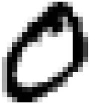

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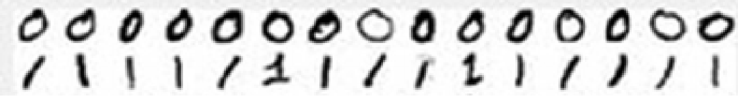

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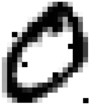

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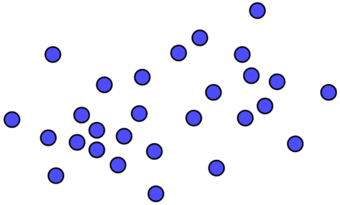

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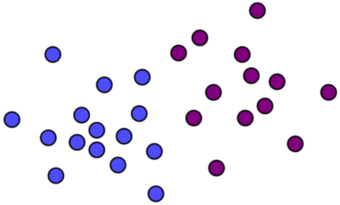

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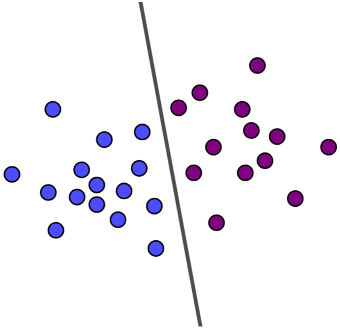

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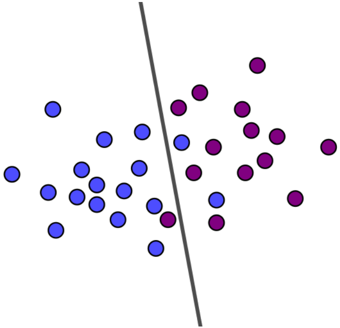

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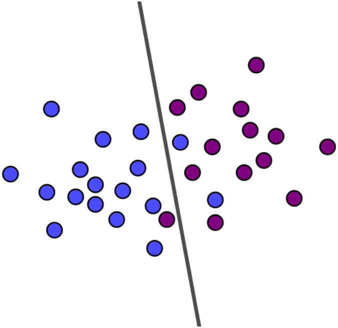

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**Learning algorithm** $\mathcal{A}$ **with sample complexity** $m$: when given a sample $S$ of size $\geq m$, $\mathcal{A}$ outputs a hypothesis that has low error w.h.p. over $S$.

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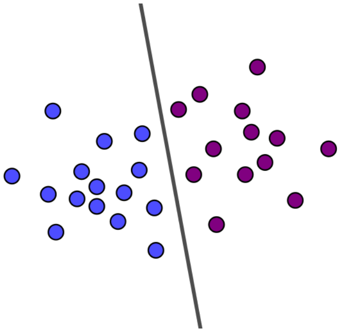

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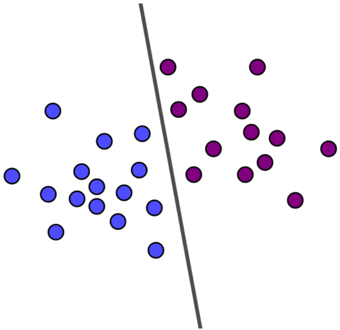

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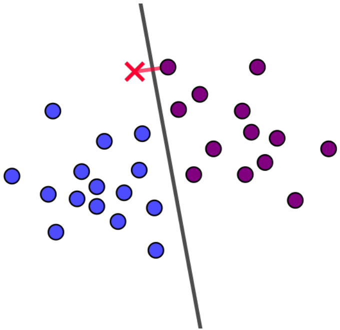

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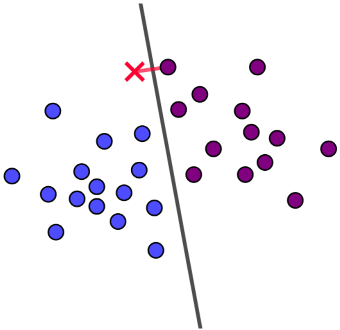

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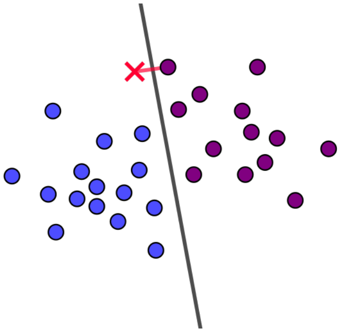

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**Question:** How do we define a misclassification?

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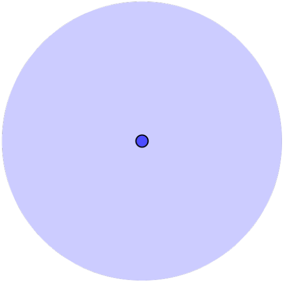

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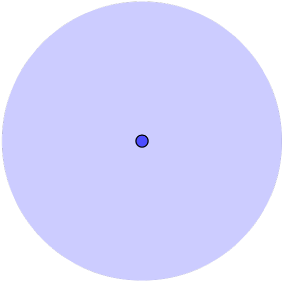

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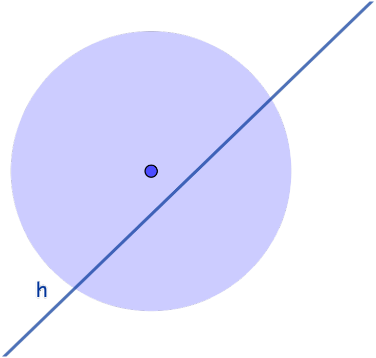


