Tractable Extensions of the Description Logic $\mathcal{EL}$ with Numerical Datatypes

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Abstract. We consider extensions of the lightweight description logic (DL) $\mathcal{EL}$ with numerical datatypes such as naturals, integers, rationals and reals equipped with relations such as equality and inequalities. It is well-known that the main reasoning problems for such DLs are decidable in polynomial time provided that the datatypes enjoy the so-called convexity property. Unfortunately many combinations of the numerical relations violate convexity, which makes the usage of these datatypes rather limited in practice. In this paper, we make a more fine-grained complexity analysis of these DLs by considering restrictions not only on the kinds of relations that can be used in ontologies but also on their occurrences, such as allowing certain relations to appear only on the left-hand side of the axioms. To this end, we introduce a notion of safety for a numerical datatype with restrictions (NDR) which guarantees tractability, extend the $\mathcal{EL}$ reasoning algorithm to these cases, and provide a complete classification of safe NDRs for natural numbers, integers, rationals and reals.

Key words: description logic, computational complexity, datatypes

1 Introduction and Motivation

Description logics (DLs) [1] provide a logical foundation for modern ontology languages such as OWL[1] and OWL 2 [2]. $\mathcal{EL}^{++}$ [3] is a lightweight DL for which reasoning is tractable (i.e., can be performed in time that is polynomial w.r.t. the size of the input), and that offers sufficient expressivity for a number of life-sciences ontologies, such as SNOMED CT [4] or the Gene Ontology [5]. Among other constructors, $\mathcal{EL}^{++}$ supports limited usage of datatypes. In DL, datatypes (also called concrete domains) can be used to define new concepts by referring to particular values, such as strings or integers. For example, the concept $\mathsf{Human} \sqcap \exists \mathsf{hasAge}.(<, 18) \sqcap \exists \mathsf{hasName}.(=, \text{“Alice”})$ describes humans whose age is less than 18 and whose name is “Alice”. Datatypes are characterised first by the domain their values can come from and also by the relations that can be used to constrain possible values. In our example, $(<, 18)$ refers to the domain of natural numbers and uses the relation “<” to constrain possible values to

[1] `http://www.w3.org/2004/OWL`

those less than 18, while (=, "Alice") refers to the domain of strings and uses the relation "=" to constrain the value to "Alice".

In order to ensure that reasoning remains polynomial, $\mathcal{EL}^{++}$ allows only for datatypes which satisfy a condition called p-admissibility [3]. In an nutshell, this condition ensures that the satisfiability of datatype constraints can be solved in polynomial time, and that concept disjunction cannot be expressed using datatype concepts. For example, if we were to allow both $\leq$ and $\geq$ for integers, then we could express $A \sqsubseteq B \sqcup C$ by formulating the axioms $\mathsf{A} \sqsubseteq \exists \mathsf{R}.(\leq, 5)$, $\exists \mathsf{R}.(\leq, 2) \sqsubseteq \mathsf{B}$ and $\exists \mathsf{R}.(\geq, 2) \sqsubseteq \mathsf{C}$. Thus, allowing both $\leq$ and $\geq$ has the same effect as extending $\mathcal{EL}^{++}$ with disjunction, which is well known to cause intractability [3]. Similarly, we can show that p-admissibility prevents us from having both $\leq$ and = or both $\geq$ and = in the language. For this reason, the EL Profile of OWL 2, which is based on $\mathcal{EL}^{++}$, admits only equality (=) in datatype expressions.

In this paper, we demonstrate how these restrictions can be significantly relaxed without loosing tractability. As a motivating example, consider the following two axioms which might be used, e.g., in a pharmacy-related ontology:

$$\mathsf{Panadol} \sqsubseteq \exists \mathsf{contains}.(\mathsf{Paracetamol} \sqcap \exists \mathsf{mgPerTablet}.(=, 500)) \tag{1}$$

$$\begin{aligned}&\mathsf{Patient} \sqcap \exists \mathsf{hasAge}.(<, 6) \sqcap \\ &\quad \exists \mathsf{hasPrescription}.\exists \mathsf{contains}.(\mathsf{Paracetamol} \sqcap \exists \mathsf{mgPerTablet}.(>, 250)) \sqsubseteq \bot\end{aligned} \tag{2}$$

Axiom (1) states that the drug Panadol contains 500 mg of paracetamol per tablet, while axiom (2) states that a drug that contains more than 250 mg of paracetamol per tablet must not be prescribed to a patient younger than 6 years old. The ontology could be used, for example, to support clinical staff who want to check whether Panadol can be prescribed to a 3-year-old patient. This can easily be achieved by checking whether the following concept is satisfiable w.r.t. the ontology:

$$\mathsf{Patient} \sqcap \exists \mathsf{hasAge}.(=, 3) \sqcap \exists \mathsf{hasPrescription}.\mathsf{Panadol} \tag{3}$$

Unfortunately, this is not possible using $\mathcal{EL}^{++}$, because axioms (1) and (2) involve both equality (=) and inequalities (<, >), and this violates the p-admissibility restriction. In this paper we demonstrate that it is, however, possible to express axioms (1) and (2) and concept (3) in a tractable extension of $\mathcal{EL}$. A polynomial classification procedure can then be used to determine the satisfiability of (3) w.r.t. the ontology by checking if adding an axiom

$$X \sqsubseteq \mathsf{Patient} \sqcap \exists \mathsf{hasAge}.(=, 3) \sqcap \exists \mathsf{hasPrescription}.\mathsf{Panadol}$$

for some new concept name X would entail $X \sqsubseteq \bot$.

