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Ockham's razor and reasoning about information flow

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Abstract What is the minimal algebraic structure to reason about information flow? Do we really need the full power of Boolean algebras with co-closure and de Morgan dual operators? How much can we weaken and still be able to reason about multi-agent scenarios in a tidy compositional way? This paper provides some answers.

Keywords Algebraic modal logic · Galois adjoints · Reasoning about knowledge and update

1 Introduction

Systems of modal logic have been applied to disciplines of science and humanities for modeling and reasoning about concepts such as provability, time, necessity and possibility, knowledge and belief. Each such system has its own set of axioms, specifically chosen for the domain of application it models. New application domains motivate introduction of new axioms who may be stronger or weaker than their original peers. Change of axioms is also motivated by development of new mathematical methods which lead to more refined and efficient versions of the existing axioms. Introduction of different axioms and logical systems initiates practical and conceptual discussions on minimality issues, for instance whether or not the set of axioms in use is the minimal such set for the domain it promises to model, or what are the foundational structures

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we want to model in a domain and are those reflected in the logical system we use? In this paper, we aim to further elaborate on these issues for the case of Epistemic Logic.

1.1 New application domains

Epistemic logics have been used by philosophers to reason about knowledge and belief, e.g. Hintikka (1962) argues for the modal logic $S5$ where we have axioms that say our knowledge is truthful and both positively and negatively introspective. Epistemic logics have also been used by computer scientists to reason about knowledge of agents in multi-agent systems (Fagin et al. 1995; Meyer and van der Hoek 1995). This has led to numerous different variations; a dramatic one considers logics with non-monotonic modalities to provide a solution to the problem of logical omniscience. The need to moreover model the interactions among the agents and to further reason about the knowledge they acquire as a result of these interactions, has led to the development of *logics of information flow* (Fagin et al. 1995; Gerbrandy 1999; Plaza 1989; van Benthem 1989; van Der Hoek and Wooldridge 2002). Very roughly put, these logics are obtained by enriching the Epistemic logics with temporal operations. They have been extended to two sorted *dynamic epistemic* logics which also model dishonest interactions of cheating and lying in Baltag et al. (1999); Baltag and Moss (2004). In this logic, the propositional sort is an $S5$ Epistemic Logic with a new 'possibly wrong' belief modality, which is only conjunction preserving, and the dynamic sort is a PDL-style Dynamic logic, which has linear operations of sequential composition and choice on actions and the modality induced by it is not a usual closure type operator.

1.2 New mathematical models

Algebraic methods have been used by mathematical logicians to prove meta-theorems such as decidability and completeness for modal logics. For instance, Jonsson and Tarski in (1951; 1952) used Boolean Algebras with operators to prove decidability and completeness of $S2$ and $S4$. Through the work of Eilenberg and Mac Lane, algebraic systems have been generalized to categorical structures whose operational and compositional nature lend themselves to easier proof theoretic implementations of logical systems. The compositionality of the categorical approach provides more refined ways of defining operators on the logical systems. For example, the usual co-closure modalities of modal logics $S4$ and $S5$ can seen as a decomposition of a pair of adjoint maps ($f \dashv g$) whose composition $g \circ f$ will provide us with an operation which is a co-closure. The f and g maps themselves can be seen as 'weaker' modalities of the logic, in the sense that they obey less truth axioms, for example they need not in general be idempotent (axiom 4) or reflexive (axiom T). However, f is disjunction preserving and g is conjunction preserving and they relate to each other via the rule of adjunction. These equips f and g with a tidy mathematical axiomatics for reasoning about more fine-grained aspect of situations, namely those in which the main modality need not be introspective and truthful. The generality of the categorical approach enables us to systematically weaken the base propositions of modal logics and for instance work in a Heyting algebra where the negation operator is weaker than the one in the original Boolean algebras of Jonsson and Tarski, or simply in a lattice where in general no negation operator is present. We have taken advantage of these bonuses and have developed an algebraic/categorical semantics to reason about information and mis-information flow (Baltag et al. 2007; Sadrzadeh 2005). In this algebra, the propositional logic is a complete lattice and the epistemic and dynamic modalities are respectively formed from conjunction and disjunction preserving operators that are adjoints to one another.

The possibility of introducing different modal logics and different ways of defining modalities makes us wonder about, and thus bring into question, the minimal set of axioms that makes each such modality and logic a necessity for the domain they try to model. Our interest lies in the application domain of reasoning about information (and mis-information) flow where one can ask: what is a parsimonious logic of information flow that can model interactive scenarios of multi-agent systems? In other words, what is a minimal set of modal and propositional axioms that enable us to reason about knowledge and interaction of agents in these systems? Or more profoundly, what would be the philosophical implications of such a minimal logic?, what kind of concepts can the logic based on these minimal axioms reason about?, and in short, what are the *foundational structures* to reason about information flow? This paper tries to provide some answers.

We start by applying Ockham's razor to complete Boolean modal algebras. We observe that one can define weaker modal operators on these algebras, those that are only disjunction or conjunction preserving and as a result have left and right adjoints respectively. In the presence of a Boolean negation, another nice connection shows up: the De Morgan duals of these weaker modalities also become adjoints to each other. Experience shows that most applications of Epistemic logic do not need all four of these modalities and only use two of them. The traditional approaches rely on negation and work on a De Morgan dual pair of these modalities. We propose a new thesis and propose to work with an adjoint pair instead. Thus our base propositional algebra need not have a Boolean negation: it can be a Heyting Algebra or it can have no negation at all and just be a complete lattice. Interestingly enough, fixed points can be defined for our adjoint pair of modalities by closing them under composition and disjunction and we show that the fixed points of adjoint operators are also adjoints to each other.