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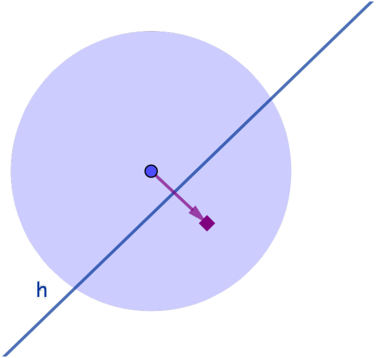


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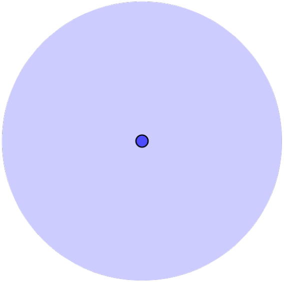

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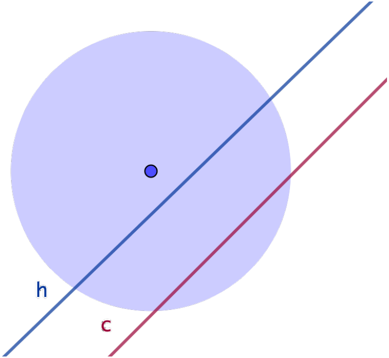


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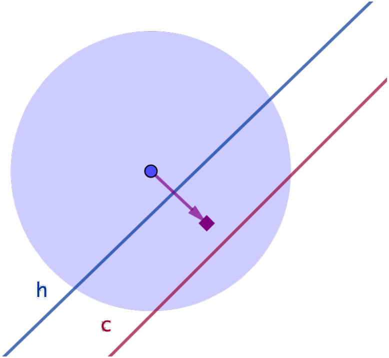


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**Robust risks:**
*Constant-in-the-ball:* probability that an adversary can perturb a point $x$ drawn from $D$ to $z$ with budget $\rho$, so that $c$ on $x$ and $h$ on $z$ differ:

$$\mathrm{R}_{\rho}^{C}(h, c) = \mathbb{P}_{x \sim D}\left(\exists z \in B_{\rho}(x) \,.\, c(x) \neq h(z)\right) \quad .$$

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*Exact-in-the-ball:* probability that an adversary can perturb a point $x$ drawn from $D$ to $z$ with budget $\rho$, so that $c$ and $h$ *disagree* on $z$:

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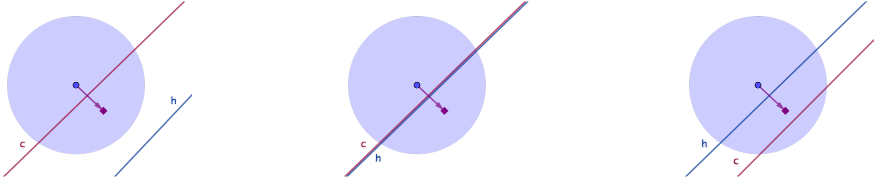


(a) $\mathsf{R}_\rho^E > 0$, $\mathsf{R}_\rho^C = 0$ , (b) $\mathsf{R}_\rho^E = 0$, $\mathsf{R}_\rho^C > 0$ , (c) $\mathsf{R}_\rho^E > 0$, $\mathsf{R}_\rho^C > 0$.

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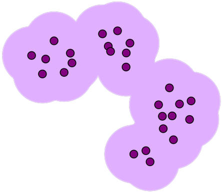

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**In our view:** adversary's power = creating perturbations that cause $c \neq h$, so we choose $\mathsf{R}^E_\rho$, despite its drawbacks.

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- We require *polynomial* sample complexity,
- It might make more sense to require *finite* sample complexity in other contexts, e.g. $\mathbb{R}^n$.

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- If $\mathcal{C}$ is non-trivial, we can find $c_1$ and $c_2$ and $x$ such that

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- Construct a distribution such that $c_1$ and $c_2$ will likely agree on a sample of size polynomial in $n$ but have $\mathrm{R}^E_\rho(c_1, c_2) = \Omega(1)$.
- Let $c \sim \mathrm{Unif}(c_1, c_2)$ before labelling the sample. Then any function we learn won't be robust against $c$ with positive probability.

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**Intuition:** input points that are close to each other cannot have vastly different probability masses.

**Examples:** uniform distribution, product distribution where the mean of each variable is bounded, etc.

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To show that MON-CONJ is not efficiently robustly learnable for $\rho(n) = \omega(\log n)$, we can show that, under the *uniform distribution*

- Choose long enough monotone conjunctions $c_1$ and $c_2$

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- Again, choose target at random before labelling.

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**Proof idea:** Two cases:

- If the target conjunction is short enough, we have learned exactly, and hence robustly.
- If the target conjunction is large enough, we can use concentration bounds to show that the adversary is unlikely to cause a label change.

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Previous computational hardness of robust learning results used:

- Another learning model (statistical query) [Bubeck et al., 2018],
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$$\underset{\text{PAC learning}}{(\mathcal{C}, \mathcal{D}, \mathcal{X})} \longrightarrow \underset{\text{Robust learning}}{(\mathcal{C}', \mathcal{D}', \mathcal{X}')}$$

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- Other learning models, e.g. when one has access to *membership queries*.

## Current and Future Work

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- More powerful learning model (e.g., membership queries).

# References I

Sébastien Bubeck, Eric Price, and Ilya Razenshteyn. *Adversarial examples from computational constraints.* arXiv preprint. arXiv:1805.10204, 2018.

Akshay Degwekar and Vinod Vaikuntanathan. *Computational limitations in robust classification and win-win results.* arXiv preprint. arXiv:1902.01086, 2019.