Local projection finite element stabilization for the generalized Stokes problem

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Abstract

We analyze pressure stabilized finite element methods for the solution of the generalized Stokes problem and investigate their stability and convergence properties. An important feature of the method is that the pressure gradient unknowns can be eliminated locally thus leading to a decoupled system of equations. Although stability of the method has been established, for the homogeneous Stokes equations, the proof given here is based on the existence of a special interpolant with additional orthogonal property with respect to the projection space. This, makes it a lot simpler and more attractive. The resulting stabilized method is shown to lead to optimal rates of convergence for both velocity and pressure approximations.

Keywords: Generalized Stokes equations, Stabilized finite elements, Local projection, convergence, error estimates.

AMS Subject Classifications: 65N12, 65N30, 65N15, 76D07.

1 Introduction

Numerical approximation of incompressible flows presents a major difficulty, namely, the need to satisfy a compatibility condition between the discrete velocity and pressure spaces ([18], [9] and [15]). This condition which has been well known since the work of Babuska and Brezzi in the 1970s prevents,

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in particular, the use of equal order interpolation spaces for the two variables, which is the most attractive choice from a computational point of view.

To overcome this difficulty, stabilized finite element methods that circumvent the restrictive $\inf - \sup$ condition have been developped for Stokes-like problems (see, [19], [14], [20], [16], and [4]). These residual-based methods represent one class of stabilized methods. They consist in modifying the standard Galerkin formulation by adding mesh-dependent terms, which are weighted residuals of the original differential equations. Although for properly chosen stabilization parameters these methods are well posed for all velocity and pressure pairs, numerical results reported by several researchers seem to indicate that these methods are sensitive to the choice of the stabilization parameters. The local stabilization suggested in [20] has some advantages in this regard. Another class of stabilized methods has been derived using Galerkin methods enriched with bubble functions (see, [1] and [3]). Alternative stabilization techniques based on a continuous penalty method have been proposed and analyzed in [11] and [10].

Recently, local projection methods that seem less sensitive to the choice of parameters and have better local conservation properties were proposed. The stabilization by projecting the pressure gradient has been analyzed in [12]. It was shown that the method is consistent in the sense that a smooth exact solution satisfies the discrete problem. Though the method may seem computationally expensive due to the nonlocal behaviour of the projection, iterative solvers were developped to make the method more efficient ([13]). Alternatively, a two-level approach with a projection onto a discontinuous finite element space of a lower degree defined on a coarser grid has been analyzed in [5], [22], and [23]. In [6] and [7], low order approximations of the oseen equations were analyzed.

In this paper, we analyze the pressure gradient stabilization method for the generalized Stokes problem. This kind of problems arise naturally in the time discretization of the unsteady Stokes problem, or the full Navier-Stokes equations by means of an operator splitting technique. Unlike the proof given by [22] and [23], where stability was shown using an inf-sup condition due to [16] and the equivalence of norms on finite dimensional spaces. Here, the stability of the pressure-gradient method is proved for arbitrary Q^k-elements, by constructing a special interpolant with additional orthogonal property with respect to the projection space. As a result, optimal rates of convergence are found for the velocity and pressure approximations. Numerical results for two-dimensional generalized Stokes flows are presented. We observe that, for the computed examples, the accuracy and the rates of convergence are as predicted by the theory.

2 Variational formulation

Let Ω be a bounded two-dimensional polygonal region, $f \in L^2(\Omega)$, σ a positive real number (typically, $\sigma = \frac{1}{\Delta t}$ where Δt is the time step in a time discretization procedure), and ν the kinematic viscosity coefficient. Then, the generalized homogeneous Stokes Problem reads
Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ satisfying:

$$\begin{aligned} \sigma\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega \\ \nabla.\mathbf{u} &= 0 \quad \text{in } \Omega \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

where, $\mathbf{V} = (H_0^1(\Omega))^d$ and $Q = L_0^2(\Omega)$, with $L_0^2(\Omega)$ denoting the set of square integrable functions with null average.
Define the forms

$$\begin{aligned} A((\mathbf{u}, p); (\mathbf{v}, q)) &= \sigma(\mathbf{u}, \mathbf{v}) + \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) - (p, \nabla.\mathbf{v}) + (q, \nabla.\mathbf{u}) \\ &\text{and} \\ F(\mathbf{v}, q) &= (\mathbf{f}, \mathbf{v}) \quad , \end{aligned} \tag{2}$$

for all $(\mathbf{v}, q) \in \mathbf{V} \times Q$, with $(.,.)$, as usual, denoting the L^2−inner product on the region Ω.
Then, the weak formulation of (1) reads in compact notation as

$$A((\mathbf{u}, p); (\mathbf{v}, q)) = F(\mathbf{v}, q) \quad , \quad \forall(\mathbf{v}, q) \in \mathbf{V} \times Q. \tag{3}$$

Let $\mathbf{V}_h$ and Q_h be finite dimensional subspaces of $\mathbf{V}$ and Q, respectively. Then, the Galerkin discrete problem reads
Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that:

$$A((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)) = F(\mathbf{v}_h, q_h) \quad , \quad \forall(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h. \tag{4}$$

Note that formulation (4) is stable only for velocity and pressure approximations satisfying the inf-sup condition (see, for example [18]).

3 Pressure gradient stabilization

Let ζ_h be a shape regular partition of the region Ω into quadrilateral elements K (see, for example [8]). Denote by h_K the diameter of element K and by h the maximum diameter of the elements $K \in \zeta_h$. The coarser mesh partition ζ_{2h} of macro-elements M is obtained by grouping sets of neighbouring four elements of ζ_h. In order to guarantee stability and converge of the following method, we assume that for elements $K \subset M \in \zeta_{2h}$ we have $h_K \sim h_M$.