On the application side, we provide new readings for our modalities: as *appearance* and *information* of agents about the reality. These are weaker than the usual *knowledge* and *belief* interpretations of Epistemic logics. But, and as we shall demonstrate, we can ask for extra conditions on them to re-gain the traditional modalities of systems K, $S4$ and $S5$. However, it would not be very straightforward to obtain our modalities from their stronger peers. We then move towards the dynamic applications and extend our minimal logic with dynamic modalities and show, by means of examples, how swiftly we can prove more fine-gained and more interesting epistemic properties of multi-agent scenarios by means of unfolding the adjunctions. Proving these properties enables us to reason about how the *information* of agents changes (not necessarily truthfully) as a result of their communication and based on their *appearances*. To model the interactions, we first add a pair of adjoint modalities to model the temporal *previous* and *next* states of the system. Then, in a second incremental step, we index these modalities with labels. The labels stand for actions of multi-agent scenarios and enable us to model what specific actions evolved the system into its next state. Finally, we observe that our index set, that is the set of interactions, is more than just a plain set and admits both a monoid and a sup-lattice structure, In short, it can be seen as a *quantale* with composition and non-deterministic choice of actions. Based on this observation, we end by proving how *epistemic systems* of Baltag et al. (2007); Sadrzadeh (2005) are obtained from our incrementally developed *real action epistemic algebras* by restricting their agents to the *optimistically paranoid* ones.

This paper can also be seen as a deductive take on the algebraic semantics of dynamic epistemic logic as presented in Baltag et al. (2007); Sadrzadeh (2005). We demonstrate how the full structure is put together operation by operation, and what new aspects of application are modeled by each operation. All along, we follow the same parsimonious strategy for both epistemic and action modalities, our strategy shows that the reliance of traditional modal and epistemic logics on negation (classical and intuitionistic) can be waived by using adjoint operators instead. The theoretical study of this minimal modal algebra, its free construction and equational theory constitutes future work.

2 Ockham's razor and reasoning about information

Intuitively, a Boolean algebra can be seen as a propositional logic in the following way

- Elements of the algebra $b_1, b_1 \in \mathcal{B}$ are logical propositions,
- their join $b_1 \vee b_2$ is the logical disjunction,
- their meet $b_1 \wedge b_2$ is the logical conjunction,
- and the partial order between them $b_1 \leq b_2$ is the logical entailment.

The definition of a complete Boolean algebra (Davey and Priestley 1990) is as follows

Definition 2.1 A *complete Boolean algebra* $\mathcal{B} = (B, \bigvee, \neg)$ is a distributive complete lattice $(B, \bigvee)$ with a negation operation, defined by the following axioms

$$b \wedge \neg b = \bot, \qquad b \vee \neg b = \top$$

A complete Boolean algebra has all joins $\bigvee_i b_i$, in particular the empty one $\bigvee \emptyset = \bot$, as a result it also has all meets $\bigwedge_i b_i$, in particular the empty one $\bigwedge \emptyset = \top$. The negation is involutive, that is $\neg\neg b = b$.

A complete Boolean algebra[1] is endowed with operators that satisfy certain properties, to obtain an algebraic [classical] modal logic. In this setting, unary operators will stand for modalities of the logic. We define a *classical modal algebra* as follows

[1] For simplicity of presentation we work with complete lattices, so that for every join preserving operator there exists a right adjoint. An alternative would be to put aside the completeness criteria and instead ask for existence of adjoints for each join (meet) preserving operator.

Definition 2.2 A *classical modal algebra* $\mathcal{B} = (B, \bigvee, \neg, f)$ is a complete Boolean algebra $(B, \bigvee, \neg)$ endowed with a join preserving operator $f : B \to B$, that is

$$f\left(\bigvee_i b_i\right) = \bigvee_i f(b_i), \quad \text{in particular } f(\bot) = \bot$$

So far we have one modality, that is the f operator, which preserves the disjunctions of the logic. But recall that in every classical modal algebra $(B, \bigvee, \neg, f)$, the join preserving operator $f : B \to B$ has a de Morgan dual $g : B \to B$ defined as

$$g(b) := \neg f(\neg b)$$

satisfying

$$g\left(\bigwedge_i b_i\right) = \bigwedge_i g(b_i), \quad \text{in particular } g(\top) = \top$$

So we obtain another modality, that is the g operator, which preserves the conjunctions of the logic. The categorical methods remind us that in a classical modal algebra $(B, \bigvee, \neg, f)$, a join preserving operator $f : B \to B$ has a meet preserving Galois right adjoint (Joyal and Tierney 1984), denoted by $f \dashv f^*$ defined as

$$f^*(b) := \bigvee \left\{ b' \in B \mid f(b') \leq b \right\}$$

Moreover and in a similar way, the de Morgan dual of f, abbreviated as g, has a join preserving Galois left adjoint, denoted by $g^* \dashv g$ and defined as

$$g^*(b) := \bigwedge \left\{ b' \in B \mid b \leq g(b') \right\}$$

So we have obtained four modalities: two De Morgan duals and two adjoints. The adjoint operators $f \dashv f^*$ and $g^* \dashv g$ satisfy the following rules

$$f(b) \leq b' \quad \text{iff} \quad b \leq f^*(b'), \qquad g^*(b) \leq b' \quad \text{iff} \quad b \leq g(b')$$

As a consequences, the following hold for the composition of adjoints

$$f(f^*(b)) \leq b, \quad b \leq f^*(f(b)), \quad g^*(g(b)) \leq b, \quad b \leq g(g^*(b))$$

There is an interesting cross-dependency between the pair of our De Morgan dual modalities (f, g) and their adjoints f^* and g^*, namely that the adjoints to the De Morgan duals are De Morgan duals of one another. In other words, in a classical modal algebra the de Morgan duality between f and g lifts to f^* and g^* as shown below

Proposition 2.3 *In a classical modal algebra* $(B, \bigvee, \neg, f)$ *the following is true*

$$f^*(b) = \neg g^*(\neg b)$$

where g is the de Morgan dual of f*, and we have* $f \dashv f^*$ *and* $g^* \dashv g$.

Proof We show $f^*(b) \leq \neg g^*(\neg b)$ and $\neg g^*(\neg b) \leq f^*(b)$. For the first inequality, start from the consequence of adjunction $f(f^*(b)) \leq b$, by anti-tonicity of negation it follows that $\neg b \leq \neg f(f^*(b))$, by involution of negation this is equivalent to $\neg b \leq \neg f(\neg\neg f^*(b))$, by de Morgan duality between f and g this is equivalent to $\neg b \leq g(\neg f^*(b))$, by the adjunction rule between g and g^* this is iff $g^*(\neg b) \leq \neg f^*(b)$, which implies $f^*(b) \leq \neg g^*(\neg b)$ by anti-tonicity of negation. Proof of the other inequality is similar. □

Thus in a complete Boolean algebra $\mathcal{B} = (B, \bigvee, \neg)$ asking for $f : B \to B$ immediately provides us with 3 other maps f^*, g, g^*, which form two pairs of adjoint operators and two pairs of de Morgan dual operators:

$$\Big(f \dashv f^*, \quad g^* \dashv g\Big), \qquad \Big(g(-) := \neg f(\neg -), \quad g^*(-) := \neg f^*(\neg -)\Big)$$

If we weaken the base algebra from a Boolean algebra BA to a Heyting algebra HA and thus obtain an intuitionistic modal algebra, a join preserving f operator gives rise to only one other operator. This is because the Intuitionistic negation defined as $\neg l := l \to \bot$, for $\to$ the left adjoint to $\wedge$ has weaker properties in connecting meets and joins, as a result g will not be meet preserving anymore. Thus by weakening the base algebra we also obtain weaker connections between the operators on the base. If we continue this weakening and reduce the base algebra to a distributive complete lattice DL, there is no negation operator present and the f map only gives rise to an f^*, the same is true in a complete lattice L. The relation between the base algebra and the operators defined on it is depicted in the table below

Negation	**De Morgan dual modalities**	**Adjoint modalities**
`Classical negation`	(BA, f, g)	$(BA, f \dashv f^*)$
	(BA, f^*, g^*)	$(BA, g^* \dashv g)$
`Intuitionistic negation`	(HA, f, g)	$(HA, f \dashv f^*)$
	–	
`No negation`	–	$(DL, f \dashv f^*)$
	–	–
`No negation`	–	$(L, f \dashv f^*)$
	–	–

The modal logic based on the algebra of the last line of the table is weaker than the modal logics based on the algebras of its above lines, in the sense that it asks for the least set of axioms from its base algebra and operators. In this case, the base algebra is a complete lattice with only one join preserving operator. We focus on this algebra as our minimal modal algebra. More formally, we have

Definition 2.4 An *adjoint modal algebra* denoted by $(L, f \dashv f^*)$ is a complete lattice L endowed with a join preserving map $f : L \to L$.

The operators f and f^* and the adjunction $f \dashv f^*$ between them can be used to define other pairs of adjoint maps on the base algebra. For example, closing both f and f^* under composition and disjunction provides us with a pair of interesting operators. These closed maps can be seen as special *fixed point* operators, which will stay adjoint to each other, via the following result

Proposition 2.5 *In any adjoint modal algebra* $(L, f \dashv f^*)$*, the following are true*

- $f^i \dashv f^{*i}, \quad \forall i \in \mathbb{N}$, *where* $f^i = \underbrace{f \cdots f}_{i}$ *stands for i times self composition of* f.
- $\bigvee_{i=1} f^i \dashv \bigwedge_{i=1} f^{*i}$

Proof $f^i \dashv f^{*i}$ is equivalent to $f^i(l) \leq l'$ iff $l \leq f^{*i}(l')$, which follows by i times applying $f(l) \leq l'$ iff $l \leq f^*(l')$. Similarly, $\bigvee_{i=1} f^i \dashv \bigwedge_{i=1} f^{*i}$ is equivalent to $\left(\bigvee_{i=1} f^i\right)(l) \leq l'$ iff $l \leq \left(\bigwedge_{i=1} f^{*i}\right)(l')$, which follows from the definitions of arbitrary meets and joins applied to item one. □

Operators of the first item above are closed under composition and can be seen as a pair of adjoint fixed point operators. Operators of the second item above are moreover closed under disjunction and conjunction respectively and can be seen as a pair of adjoint least and greatest fixed points. One can make these reflexive by starting the range of i from 0.

We make our modal algebra more suited for epistemic applications by considering a family of join preserving operators, instead of just one, and thus obtain a multi-modal algebra, defined below

Definition 2.6 A *multi-agent adjoint modal algebra* (MAMA) denoted by $(L, f_A \dashv f_A^*)_{\mathcal{A}}$ is a complete lattice L endowed with a family of join preserving maps $\{f_A\}_{A \in \mathcal{A}} : L \to L$.

We refer to this algebra as an *epistemic algebra* and provide epistemic interpretations for its modalities: $f_A(l)$ is interpreted as 'appearance of proposition l to agent A'. That is

$f_A(l)$ is all the propositions that *appear* to agent A as possible or true when in reality l is true.