We then define the equal order continuous finite element spaces

$$\mathbf{V}_h = V_h^2 = \left\{ \mathbf{v} \in \left(H_0^1(\Omega)\right)^2 : \ \mathbf{v}\mid_K \in \left(Q_h^k(K)\right)^2, \ \forall K \in \zeta_h \right\}$$
$$Q_h = \left\{ q \in H^1(\Omega) : \ q\mid_K \in Q_h^k(K), \ \forall K \in \zeta_h \right\} \tag{5}$$

where Q_h^k denotes the standard continuous isoparametric finite element functions defined by means of a mapping from a reference element. On the reference quadrilateral the approximation functions are polynomials of degree less than or equal to k in each variable. We shall also use P_h^k to denote the space of polynomials of degree less than or equal to k over ζ_h.
Additionally, we define the pressure-gradient finite element space by

$$\mathbf{Y}_{2h} = Y_{2h}^2 = \bigoplus_{M \in \zeta_{2h}} (Q_{2h}^{k-1,disc}(M))^2. \tag{6}$$

where $Q_{2h}^{k-1,disc}$(respectively $P_{2h}^{k,disc}$) denote the finite element spaces of discontinuous functions across elements of ζ_{2h}.
Define the local projection operator $\pi_M : L^2(M) \to Q_{2h}^{k-1}(M)$ by

$$(w - \pi_M w, \phi)_M = 0, \quad \forall \phi \in Q_{2h}^{k-1}(M) \tag{7}$$

which generates the global projection $\pi_h : L^2(\Omega) \to Y_{2h}$ defined by

$$(\pi_h w)\mid_M = \pi_M(w\mid_M), \quad \forall M \in \zeta_{2h}\ , \forall w \in L^2(\Omega). \tag{8}$$

The fluctuation operator $\kappa_h : L^2(\Omega) \to L^2(\Omega)$ is given by

$$\kappa_h = id - \pi_h \tag{9}$$

where, id denotes the identity operator on $L^2(\Omega)$. For simplicity, we shall use the same notation id, π_M, π_h, and κ_h for vector-valued functions. Thus, $\kappa_h \nabla p$ is to be inderstood as acting on each component of ∇p seperately.

Now, we are ready to introduce the stabilizing term

$$S(p_h; q_h) = \sum_{K \in \zeta_h} \alpha_K \left(\kappa_h \nabla p_h, \nabla q_h\right)_K = \sum_{K \in \zeta_h} \alpha_K \left(\kappa_h \nabla p_h, \kappa_h \nabla q_h\right)_K \tag{10}$$

where α_K are element parameters that depend on the local mesh size.

Thus, our stabilized discrete problem reads as: Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$A((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)) + S(p_h; q_h) = F(\mathbf{v}_h, q_h) \ , \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h. \quad (11)$$

This can be written component-wise as: Find $(\mathbf{u}_h, p_h, \lambda_h) \in \mathbf{V}_h \times Q_h \times \mathbf{Y}_{2h}$ such that

$$\begin{aligned} \sigma(\mathbf{u}_h, \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h, \nabla . \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \ , & \quad \forall \mathbf{v}_h \in \mathbf{V}_h \\ \sum_K \alpha_K (\nabla p_h, \nabla q_h) - \sum_K \alpha_K (\lambda_h, \nabla q_h) - (q_h, \nabla . \mathbf{u}_h) = 0 \quad , & \quad \forall q_h \in Q_h \\ - \sum_K \alpha_K (\nabla p_h, \xi_h) + \sum_K \alpha_K (\lambda_h, \xi_h) = 0 \quad , & \quad \forall \xi_h \in \mathbf{Y}_{2h} \end{aligned} \quad (12)$$

where, λ_{2h} is the local L^2-projection of ∇p_h onto a discrete space $\mathbf{Y}_{2h}$.
In order to investigate the properties of the bilinear form $A((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)) + S(p_h; q_h))$ on the product space $\mathbf{V}_h \times Q_h$, we introduce the mesh dependent norm

$$\|(\mathbf{v}_h, q_h)\|^2 = \sigma \|\mathbf{v}_h\|^2_{0,\Omega} + \nu |\mathbf{v}_h|^2_{1,\Omega} + (\sigma + \nu) \|q_h\|^2_{0,\Omega} + S(q_h; q_h). \quad (13)$$

3.1 Stability

The main idea in the analysis of local projection methods is the construction of an interpolation operator $j_h : H^1(\Omega) \to Y_{2h}$ with $j_h v \in H^1_0(\Omega)$ for all $v \in H^1_0(\Omega)$, satisfying the usual approximation property

$$\|v - j_h v\|_{0,K} + h_K |v - j_h v|_{1,K} \leq C h^s_K \|v\|_{s,w(K)} \ , \ \forall v \in H^s(w(K)), \ 1 \leqslant s \leqslant k+1 \quad (14)$$

where $w(K)$ denotes a certain local neighbourhood of K.
With the additional orthogonal property

$$(v - j_h v, \phi_h) = 0 \quad , \quad \forall \phi_h \in Y_{2h}, \ \forall v \in H^1(\Omega), \quad (15)$$

Lemma 1 *Let $i_h : H^1(\Omega) \to V_h$ be an interpolation operator such that $i_h v \in H^1_0(\Omega)$ for all $v \in H^1_0(\Omega)$ with the error estimate*

$$\|v - i_h v\|_{0,K} + h_K |v - i_h v|_{1,K} \leq C h^s_K \|v\|_{s,w(K)} \ , \ \forall v \in H^s(\Omega), \ 1 \leqslant s \leqslant k+1 \quad (16)$$

Further, assume that the local inf-sup condition

$$\inf_{q_h \in Y_{2h}(K)} \sup_{v_h \in V_h(K)} \frac{(v_h, q_h)_K}{\|v_h\|_{0,K} \|q_h\|_{0,K}} \geqslant \beta_1 \tag{17}$$

holds for all $K \in \zeta_{2h}$*, with a positive constant* β_1 *independent of the mesh size. Then, there exists an interpolation operator* $j_h : H^1(\Omega) \to Y_{2h}$ *with the properties (14) and (15).*

Proof. For the construction of the interpolation operator j_h we refer to Theorem 2.2 in ([21]). ■

Remark 2 *Note that condition (17) can be checked using Stenberg's technique on macro-elements* $M \in \zeta_{2h}$ *which are equivalent to a reference element* $\widehat{M}$*. The* $\inf - \sup$ *condition holds if the the null space* N_M *is such that*

$$N_M = \left\{ q_h \in Y_{2h}(M) : (v_h, q_h)_M = 0, \ \forall v_h \in V_h(M) \cap H_0^1(M) \right\} = \{0\} \, . \tag{18}$$

Note also that the fluctuation operator κ_h satisfies the approximation property

$$\|\kappa_h q\|_{0,M} \leq C h_M^l \, |q|_{l,M} \, , \ \forall q \in H^l(M), \forall M \in \zeta_{2h}, \, 0 \leqslant l \leqslant k. \tag{19}$$

Since, The L^2- local projection $\pi_M : L^2(M) \to Y_{2h}(M)$ becomes the identity for the space $Q^{k-1}(M) \subset H^l(M)$, and the kernel of κ_h contains $P^{k-1}(M) \subset Q^{k-1}(M)$. Then, the Bramble-Hilbert Lemma gives the approximation properties stated in assumption (19).