Here are some explanatory examples of our notion of *appearance*:

- If $f_A(l) = l$ then the appearance of agent A about reality is the reality itself, so A's appearance is totally compatible with reality.
- If $f_A(l) = \top$ then all the propositions of the logic appear as possible to agent A, in other words, he has no clue about what is going on in reality.

- If $l \leq f_A(l)$, for instance when $f_A(l) = l \vee l'$, then reality appears as possible to agent A, although he cannot be sure about it, since l' also appears equally possible to him.

In each case above, we can also talk about *information* of agent A, in the following lines

- If $f_A(l) = l$ then A's information about reality is the reality itself, so A is well informed or has truthful information.
- If $f_A(l) = \top$ then A has no information at all about reality.
- If $f_A(l) = l \vee l'$, then A's information about reality includes the reality, but is weaker than it.

Based on the above intuitions, we use the left adjoint $f_A^*(l)$ to define our notion of *information* and read it as 'information of agent A about proposition l', or more propositionally as follows

$f_A^*(l)$ is read as 'agent A is informed that proposition l holds'.

Now we can apply the adjunction rule

$$f_A(l) \leq l \quad \text{iff} \quad l \leq f_A^*(l')$$

to produce equivalent information formulae for each appearance case above:

- If $f_A(l) = l$ then $l \leq f_A^*(l)$, so l implies that A has truthful information about l.
- If $f_A(l) = \top$ then $l \leq f_A^*(\top)$, so l implies that A has no information about l.
- If $f_A(l) = l \vee l'$, then $l \leq f_A^*(l \vee l')$, so l implies that A is informed that either l or another proposition l' hold in reality.

For more examples on appearances consider the following comparisons

- If $f_A(l) \leq f_B(l)$ then agent B is more uncertain about l than agent A, since more propositions appear as possible to him. So we can say that A is more informed or has more information about proposition l than agent B. An example would be when $f_B(l) = l \vee l'$ where as $f_A(l) = l$, clearly $l \leq l \vee l'$ and $l \vee l'$ stands for two possibilities for agent B as opposed to the only one possibility, that is l, for agent A.
- If $f_A(l) \leq f_A(l')$, then agent A is more uncertain about l' than about l, thus he is more informed about l than about l'. An example would be when $f_A(l) = l$ and $f_A(l') = l \vee l'$, so whenever l is true in reality, A is informed that this is the case, but when l' is true in reality, his information does not tell anything useful to him, since he cannot distinguish between l and l', both appear to him as equivalently possible.

The above notions of *appearance* and *information* are based on weaker modalities than those of the usual Epistemic logics. They provide new readings for the modalities; readings that stand for new concepts that Epistemic logics did not account for before. However, in our weaker system, we can define the stronger epistemic notions of other logics. For instance, the knowledge modality of system K can now be described as 'truthful information' and defined by

$$K_A(l) := f_A^*(l) \wedge l$$

So we have

$K_A(l)$ is read as 'A has truthful information that l'.

If the appearance maps are weakly idempotent and decreasing (i.e. weak co-closures), then one obtains the knowledge of system $S4$.

Proposition 2.7 *In a MAMA $(L, f_A \dashv f_A^*)_{\mathcal{A}}$, if we have $f_A(l) \leq l$ and $f_A f_A(l) \leq f_A(l)$ then the following hold*

- $f_A^*(l) \leq f_A^* f_A^*(l)$
- $K_A(l) \leq K_A K_A(l)$ *and* $K_A(l) = l$, *for* $K_A(l) := f_A^*(l) \wedge l$

Proof For the first one, by the corollary of adjunction we have $f_A f_A^*(l) \leq l$, from this by weak idempotence of f_A and transitivity we have $f_A f_A f_A^*(l) \leq f_A f_A^*(l) \leq l$ and thus it follows that $f_A f_A f_A^*(l) \leq l$, which by adjunction is equivalent to $f_A^*(l) \leq f_A^* f_A^*(l)$. For the second one, we have to show $f_A^*(l) \wedge l \leq f_A^*(f_A^*(l) \wedge l) \wedge (f_A^*(l) \wedge l)$, which is equivalent to $f_A^*(l) \wedge l \leq f_A^*(f_A^*(l) \wedge l)$, that is $f_A^*(l) \wedge l \leq f_A^* f_A^*(l) \wedge f_A^*(l)$, which follows from item one and the adjunction equivalence of decreasing property of f_A, that is $l \leq f_A^*(l)$. The third one easily follows from $f_A(l) \leq l$, which is by adjunction equivalent to $l \leq f_A^*(l)$ and by definition of meet we obtain $l \wedge f_A^*(l) = l$, which is nothing but $K_A(l) = l$. □

When L is a complete Boolean algebra, belief is defined as the de Morgan dual of K_A, that is $B_A(l) := \neg K_A(\neg l)$. In this setting, knowledge of the system $S5$ is obtained by asking for the weak idempotence and decreasing of appearance maps.

Proposition 2.8 *In a MAMA $(L, f_A \dashv f_A^*)_{\mathcal{A}}$, if L is a complete Boolean algebra $(L, \bigvee, \neg)$ and we have $f_A(l) \leq l$ and $f_A f_A(l) \leq f_A(l)$ then it follows that*

- $\neg K_A \leq K_A(\neg K_A(l))$
- $K_A(l) = l$.

Proof Similar to the proof of proposition 2.7.