Remark 3 *The justification that the pair* $V_h/Y_{2h} = Q_h^k/Q_{2h}^{k-1,disc}$*, for* $k \geqslant 1$*, satisfy (17) follows from (18) using the one-to-one property of the mapping* $F_M : \widehat{M} \to M$ *combined with a positive bilinear function corresponding to the central node of* $\widehat{M}$ *(see, [21] and [17]). Further, using the same argument we can show that* $V_h/Y_{2h} = Q_h^k/P_{2h}^{k-1,disc}$ *gives also a stable approximation.*

Theorem 4 *Let properties (14), (15), and (19) hold and the parameters* α_K *be such that* $\alpha_K = Ch_K^2$ *for each element* $K \in \zeta_h$*. Then, the bilinear form of the pressure-gradient stabilized method satisfies*

$$\sup_{\substack{(\mathbf{w}_h, r_h) \in V_h \times Q_h \\ (\mathbf{w}_h, r_h) \neq 0}} \frac{A((\mathbf{v}_h, q_h) \, ; (\mathbf{w}_h, r_h)) + S(q_h; r_h)}{\|(\mathbf{w}_h, r_h)\|} \geq \beta \, \|(\mathbf{v}_h, q_h)\|$$

for some positive constant β *independent of the mesh parameter* h*.*

Proof. Let $(\mathbf{v}_h,q_h) \in \mathbf{V}_h \times Q_h$ we have:

$$A((\mathbf{v}_h, q_h);(\mathbf{v}_h, q_h))+S(q_h;q_h) = \sigma \left\|\mathbf{v}_h\right\|_{0,\Omega}^2+\nu \left|\nabla \mathbf{v}_h\right|_{1,\Omega}+\sum_{K\in\zeta_h} \alpha_K \left\|\kappa_h \nabla q_h\right\|_{0,K}^2 . \tag{20}$$

Further, the continuous inf − sup condition implies the existence of $\mathbf{v}_{q_h} \in \mathbf{V}$ (see, [18]) satisfying

$$(q_h, \nabla \cdot \mathbf{v}_{q_h}) = \left\|q_h\right\|_{0,\Omega}^2 \quad \text{with} \quad \left\|\mathbf{v}_{q_h}\right\|_{1,\Omega} \leqslant \left\|q_h\right\|_{0,\Omega} . \tag{21}$$

Let $\widetilde{\mathbf{v}}_{q_h} = j_h \mathbf{v}_{q_h}$, then

$$\begin{aligned} A((\mathbf{v}_h, q_h);(-\widetilde{\mathbf{v}}_{q_h},0)) + S(q_h;0) &= -\sigma(\mathbf{v}_h, \widetilde{\mathbf{v}}_{q_h}) - \nu(\nabla \mathbf{v}_h, \nabla \widetilde{\mathbf{v}}_{q_h}) \\ &+ (\nabla \cdot \widetilde{\mathbf{v}}_{q_h}, q_h) \\ &= -\sigma(\mathbf{v}_h, \widetilde{\mathbf{v}}_{q_h}) - \nu(\nabla \mathbf{v}_h, \nabla \widetilde{\mathbf{v}}_{q_h}) \\ &+ \left\|q_h\right\|_{0,\Omega}^2 - (\nabla \cdot (\mathbf{v}_{q_h} - \widetilde{\mathbf{v}}_{q_h}), q_h). \end{aligned} \tag{22}$$

Integrating by parts the fourth term on the right hand of (22), and using properties (14) and (15) we obtain

$$\begin{aligned} \left|(\nabla \cdot (\mathbf{v}_{q_h} - \widetilde{\mathbf{v}}_{q_h}), q_h)\right| &= \left|(\mathbf{v}_{q_h} - \widetilde{\mathbf{v}}_{q_h}, \nabla q_h)\right| = \left|(\mathbf{v}_{q_h} - \widetilde{\mathbf{v}}_{q_h}, \kappa_h \nabla q_h)\right| \\ &\leqslant \left(\sum_{K\in\zeta_h} \alpha_K^{-1} \left\|\mathbf{v}_{q_h} - \widetilde{\mathbf{v}}_{q_h}\right\|_{0,K}^2\right)^{\frac{1}{2}} \left(\sum_{K\in\zeta_h} \alpha_K \left\|\kappa_h \nabla q_h\right\|_{0,K}^2\right)^{\frac{1}{2}} \\ &\leqslant C \left(\sum_{K\in\zeta_h} \alpha_K \left\|\kappa_h \nabla q_h\right\|_{0,K}^2\right)^{\frac{1}{2}} \left\|\mathbf{v}_{q_h}\right\|_{1,\Omega} \end{aligned}$$

i.e.

$$\left|(\nabla \cdot (\mathbf{v}_{q_h} - \widetilde{\mathbf{v}}_{q_h}), q_h)\right| \leqslant C_1 \left(\sum_{K\in\zeta_h} \alpha_K \left\|\kappa_h \nabla q_h\right\|_{0,K}^2\right)^{\frac{1}{2}} \left\|q_h\right\|_{0,\Omega} . \tag{23}$$

The first two terms in (22) are estimated by

$$\begin{aligned} -\sigma(\mathbf{v}_h, \widetilde{\mathbf{v}}_{q_h}) - \nu(\nabla \mathbf{v}_h, \nabla \widetilde{\mathbf{v}}_{q_h}) &\geqslant -\sigma \left\|\mathbf{v}_h\right\|_{0,\Omega} \left\|\widetilde{\mathbf{v}}_{q_h}\right\|_{0,\Omega} - \nu \left|\mathbf{v}_h\right|_{1,\Omega} \left|\widetilde{\mathbf{v}}_{q_h}\right|_{1,\Omega} \\ &\geqslant -\max(\sigma, \nu)(\left\|\mathbf{v}_h\right\|_{0,\Omega} + \left|\mathbf{v}_h\right|_{1,\Omega}) \left\|q_h\right\|_{0,\Omega} . \end{aligned}$$

Therefore, using Young's inequality, we obtain

$$A((\mathbf{v}_h,q_h);(\widetilde{\mathbf{v}}_{q_h},0))+S(q_h;0)\geqslant -\max(\sigma,\nu)(\frac{1}{2\delta}\left\|\mathbf{v}_h\right\|_{0,\Omega}^2+\frac{\delta}{2}\left\|q_h\right\|_{0,\Omega}^2 \\ +\frac{1}{2\delta}\left|\mathbf{v}_h\right|_{1,\Omega}^2+\frac{\delta}{2}\left\|q_h\right\|_{0,\Omega}^2)+\left\|q_h\right\|_{0,\Omega}^2 \\ -\frac{\delta C}{2}\left\|q_h\right\|_{0,\Omega}^2-\frac{C}{2\delta}\sum_{K\in\zeta_h}\alpha_K\left\|\kappa_h\nabla q_h\right\|_{0,K}^2$$

i.e.

$$A((\mathbf{v}_h,q_h);(-\widetilde{\mathbf{v}}_{q_h},0))+S(q_h;0)\geqslant -C_1\left\|\mathbf{v}_h\right\|_{0,\Omega}^2-C_1\left|\mathbf{v}_h\right|_{1,\Omega}^2+C_2\left\|q_h\right\|_{0,\Omega}^2 \\ -C_3\sum_{K\in\zeta_h}\alpha_K\left\|\kappa_h\nabla q_h\right\|_{0,K}^2 \tag{24}$$

when we choose $0<\delta<1/(\max(\sigma,\nu)+\dfrac{C}{2})$.

Also,

$$A((\mathbf{v}_h,q_h);(\mathbf{v}_h,q_h))+S(q_h;q_h)=\sigma\left\|\mathbf{v}_h\right\|_{0,\Omega}^2+\nu\left|\mathbf{v}_h\right|_{1,\Omega}^2+\sum_{K\in\zeta_h}\alpha_K\left\|\kappa_h\nabla q_h\right\|_{0,K}^2. \tag{25}$$

Let $(\mathbf{w}_h,r_h)=(\mathbf{v}_h-\delta\widetilde{\mathbf{v}}_{q_h},q_h)$; combining (24) and (25) gives

$$A((\mathbf{v}_h,q_h);(\mathbf{w}_h,r_h))+S(q_h;r_h)=A((\mathbf{v}_h,q_h);(\mathbf{v}_h,q_h))+S(q_h;q_h) \\ +\delta A((\mathbf{v}_h,q_h);(-\widetilde{\mathbf{v}}_{q_h},0)) \\ \geq(\sigma-\delta C_1)\left\|\mathbf{v}_h\right\|_{0,\Omega}^2+(\nu-\delta C_1)\left|\mathbf{v}_h\right|_{1,\Omega}^2 \\ +C_2\left\|q_h\right\|_{0,\Omega}^2+(1-\delta C_3)\sum_{K\in\zeta_h}\alpha_K\left\|\kappa_h\nabla q_h\right\|_{0,K}^2$$

i.e.

$$A((\mathbf{v}_h,q_h);(\mathbf{w}_h,r_h))+S(q_h;r_h)\geq C\left\|(\mathbf{v}_h,q_h)\right\|^2 \tag{26}$$

when we choose $0<\delta<\min\left\{1/(\max(\sigma,\nu)+\dfrac{C}{2}),\sigma/C_1,\nu/C_1,1/C_3\right\}$.

The norm of $(\mathbf{w}_h,r_h)$ is estimated by

$$\left\|(\mathbf{w}_h,r_h)\right\|^2\leqslant\sigma\left(\left\|\mathbf{v}_h\right\|_{0,\Omega}+\delta\left\|\widetilde{\mathbf{v}}_{q_h}\right\|_{0,\Omega}\right)^2+\nu\left(\left|\mathbf{v}_h\right|_{1,\Omega}+\delta\left|\widetilde{\mathbf{v}}_{q_h}\right|_{1,\Omega}\right)^2 \\ +(\sigma+\nu)\left\|q_h\right\|_{0,\Omega}^2+\sum_{K\in\zeta_h}\alpha_K\left\|\kappa_h\nabla q_h\right\|_{0,K}^2 \\ \leqslant\sigma(\left\|\mathbf{v}_h\right\|_{0,\Omega}+\delta\left\|q_h\right\|_{0,\Omega})^2+\nu(\left|\mathbf{v}_h\right|_{1,\Omega}+\delta\left\|q_h\right\|_{0,\Omega})^2 \\ +(\sigma+\nu)\left\|q_h\right\|_{0,\Omega}^2+\sum_{K\in\zeta_h}\alpha_K\left\|\kappa_h\nabla q_h\right\|_{0,K}^2.$$

Hence, using Young inequality, we obtain

$$\|(\mathbf{w}_h, r_h)\|^2 \leqslant 2(1+\delta)^2(\sigma \|\mathbf{v}_h\|_{0,\Omega}^2 + \nu |\mathbf{v}_h|_{1,\Omega}^2) + \left[2(1+\delta)^2+1\right](\sigma+\nu)\|q_h\|_{0,\Omega}^2 + \sum_{K\in\zeta_h} \alpha_K \|\kappa_h \nabla q_h\|_{0,K}^2 . \quad (27)$$

i.e.

$$\|(\mathbf{w}_h, r_h)\|^2 \leqslant \left[2(1+\delta)^2+1\right]\|(\mathbf{v}_h, q_h)\|^2 \quad (28)$$

Thus, (26) and (28) yield the required stability result

$$\sup_{\substack{(\mathbf{w}_h, r_h)\in V_h\times Q_h \\ (\mathbf{w}_h,, r_h)\neq 0}} \frac{A((\mathbf{v}_h, q_h);(\mathbf{w}_h, r_h)) + S(q_h; r_h)}{\|(\mathbf{w}_h, r_h)\|} \geq \beta \|(\mathbf{v}_h, q_h)\| . \quad (29)$$

■

Remark 5 *For Stokes flow ($\sigma \to 0$), $\alpha_K = h^2$ has proven to be a good choice for the stabilization parameter ([5]). In addition, the analysis given in ([2]) reveals that for the current problem $\alpha_K = \frac{\sigma h^2}{\nu}$ is a reasonable choice because it takes into account the effect of the zero term.*

Note that the above theorem guaranties unique solvability of the stabilized discrete problem (11). However, unlike the residual-based stabilization schemes ([19], [16]), here, we do not have Galerkin orthogonality. As a consequence we need to estimate the consistency error.

Lemma 6 *Assume that the fluctuation operator κ_h satisfies the approximation property (19). Let $(\mathbf{u}, p) \in \mathbf{V} \times (Q \cap H^l(\Omega))$, $0 \leqslant l \leqslant k$, be the solution of the generalized Stokes problem (3) and $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ the solution of the stabilized problem (11). Then, the consistency error can be estimated by*

$$A((\mathbf{u}-\mathbf{u}_h, p-p_h);(\mathbf{v}_h, q_h)) \leqslant C \left(\sum_{K\in\zeta_h} \alpha_K h_K^{2l-2} |p|_{l,K}^2\right)^{\frac{1}{2}} \|(\mathbf{v}_h, q_h)\|$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$.