Similar to proposition 2.5, we define adjoint fixed points for our indexed modalities as below

Proposition 2.9 *In any MAMA $(L, f_A \dashv f_A^*)_{\mathcal{A}}$ for $\beta \subseteq \mathcal{A}$ the following are true*

- $f_\beta \dashv f_\beta^*$, *for* $f_\beta := \bigvee_{B \in \beta} f_B$ *and* $f_\beta^* := \bigwedge_{B \in \beta} f_B^*$.
- $f_\beta^i \dashv f_\beta^{*i}$
- $\bigvee_{i=1} f_\beta^i \dashv \bigwedge_{i=1} f_\beta^{*i}$

Proof For the first direction of the first one assume $f_\beta(l) \leq l'$, by definition of join it follows that $f_B(l) \leq l'$ for all $B \in \beta$, by adjunction this is iff $l \leq f_B^*(l')$ for all $B \in \beta$, by definition of meet it follows that $l \leq f_\beta^*(l')$. Proof of the other direction is similar. The second one follows from i times unfolding the first one, and the third one from the first two. □

These group maps have sensible interpretations in an epistemic context, for example $f_\beta(l)$ can be read as 'the appearance of l to all the agents in group β', similarly $f_\beta^*(l)$ can be read as 'the shared information of agents in β about l', or more propositionally as

$f_\beta^*(l)$ is read as 'all the agents in β are informed that l holds'.

The former contains the collection or the union of appearances of agents in β about the same proposition l, and can be read as 'accumulated appearance'. The latter contains the common part or the intersection of information of agents in β about the same proposition l and can be read as 'shared information'. Closing the *shared information* under composition and conjunction (the third item in proposition 2.9) provides us with the notion of *common information*, which is the infinite nested information of agents about one another's information:

$\bigwedge_{i=1} f_\beta^{*\,i}(l)$ is read as 'all the agents in group β are informed that l, and are also informed that everyone in the group is informed that l, and so on $\cdots$'.

The notion of 'common knowledge among the group β' in system K is obtained by starting the index i from 0 rather than 1, that is

$$CK_\beta := \bigwedge_{i=0} f_\beta^{*\,i} = l \wedge \bigwedge_{i=1} f_\beta^{*\,i}$$

In other words, common knowledge among agents in group β can be defined in terms of their common information as follows

Agents in β have common knowledge that l iff they have truthful common information that l.

Although, weaker than the knowledge and belief modalities of Epistemic logics, appearance and information modalities can also be used to model epistemic applications and to prove weaker properties about them. However, and as we will see in the next section, this weakness becomes a necessity while reasoning about mis-information.

Example 2.10 Consider the following coin toss scenario: in front of agents A and B, agent C throws a coin and covers it in his palm. We consider a MAMA containing propositions $H, T \in L$. Appearances are set according to uncertainty of agents

$$f_A(H) = f_A(T) = H \vee T$$

We show that the information A, B and C have is that the coin is either heads or tails, for example

$$H \leq f_A^*(H \vee T)$$
$$H \leq f_A^* \, f_B^*(H \vee T)$$
$$H \leq f_B^* \, f_A^* \, f_B^*(H \vee T)$$

Consider the second property, by the adjunction rule it holds iff we have $f_A(H) \leq f_B^*(H \vee T)$, by assumptions on f_A this is equivalent to $H \vee T \leq f_B^*(H \vee T)$. By the adjunction rule this holds iff we have $f_B(H \vee T) \leq H \vee T$, now since f_B is join preserving this is equivalent to $f_B(H) \vee f_B(T) \leq H \vee T$, which is, by assumptions on f_B, equivalent to $(H \vee T) \vee (H \vee T) \leq H \vee T$, which holds by the definition of $\vee$. The proofs of other cases are similar.

3 Ockham's razor and reasoning about flow of information

To reason about flow of information, we add another modality to our epistemic algebra: the action modality. This will enable us to prove more properties about scenarios: before we were able to prove that agents have some information, now we can show how their acquired this information, that is how their initial information got updated as a result of some communication action taking place among them. Epistemic algebras could only reason about information, the question is how to enrich them in a minimal way such that they can also reason about communication.

$$\underbrace{\text{Information} \qquad \text{Communication}}$$
$$\Downarrow$$
$$\underbrace{\text{Epistemics} \qquad \text{Dynamics}}$$
$$(L, f_A \dashv f_A^*)_{\mathcal{A}} \qquad ??$$

The resulting logic is obtained by endowing our MAMA with a new operator to stand for dynamics. The reasoning power of this algebra is increased by asking the new operator to weakly permute with the existing epistemic operators. The definition of the new algebra is as follows

Definition 3.1 A *temporal epistemic algebra* denoted by $(L, f_A \dashv f_A^*, h \dashv h^*)_{\mathcal{A}}$ is a multi-agent adjoint modal algebra $(L, f_A \dashv f_A^*)_{\mathcal{A}}$ endowed with a join preserving map $h\colon L \to L$, such that the following permutation holds

$$f_A \, h(l) \leq h \, f_A(l)$$

We read $h^*(l)$ as

> 'In the next state of the system l holds'.

Hence $h^* f_A^*(l)$ is read as

‘In the next state of the system agent A gets informed that l holds’.

Similarly, $h(l)$ is read as ‘in the previous state of the system l held’. The permutation axiom of the algebra is a weak permutation between the two operators and demonstrates a preservation or no-miracle condition on the information: if an agent obtains some information in the next state of the system, it should be the case that this information existed in the system previously, thus the acquired information should somehow be implied by the previous information. In other words, information is not generated and cannot be destroyed freely and without a cost, it can only be accumulated. This axiom is similar to the *appearance-update* axiom of Baltag et al. (2007); Sadrzadeh (2005) and also corresponds to a weaker version of the *action-knowledge* axiom of Baltag and Moss (2004); Baltag et al. (1999). A similar axiom can also be found in the Epistemic Temporal Logic of Fagin et al. (1995) in the name of *perfect recall*.

According to proposition 2.9, a temporal fixed point operator can be defined as $\bigwedge_i h^{*i}(l)$ and interpreted as follows

‘Eventually in some future state of the system proposition l holds’.

Rather than reasoning about whether an information property holds in the next state of the system, it would be more sensible to name and reason about the action that led the system to its next state, the action that caused the information property to hold in the next state of the system. To do so, we endow our temporal epistemic algebra with a family of operators that are indexed over a set of actions. The passage from one temporal operator to a family of action operators is similar to the passage from the mono-modal epistemic algebras to the multi-agent ones. The new setting is defined as follows

Definition 3.2 An *action epistemic algebra* denoted by $(L, f_A \dashv f_A^*, h_a \dashv h_a^*)_{\mathcal{A}, Act}$ is a multi-agent adjoint modal algebra $(L, f_A \dashv f_A^*)_{\mathcal{A}}$ endowed with a family of join preserving maps $\{h_a\}_{a \in Act} : L \to L$, such that

$$f_A\, h_a(l) \leq h_a\, f_A(l)$$

We read $h_a^*(l)$ as

‘After action a proposition l holds’.

Similarly, $h_a^*\, f_A^*(l)$ is read as

‘After action a agent A gets informed that l holds’.

The fixed point of the action operator $\bigwedge_i h_\alpha^{*i}$, for $\alpha \subseteq Act$ is interpreted as

‘Eventually after the actions in α proposition l holds’.

We end this section by describing two restrictions that will bring our algebras closer to the specific application domain in mind. These restrictions have been introduced and discussed in detail in the algebra of Baltag et al. (2007); Sadrzadeh (2005), and correspond to similar restrictions in the dynamic epistemic logic of Baltag and Moss (2004); Baltag et al. (1999). The main point is that the actions that we are interested in reasoning about are the communication actions that take place in epistemic scenarios. These do not change the facts of the world and are of the form of announcements to a group of agents of a propositional or epistemic content. In order to model them, we ask for the following two axioms, for $a \in CAct \subseteq Act$ and $\phi \in \Phi \subseteq L$

$$\begin{aligned} l \in ker(a) \quad &\text{iff} \quad h_a(l) \leq \bot \\ l \leq \phi \quad &\text{iff} \quad h_a(l) \leq \phi \end{aligned}$$

We refer to $CAct$ as the communication actions and to Φ as the 'facts' of the system. The first axiom says that each communication action $a \in CAct$ has a kernel $ker(a)$, which stands for its 'co-content', that is all the propositions to which the action cannot be applied. The second axiom says that if a proposition l entails a fact ϕ, that is $l \leq \phi$ then a communication actions a does not have any effect on this entailment, that is $h_a(l) \leq \phi$.

Example 3.3 As an example, consider again the coin toss scenario where agent C uncovers the coin and announces: 'the coin is heads'. The announcement is a communication action $a \in CAct$ that appears as it is to all the agents since it is a public action, so $f_A(a) = f_B(a) = f_C(a) = a$. The kernel of this action is T, since it cannot apply when the coin has come down tails. The set of facts in this scenario is $\{H, T\}$.

We want to show that after this announcement the uncertainty of agents gets waived and for instance A will acquire information that the coin is heads, that is

$$H \leq h_a^* f_A^*(H)$$

By the adjunction rule on h_a^* this holds iff $h_a(H) \leq f_A^*(H)$, by the adjunction rule on f_A^*, this holds iff $f_A h_a(H) \leq H$. By the no-miracle axiom it suffices to show $h_a f_A(H) \leq H$. By the assumptions on $f_A(H)$ this is equivalent to $h_a(H \vee T) \leq H$. Since h_a is join preserving this is equivalent to showing $h_a(H) \vee h_a(T) \leq H$. By definition of $\vee$ in a lattice it suffices to show the following two case

$$\begin{cases} h_a(H) \leq H \\ h_a(T) \leq H \end{cases}$$

The first case follows since H is a fact and thus $h_a(H) = H$ and in a partial order we have that $H \leq H$. The second case follows since $T \in ker(a_H)$, which means $a_H(T) = \bot$, thus $\bot \leq H$. Other nested information properties such as the following ones are proved in a similar fashion

$$H \leq h_a^* f_A^* f_C^*(H), \quad H \leq h_a^* f_A^* f_B^*(H)$$

4 Ockham's razor and reasoning about flow of mis-information

In the closer-to-real life versions of the scenarios of multi-agent systems, agents are not always honest and thus communication actions are not always truthful. We would like to be able to reason about these scenarios and model the cheating and lying actions of dishonest agents. In order to do this, and following the approaches of Baltag et al. (2007); Baltag and Moss (2004); Baltag et al. (1999); Sadrzadeh (2005), we introduce epistemic structure on actions. Similar to the epistemic structure on propositions, these will stand for 'appearances of agents about actions'. Also similar to the epistemic structure on propositions, these are added by endowing the set of actions Act with a family of appearance maps $f'_A \colon Act \to Act$, one for each agent. The new algebras are defined below

Definition 4.1 A *real action epistemic algebra* denoted by $(L, f_A \dashv f_A^*, h_a \dashv h_a^*)_{\mathcal{A},(Act, f'_A \dashv (f'_A)^*)_{\mathcal{A}}}$ is an action epistemic algebra $(L, f_A \dashv f_A^*, h_a \dashv h_a^*)_{\mathcal{A},Act}$ where the set of actions Act is endowed with a family of join preserving maps $\{f'_A\}_{A\in\mathcal{A}} \colon Act \to Act$, and we have

$$f_A\, h_a(l) \leq h_{f'_A(a)}\, f_A(l)$$

The no-miracle axiom now becomes a no-miracle axiom up to the appearance of actions, that is if an agent acquires new information after an action, this information is based on the state of the system before the action and also the appearance of the agent about that action.