Proof. Subtracting (3) from (11) we obtain

$$A((\mathbf{u}_h, p_h);(\mathbf{v}_h, q_h)) + S(p_h; q_h) - A((\mathbf{u}, p);(\mathbf{v}_h, q_h)) = 0 \ , \quad (30)$$

which implies that

$$A((\mathbf{u}-\mathbf{u}_h, p-p_h)\,;(\mathbf{v}_h,q_h)) = S(p_h;q_h). \tag{31}$$

Using the approximation property (19) of the fluctuation operator κ_h we obtain

$$\|\kappa_h \nabla p_h\|_{0,M} \leqslant C h_M^{l-1} |\nabla p_h|_{l-1,M} \leqslant C h_M^{l-1} |p_h|_{l,M} \leqslant \widetilde{C} h_K^{l-1} |p_h|_{l,K}\,.$$

Hence,

$$\begin{aligned} S(p_h;q_h) &\leqslant \left(\sum_{K\in\zeta_h} \alpha_K \|\kappa_h \nabla p_h\|_{0,K}^2\right)^{\frac{1}{2}} \left(\sum_{K\in\zeta_h} \alpha_K \|\kappa_h \nabla q_h\|_{0,K}^2\right)^{\frac{1}{2}} \\ &\leqslant \widetilde{C} \left(\sum_{K\in\zeta_h} \alpha_K h_K^{2l-2} |p_h|_{l,K}^2\right)^{\frac{1}{2}} \|(\mathbf{v}_h,q_h)\|\,. \end{aligned} \tag{32}$$

from which the result of the Lemma follows. ■

3.2 Error Analysis

As a consequence of the above stability and consistency results we obtain the following error estimate.

Theorem 7 *Assume that the solution $(\mathbf{u}, p)$ of (3) belongs to $\mathbf{V}\cap(H^{s+1}(\Omega))^2\times(Q\cap H^l(\Omega))$, $1\le s,l\le k$. Then, the following error estimate holds*

$$\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega} + \|p-p_h\|_{0,\Omega} \le C(h^s \|\mathbf{u}\|_{s+1,\Omega} + h^l \|p\|_{l,\Omega}).$$

Where, C is a positive constant independent of h.
Proof. *Let $\widetilde{\mathbf{u}}_h = j_h\mathbf{u}$ and $\widetilde{p}_h = i_h p$ be the interpolants of the velocity and pressure, respectively. Then, Theorem 4 implies the existence of $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$ such that*

$$\|(\mathbf{v}_h, q_h)\| \leqslant C \tag{33}$$

with

$$\|\widetilde{\mathbf{u}}_h - \mathbf{u}_h\|_{1,\Omega} + \|\widetilde{p}_h - p_h\|_{0,\Omega} \leqslant \frac{3}{\min\left\{\sigma^{\frac{1}{2}}, \nu^{\frac{1}{2}}\right\}} \|(\widetilde{\mathbf{u}}_h - \mathbf{u}_h, \widetilde{p}_h - p_h)\|$$

with the right hand side satisfying

$$\begin{aligned}\|(\widetilde{\mathbf{u}}_h - \mathbf{u}_h, \widetilde{p}_h - p_h)\| &\leqslant \frac{1}{\widetilde{\beta}} \frac{A((\widetilde{\mathbf{u}}_h - \mathbf{u}_h, \widetilde{p}_h - p_h) ; (\mathbf{v}_h, q_h)) + S(\widetilde{p}_h - p_h; q_h)}{\|(\mathbf{v}_h, q_h)\|} \\ &\leqslant \frac{1}{\widetilde{\beta}} \frac{A((\widetilde{\mathbf{u}}_h - \mathbf{u}, \widetilde{p}_h - p) ; (\mathbf{v}_h, q_h)) + S(\widetilde{p}_h - p; q_h)}{\|(\mathbf{v}_h, q_h)\|} \\ &+ \frac{1}{\widetilde{\beta}} \frac{A((\mathbf{u} - \mathbf{u}_h, p - p_h) ; (\mathbf{v}_h, q_h)) + S(p - p_h; q_h)}{\|(\mathbf{v}_h, q_h)\|}.\end{aligned} \tag{34}$$

Consequently, the consistency estimate of the method implies

$$\frac{A((\mathbf{u} - \mathbf{u}_h, p - p_h) ; (\mathbf{v}_h, q_h)) + S(p - p_h; q_h)}{\|(\mathbf{v}_h, q_h)\|} \leqslant Ch^l \|p\|_{l,\Omega} . \tag{35}$$

The Galerkin terms of $A((\widetilde{\mathbf{u}}_h - \mathbf{u}, \widetilde{p}_h - p) ; (\mathbf{v}_h, q_h)) + S(\widetilde{p}_h - p; q_h)$ *can be estimated using the approximation properties of* j_h *and* i_h*. Hence, we get*

$$\begin{aligned}\sigma(\widetilde{\mathbf{u}}_h - \mathbf{u}, \mathbf{v}_h) &\leqslant \sigma \|\widetilde{\mathbf{u}}_h - \mathbf{u}\|_{0,\Omega} \|\mathbf{v}_h\|_{0,\Omega} \leqslant C\sigma h^{s+1} |\mathbf{u}|_{s+1,\Omega} \|(\mathbf{v}_h, q_h)\| , \\ \nu(\nabla (\widetilde{\mathbf{u}}_h - \mathbf{u}) , \nabla \mathbf{v}_h) &\leqslant \nu |\widetilde{\mathbf{u}}_h - \mathbf{u}|_{1,\Omega} |\mathbf{v}_h|_{1,\Omega} \leqslant C\nu h^s |\mathbf{u}|_{s+1,\Omega} \|(\mathbf{v}_h, q_h)\| , \\ |(p - \widetilde{p}_h, \nabla \cdot \mathbf{v}_h)| &\leqslant C \|p - \widetilde{p}_h\|_{0,\Omega} |\mathbf{v}_h|_{1,\Omega} \leqslant Ch^l |p|_{l,\Omega} \|(\mathbf{v}_h, q_h)\| .\end{aligned} \tag{36}$$

The fourth Galerkin term is estimated by applying the orthogonality property of j_h*. Then, using* $\alpha_K = Ch_K^2$ *we get*

$$\begin{aligned}|(\nabla \cdot (\widetilde{\mathbf{u}}_h - \mathbf{u}), q_h)| &= |(\widetilde{\mathbf{u}}_h - \mathbf{u}, \nabla q_h)| = |(\widetilde{\mathbf{u}}_h - \mathbf{u}, \kappa_h \nabla q_h)| \\ &\leqslant \left(\sum_{K \in \zeta_h} \alpha_K^{-1} \|\widetilde{\mathbf{u}}_h - \mathbf{u}\|_{0,K}^2 \right)^{\frac{1}{2}} \left(\sum_{K \in \zeta_h} \alpha_K \|\kappa_h \nabla q_h\|_{0,K}^2 \right)^{\frac{1}{2}} \\ &\leqslant C \left(\sum_{K \in \zeta_h} \frac{h_K^2}{\alpha_K} h_K^{2s} \|\mathbf{u}\|_{s+1,K}^2 \right)^{\frac{1}{2}} \left(\sum_{K \in \zeta_h} \alpha_K \|\kappa_h \nabla q_h\|_{0,K}^2 \right)^{\frac{1}{2}}\end{aligned}$$

i.e.