Example 4.2 Consider the coin toss scenario, we show that if C's announcement was not honest and he lied about the face of the coin, that is announced heads when he saw tails, A and B, who did not notice and neither suspect the lying, will acquire wrong information. The lying action is a communication action $\overline{a} \in CAct$ that appears as it is to the announcer C, that is $f'_C(\overline{a}) = \overline{a}$, but since A and B do not suspect it they think it is an honest announcement that is $f'_A(\overline{a}) = f'_B(\overline{a}) = a$. The kernel of the lying action is H since it could not be a lie if the coin had actually landed heads. In this lying scenario we can show, for example, the following properties

$$H \leq h^*_{\overline{a}} f^*_C(T), \quad H \leq h^*_{\overline{a}} f^*_A(H), \quad H \leq h^*_{\overline{a}} f^*_A f^*_C(H)$$

Properties of this and other examples, such as the muddy children puzzle with cheating and lying, are proved using the same strategy as in the honest versions demonstrated. In the muddy children one needs to repeat the kernel argument for the number of dirty children in the puzzle minus 2.

Consider the third property, by adjunction on $h^*_{\overline{a}}$, f^*_A and f^*_C respectively, it is equivalent to $f_C\, f_A\, h_{\overline{a}}(H) \leq H$. By the no-miracle axiom between f_A and $h_{\overline{a}}$ it suffices to show

$$f_C\, h_{f'_A(\overline{a})}\, f_A(H) \leq H$$

which is equivalent to $f_C\, h_a\, f_A(H) \leq H$ since $f'_A(\overline{a}) = a$. By the no-miracle axiom this time between f_C and h_a, it suffices to show $h_{f'_C(a)}\, f_C\, f_A(H) \leq H$, which is equivalent to $h_a f_C f_A(H) \leq H$ since $f'_C(a) = a$. We substitute values for f_A and

f_C and need to show $h_a(H \vee T) \leq H$. By distributivity and definition of join this is obtained by showing two cases

$$\begin{cases} h_a(H) \leq H \\ h_a(T) \leq H \end{cases}$$

The second case holds since T is in the kernel of a, and thus $h_a(T) = \bot \leq H$, the first case follows similar to the previous example and by preservation of facts.

Two observations are in place here:

- The family of indexed unary maps $\{h_a\}_{a \in Act} \colon L \to L$ is equivalent to the binary operation

$$h \colon L \times Act \to L\,.$$

- There is some implicit structure on the set of actions: they can be sequentially composed $a \bullet a'$, non-deterministically chosen $a \vee a'$, and there is a neutral action 1 in which nothing happens $1 \bullet a = a \bullet 1 = a$. Assuming the existence of all the choices (joins) and their distributivity over the composition, permits us to form a *quantale* of actions $\mathcal{Q} = (Q, \bigvee, \bullet, 1)$[2].

The index sets of a real action epistemic algebra make the structure a bit too crowded, especially the index set of actions which is itself indexed over the set of agents. The situation can be improved by considering instead two separate multi-agent adjoint modal algebras: one for the propositions $(L, f_A \dashv f_A^*)_{\mathcal{A}}$ and another one for the actions $(Act, f'_A \dashv (f')_A^*)_{\mathcal{A}}$ where the latter acts on the former via the binary counterpart of the h_a operators, that is via the binary operation of $h \colon L \times Act \to L$. It is easy to show that the equivalence mentioned in the first observation above lifts to one between a real action epistemic algebra and these two MAMA's. Formally speaking we have

Proposition 4.3 *A real action epistemic algebra*

$$(L, f_A \dashv f_A^*, h_a \dashv h_a^*)_{\mathcal{A}, (Act, f'_A \dashv (f'_A)^*)_{\mathcal{A}}}$$

is equivalent to

$$\big((L, f_A \dashv f_A^*)_{\mathcal{A}}, (Act, f'_A \dashv (f')_A^*)_{\mathcal{A}}, h\big)$$

whenever $h \colon L \times Act \to L$ *is the binary equivalent of* $\{h_a\}_{a \in Act} \colon L \to L$.

Proof Follows directly from the equivalence of observation 1 above. In particular the join preservation of h_a, that is $h_a(\bigvee_i l_i) = \bigvee_i h_a(l_i)$ lifts to $h(\bigvee_i l_i, a) = \bigvee_i h(l_i, a)$ and the permutation between h_a and f_A, that is $f_A h_a(l) \leq h_{f'_A(a)} f_A(l)$ lifts to $f_A h(l, a) \leq h(f_A(l), f'_A(a))$. □

[2] This can be, for instance, the powerset of the free monoid generated on Act, that is $\mathcal{P}(Act^*)$.