$$|(\nabla \cdot (\widetilde{\mathbf{u}}_h - \mathbf{u}), q_h)| \leqslant Ch_K^s \|\mathbf{u}\|_{s+1,K} \|(\mathbf{v}_h, q_h)\| . \tag{37}$$

The stability term is estimated using the L_2-*stability of the fluctuation operator* κ_h*, the approximation properties of* i_h *and* $\alpha_K = Ch_K^2$*, hence we* *obtain*

$$
\begin{aligned}
S(\widetilde{p}_h - p; q_h) &= \sum_{K\in\zeta_h} \alpha_K(\kappa_h\nabla(\widetilde{p}_h - p), \kappa_h\nabla q_h) \\
&\leqslant \left(\sum_{K\in\zeta_h} \alpha_K \left\|\kappa_h\nabla(\widetilde{p}_h - p)\right\|_{0,K}^2\right)^{\frac{1}{2}} \left(\sum_{K\in\zeta_h} \alpha_K \left\|\kappa_h\nabla q_h\right\|_{0,K}^2\right)^{\frac{1}{2}} \\
&\leqslant C_1\left(\sum_{K\in\zeta_h} C_2 h_K^2\, h_K^{2l-2} \left\|p\right\|_{l,w(K)}^2\right)^{\frac{1}{2}} \left\|(\mathbf{v}_h, q_h)\right\|
\end{aligned}
$$

i.e.

$$S(\widetilde{p}_h - p; q_h) \leqslant C h_K^l \left\|p\right\|_{l,\Omega} \left\|(\mathbf{v}_h, q_h)\right\|. \tag{38}$$

Thus, using (35), (36), (37), and (38) we obtain the required error estimate

$$\left\|\mathbf{u} - \mathbf{u}_h\right\|_{1,\Omega} + \left\|p - p_h\right\|_{0,\Omega} \leq C(h^s \left\|\mathbf{u}\right\|_{s+1,\Omega} + h^l \left\|p\right\|_{l,\Omega}).$$

■

Remark 8 *We note that because of the compatibility of the* $Q_h^k/P_{2h}^{k-1,disc}$ *approximation ([9]) the stability of (11) and the above error estimates hold also for such approximation.*

3.3 Computational aspects

The discretization of (5) leads to the linear system

$$\begin{bmatrix} A & B^T & 0 \\ B & S_1 & S_2^T \\ 0 & S_2 & S_3 \end{bmatrix} \begin{bmatrix} U \\ P \\ \widetilde{P} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ 0 \end{bmatrix} \tag{39}$$

Where, U, P, and $\widetilde{P}$ denote the vectors containing the nodal values of velocity, pressure and pressure-gradient, respectively. The matrices B and B^T denote the divergence and gradient matrices, where the rows and columns associated to the prescribed velocity values have been omitted. The matrices S_1 and S_3 denote the pressure Laplacian and scaled mass matrix, while S_2 and S_2^T represent the pressure-gradient projection divergence and gradient matrices. Here, the matrices S_1, S_2 and S_3 depend on the values of the mesh parameter $\alpha = \{\alpha_K : K \in \zeta_h; \alpha_K > 0\}$. The vectors F_1 and F_2 represent the discretization of the right hand side terms and eventual contributions from inhomogeneous boundary conditions.

Remark 9 *Since the functions of* $\mathbf{Y}_{2h}$ *are discontinuous on* Ω*, the formulation given in (5) leads to a decoupled system of equations for which the pressure gradient unknowns can be eliminated locally.*

In fact, integration of (5) on a patch $e \in \zeta_{2h}$ leads to the local algebraic linear system

$$\begin{bmatrix} A_1^e & 0 & B_1^{eT} & 0 \\ 0 & A_2^e & B_2^{eT} & 0 \\ B_1^e & B_2^e & S_1^e & S_2^{eT} \\ 0 & 0 & S_2^e & S_3^e \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ P \\ \widetilde{P} \end{bmatrix} = \begin{bmatrix} F_1^e \\ F_2^e \\ 0 \\ 0 \end{bmatrix} \tag{40}$$

where, U_1 and U_2 denote vectors containing the the first component and the second component nodal values of the velocity field, respectively. As above, P and $\widetilde{P}$ denote the corresponding pressure and pressure gradient nodal values, respectively.
Let $\{\varphi_i\}_{i=1}^{N_u^e}$, $\{\psi_i\}_{i=1}^{N_p^e}$ and $\left\{\widetilde{\psi}_i\right\}_{i=1}^{N_{\widetilde{p}}^e}$ be the local basis functions on the element $e \in \zeta_{2h}$ for $\mathbf{V}_h$, Q_h, and $\mathbf{S}_h$, respectively. The matrices A_1^e, A_2^e, B_1^e, B_2^e, S_1^e, S_2^e, and S_3^e are given by

$$(A_1^e)_{ij} = (A_2^e)_{ij} = \int_e (\sigma\varphi_i\varphi_j + \nu\nabla\varphi_i.\nabla\varphi_j)dx, \quad (B_1^e)_{ij} = -\int_e \psi_i \frac{\partial\varphi_j}{\partial x}dx,$$
$$(B_2^e)_{ij} = -\int_e \psi_i \frac{\partial\varphi_j}{\partial y}dx, \quad (S_1^e)_{ij} = \sum_{K\in e}\alpha_K \int_e \nabla\psi_i.\nabla\psi_j dx,$$
$$(S_2^e)_{ij} = -\sum_{K\in e}\alpha_K \int_e \nabla\psi_i.\widetilde{\psi}_j dx, \text{ and } (S_3^e)_{ij} = \sum_{K\in e}\alpha_K \int_e \widetilde{\psi}_i.\widetilde{\psi}_j dx.$$

Because the pressure gradient terms arise only locally, elimination (rather like static condensation) can be used on the $2h$ macroelement to yield the reduced local system.

$$\begin{bmatrix} A_1^e & 0 & B_1^{eT} \\ 0 & A_2^e & B_2^{eT} \\ B_1^e & B_2^e & S_1^e - S_2^{eT} S_3^{e-1} S_2^e \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ P \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ 0 \end{bmatrix}. \tag{41}$$

Assembly of the local matrices (41) leads to a global system of the form

$$\begin{bmatrix} A & B^T \\ B & \widehat{S} \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}. \tag{42}$$

Where $\widehat{S}$ is assembled from the matrices $S_1^e - S_2^{eT} S_3^{e-1} S_2^e$.
The work $\widehat{S}$ now plays a similar role in consideration of linear algebra solution algorithms to that associated with basic least squares terms in reduced-based stabilization: see [15] Section 5.5.2.