So far we are able to reason about the information acquired by agents as a result of atomic actions taking place among them. The information acquired by composition of actions can also be taken care of by composing the action operators, for example $h_a h_b f_A^*(l)$ says that after doing action a followed by action b, agent A is informed that l. What is missing is reasoning about the information after a choice of actions, for example to express what an agent would acquire if either action a or action b take place. If we move from the plain set of actions Act to the quantale of actions $(Act, \bigvee, \bullet, 1)$, we obtain an algebraic structure on the actions which enables us to as well reason about non-deterministic choices of actions. What will happen to the appearance maps? For example, given the appearance of atomic actions a, b in Act, what would be the appearance of the choice of actions $a \vee b$, and their composition $a \bullet b$? The most natural and neutral way of extending appearance maps to choice of actions is point-wisely, that is making the appearance of the non-deterministic choice be equal to the choice of the appearances

$$f'_A\left(\bigvee_i a_i\right) = \bigvee_i f'_A(a_i)$$

How about with regard to the sequential composition and its unit? This depends on what kind of agents do we want to model. For instance, we may decide to consider it possible for our agents to be *paranoid*, that is, when nothing is happening in reality, it appears to them that something is happening. In this case, we do not need to ask for any extra inequalities between 1 and $f'_A(1)$. Since, for example, $f'_A(1)$ can be equal to any action a, and in general there is no order relation between 1 and an arbitrary action a. However, this may be a bit too strong of an assumption, we can weaken it by asking the agents to be *optimistically paranoid*. That is, when nothing is happening in reality, it appears to them that either nothing is happening or something is happening. In other words, we ask that appearance to all agents of the action in which nothing happens always include it, that is

$$1 \leq f'_A(1)$$

It is then easy to show (see Baltag et al. 2007; Sadrzadeh 2005) that this inequality will lead us to an inequality on the appearance of a sequential composition, that is $f'_A(a \bullet b) \leq f'_A(a) \bullet f'_A(b)$. There is a third possibility and that is when the agents are not paranoid at all. In other words, whenever nothing is happening in reality, it appears to them that nothing is happening. So their appearance of 1 is equal to 1

$$1 = f'_A(1)$$

This inequality will force the appearance of the sequential composition to be equal to the sequential composition of the appearances, that is. It will also force our permutation axiom to be equality rather than inequality, that is

$$f'_A(a \bullet b) = f'_A(a) \bullet f'_A(b), \quad f_A h_a(l) = h_{f'_A(a)} f_A(l)$$

However, since the goal of this paper is to stay minimal in the axioms of the algebra and that inequality is weaker than equality, it is reasonable to work with the inequality versions of axioms and assume that our agents are *optimistically paranoid*. A more detailed discussion of these and other attitudes of agents and their relation to axioms of the algebra is well in place, but out of the limits of the current paper.

Let us end by defining the notion of a quantale endowed with appearance maps for optimistically paranoid agents and show how it will help us relate our system to the other algebra of information and mis-information flow.

Definition 4.4 An *epistemic quantale* denoted by $(Q, f'_A \dashv f'^*_A)_\mathcal{A}$ is a multi-agent adjoint modal algebra where Q is a quantale and moreover we have

$$1 \leq f'_A(1)\,, \quad f'_A(a \bullet b) \leq f'_A(a) \bullet f'_A(b)$$

The *epistemic systems* of Baltag et al. (2007); Sadrzadeh (2005) are obtained from the two sorted structure of proposition 4.3 as follows

Proposition 4.5 *The pair* $\big((L, f_A \dashv f^*_A)_\mathcal{A}, (Q, f'_A \dashv f'^*_A)_\mathcal{A}, h\big)$ *is an* epistemic system *whenever* $(L, f_A \dashv f^*_A)_\mathcal{A}$ *is a MAMA,* $(Q, f'_A \dashv f'^*_A)_\mathcal{A}$ *is an epistemic quantale, the pair is equivalent to a real action epistemic algebra* $(L, f_A \dashv f^*_A, h_a \dashv h^*_a)_{\mathcal{A},(Act, f'_A \dashv (f'_A)^*)_\mathcal{A}}$ *and moreover h satisfies the following*

$$h\left(l, \bigvee_i a_i\right) = \bigvee_i h(l, a_i), \quad h(l, 1) = l, \quad h(l, a \bullet b) = h(h(l, a), b)$$

Proof Follows from definition 4.5 and proposition 4.3. □

5 Conclusion

We have presented a minimal algebraic modal logic where modalities are not necessarily positively or negatively introspective, that is they do not in general obey axioms 4 and 5 of modal logic, neither are they in general truthful, that is obey axiom T. The only condition on them is preservation of disjunctions or conjunctions of their base propositional setting. The propositional setting is also weak: it has neither implication nor negation, and is not necessarily distributive. Lack of negation means that our modalities are not de Morgan duals, but they are connected to each other in a weaker sense and as adjoints. These minimal modalities can be interpreted as new modes such as 'information' and 'appearance', from which belief and truthful knowledge can be derived. We have defined fixed point operators for these modalities such that the pair of fixed points are also adjoints. The applicability of our logic is demonstrated via examples of epistemic scenarios. This weak setting can be extended to also model the flow of information, be it caused by the passage of time or by application of actions. Actions can have some extra structure on them to model cheating and lying, sequential composition and non-deterministic choice. All of these can be modularly added to the weak modal algebra we started with. At the end, we show how restricting our agents to the *optimistically paranoid* ones allows us to obtain the structure of an *epistemic system*, developed in previous work as the algebraic semantics of Dynamic Epistemic Logic.

One needs to study the universal algebraic properties of our weak modal algebras in the lines of Gehrke et al. (2005); that if they have an equational theory, how can they be freely generated, what does their relational semantics look like, how to develop a Stone-like duality for them, etc. One possible challenge might lie in the rule of adjunction $f(l) \leq l'$ iff $l \leq f^*(l')$, which is not an equation. The equations are obtained from composing the adjoints, for example $f \circ f^*(l) \leq l$ and $(f \circ f^*)^2 = f \circ f^*$, but those are not of rank 1.

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