4 Numerical Results

In this section, numerical results for two-dimensional generalized Stokes flows are presented. The performance of the $Q_h^1 - Q_h^1$ velocity-pressure approximation is assessed for $\alpha_K = \frac{\sigma h^2}{\nu}$. The velocity and pressure norms displayed confirm the convergence rates predicted by Theorem 3. For both problems an SOR preconditioned MINRES code is used to solve the algebraic linear system obtained by elimination of the pressure-gradient unknowns. More efficient preconditioned iterative linear solvers will be the subject of future work.

4.1 Test 1 Problem

The first problem consists in solving a generalized Stokes problem in the unit square $[0,1] \times [0,1]$, with exact solution:

$$\mathbf{u}(x,y) = (u_x, u_y)^T \ ; \ p(x,y) = x - x^2$$
$$\text{with} \ \ u_x = 2x^2(1-x)^2 y(1-y)(1-2y) \ , \ \ u_y = -2x(1-x)(1-2x)y^2(1-y)^2.$$

Numerical results obtained for $\sigma = 1$ and $\nu = 1, 10^{-2}, 10^{-3}$, and 10^{-4}, respectively, are displayed in figures 1-2. These results indicate that the error norms $\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$ and $\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}$ converge at the predicted rates, while $\|p - p_h\|_{0,\Omega}$ seems to converge one degree higher than predicted. Superconvergence results were also reported by [13] for L_2 and H_1 norms using both triangular (P^1 and P^2) and quadrilateral (Q^1 and Q^2) elements for the global pressure gradient projection method. This behaviour is believed to be due to the symmetry of the problem. In Figure 3 we have also displayed the pressure contours. It is observed that for $\sigma = 1, \nu = 1$ and $\nu = 10^{-4}$ there no oscillations in the pressure solution and we get the expected vertical isobars.

4.2 Lid-driven cavity flow

Next, we address the lid-driven cavity problem, with domain Ω as before, $\mathbf{f} = \mathbf{0}$. Our aim here is to assess the performance of the method using a graded mesh near $x = 0$, $x = 1$, $y = 0$, and $y = 1$. We impose a leaky boundary condition, that is for $x, y \in [0, 1] : u_x(0, y) = u_x(1, y) = 0$, $u_x(x, 0) = 0$, and $u_x(x, 1) = 1$. Numerical results are obtained for $\nu = 1$ and $\nu = 10^{-4}$, both using $\sigma = 1$. Streamlines and elevations for the pressure field are displayed in figures 4-5. We observe that there are no oscillations for the pressure for both cases, which shows that the method treats well the inf-sup condition and the boundary layer for the reaction dominated regime. Further, the streamlines of figure 4 indicate that for $\sigma = 1$ and $\nu = 1$ the flow is essentially a Stokes-like flow with small counter-rotating recirculations appearing at the bottom two corners which is in agreement with similar results found in the literature (see, for example [15]). While for $\sigma = 1$ and $\nu = 10^{-4}$ we observe that a second circulation starts appearing at the bottom of the cavity.

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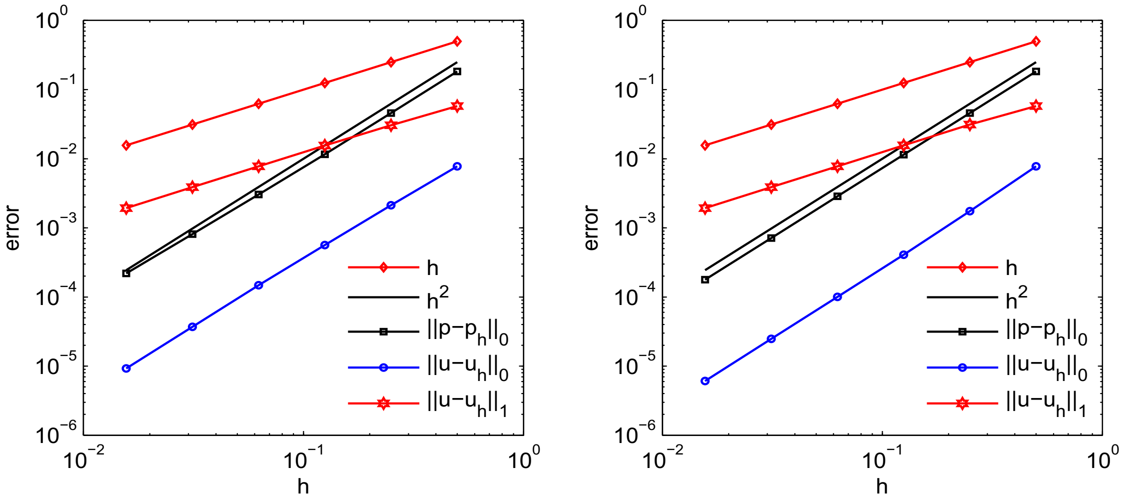

Figure 1: Rates of convergence for $\sigma = 1$, $\nu = 1$ (left), and $\nu = 10^{-2}$ (right).

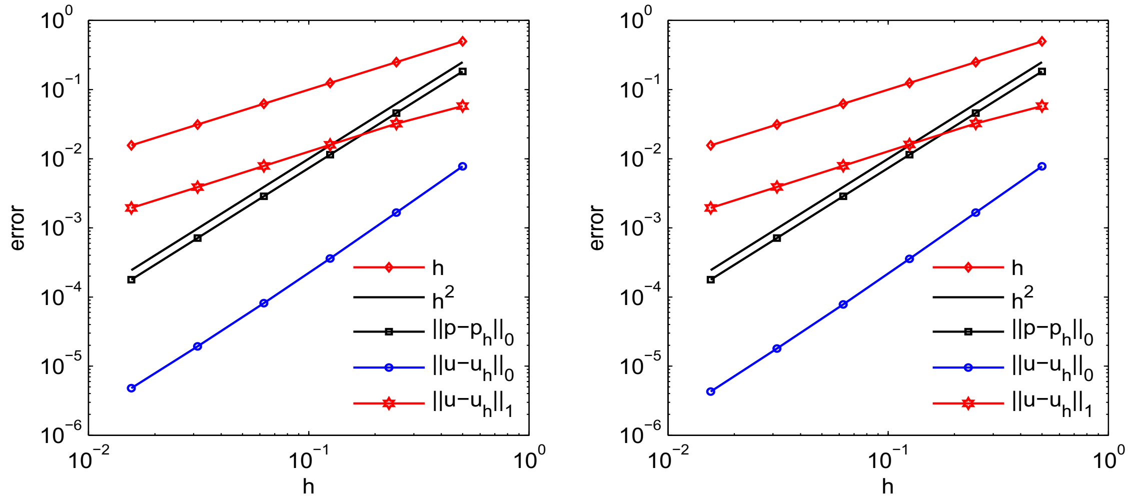

Figure 2: Rates of convergence for $\sigma = 1$, $\nu = 10^{-3}$ (left), and $\nu = 10^{-4}$ (right).

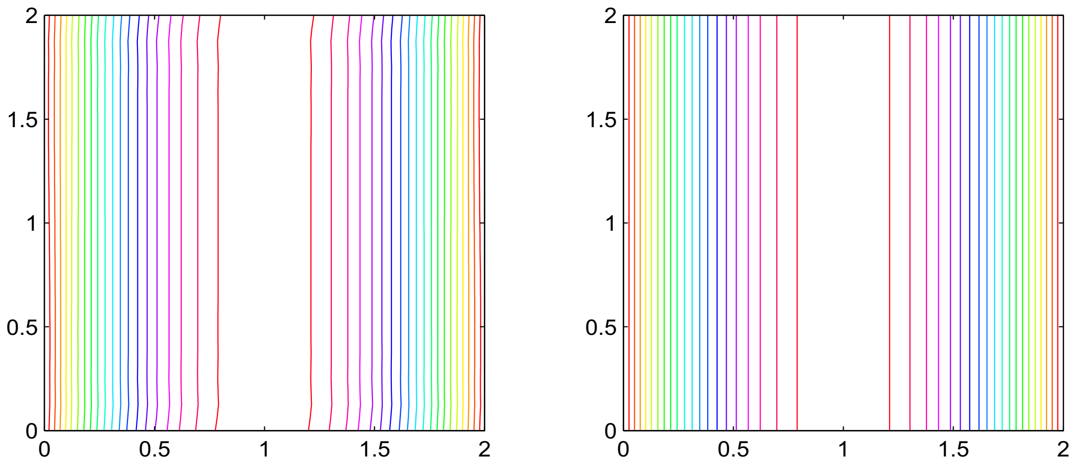

Figure 3: Pressure contours, for $\sigma = 1$, $\nu = 1$ (left), and $\nu = 10^{-4}$ (right).

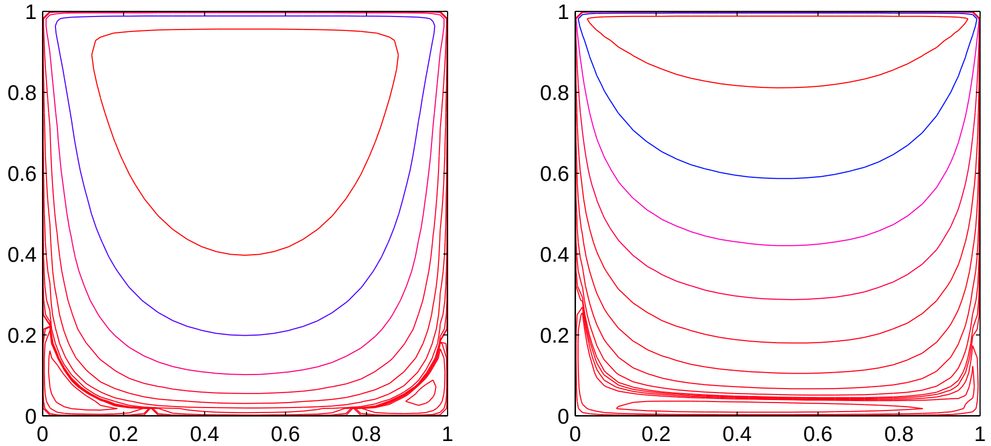

Figure 4: Exponential distributed streamline plot for $\sigma = 1$, $\nu = 1$ (left), and $\nu = 10^{\;4}$ (right).

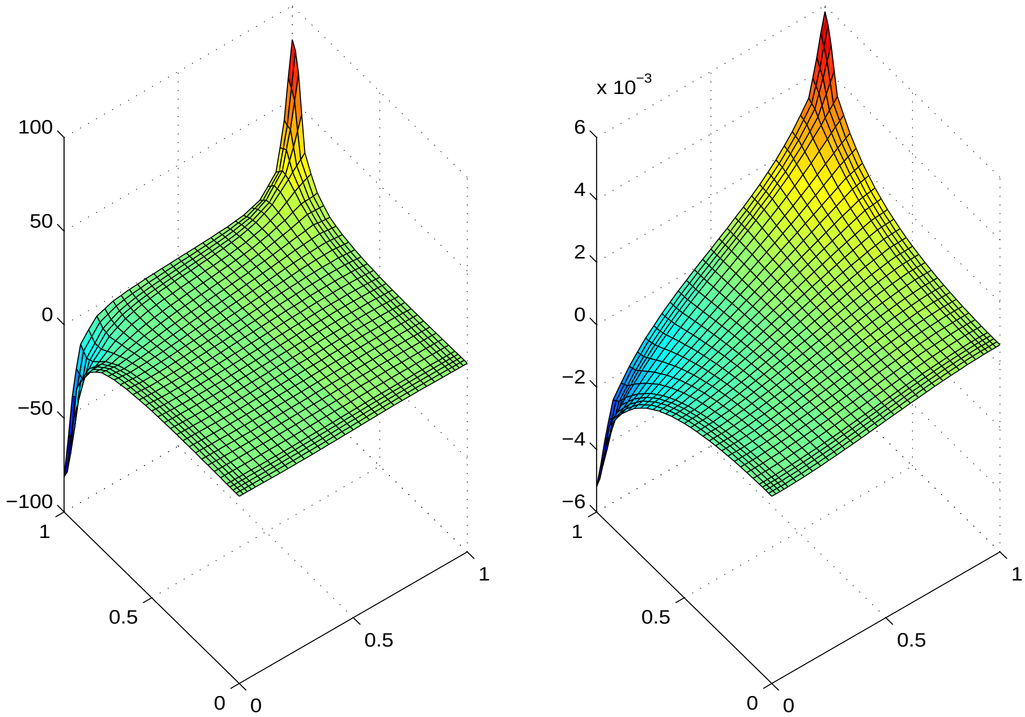

Figure 5: Elevation of the pressure field for $\sigma = 1$, $\nu = 1$ (left), and $\nu = 10^{-4}$ (right).