Our idea is based on the intuition that equality in (1) and (3) serves a different purpose than inequalities do in (2). Equality in (1) and (3) is used to state a *fact* (the content of a drug and the age of a patient) whereas inequalities in (2) are used to trigger a *rule* (what happens if a certain quantity of drug is prescribed to a patient of a certain age). In other words, equality is used *positively* and inequalities are used *negatively.* It seems reasonable to assume that positive usages of datatypes will typically involve equality since a fact can usually be precisely stated. On the other hand, it seems reasonable to assume that negative occurrences of datatypes will typically involve equality as well as inequalities since a rule usually applies to a range of situations. In this paper, we make a fine-grained study of datatypes in $\mathcal{EL}$ by considering restrictions not only on the kinds of relations included in a datatype, but also on whether the relations can be used positively or negatively.

The main contributions of this paper can be summarised as follows:

1. We introduce the notion of a *Numerical Datatype with Restrictions* (*NDR*) that specifies the domain of the datatype, the datatype relations that can be used positively and the datatype relations that can be used negatively.
2. We extend the $\mathcal{EL}$ reasoning algorithm [3] to provide a polynomial reasoning procedure for an extension of $\mathcal{EL}$ with NDRs, and we prove that this procedure is sound for any NDR.
3. We introduce the notion of a *safe NDR*, show that every extension of $\mathcal{EL}$ with a safe NDR is tractable, and prove that our reasoning procedure is complete for any safe NDR.
4. Finally, we provide a complete classification of safe NDRs for the cases of natural numbers, integers, rationals and reals. Notably, we demonstrate that the numerical datatype restrictions can be significantly relaxed by allowing arbitrary numerical relations to occur negatively—not only equality as currently specified in the OWL 2 EL Profile. As argued earlier, this combination is of particular interest to ontology engineering, and is thus a strong candidate for the next extension of the EL Profile in OWL 2.

This work is based on a Master's thesis [6].

2 Preliminaries

In this section we introduce an extension of $\mathcal{EL}^{\perp}$ [3] with numerical datatypes which we denote by $\mathcal{EL}^{\perp}(\mathcal{D})$. In the DL literature the notion of a datatype is better known as a concrete domain [7]; we call them datatypes to be more consistent with OWL 2 [2]. The syntax of $\mathcal{EL}^{\perp}(\mathcal{D})$ uses a set of *concept names* N_C, a set of *role names* N_R and a set of *feature names* N_F. $\mathcal{EL}^{\perp}(\mathcal{D})$ is parametrised with a *numerical domain* D, such that $\mathcal{D} \subseteq \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. N_C, N_R and N_F are countably infinite sets and, additionally, pairwise disjoint.

Definition 1 ($\mathcal{D}$-datatype restriction). *We call (s, y), where $y \in \mathcal{D}$ and $s \in \{<, \leq, >, \geq, =\}$, a* $\mathcal{D}$-datatype restriction *(or simply a* datatype restriction *if the domain $\mathcal{D}$ is clear from the context). Given a domain $\mathcal{D}$, a $\mathcal{D}$-datatype restriction $r = (s, y)$ and an $x \in \mathcal{D}$, we say that x satisfies r and we write $r(x)$ iff $(x, y) \in s$, where s is interpreted as the corresponding standard relation on real numbers.*

Table 1. Concept descriptions in $\mathcal{EL}^{\bot}(\mathcal{D})$

Name	Syntax	Semantics
Concept name	C	$C^{\mathcal{I}}$
Top	$\top$	$\Delta^{\mathcal{I}}$
Bottom	$\bot$	$\emptyset$
Conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
Existential restriction	$\exists R.C$	$\{x \in \Delta^{\mathcal{I}} \mid \exists y \in \Delta^{\mathcal{I}} :\ (x, y) \in R^{\mathcal{I}} \land y \in C^{\mathcal{I}}\}$
Datatype restriction	$\exists F.r$	$\{x \in \Delta^{\mathcal{I}} \mid \exists v \in \mathcal{D} :\ (x, v) \in F^{\mathcal{I}} \land r(v)\}$

Intuitively, a datatype restriction is used to specify a subset of the numerical domain so that one can form new concepts that refer to elements of this subset. The set of concepts is recursively defined using the constructors listed in the middle column of Table 1, where C and E are concepts, $R \in N_R$, $F \in N_F$ and r is a $\mathcal{D}$-datatype restriction. We typically use the capital letters A, B to refer to concept names and the capital letters C, E or F to refer to concepts. We also set the abbreviations $N_C^{\top} = N_C \cup \{\top\}$ and $N_C^{\top,\bot} = N_C \cup \{\top, \bot\}$.

An *axiom* α *in* $\mathcal{EL}^{\bot}(\mathcal{D})$ or simply *an axiom* α is an expression of the form $C \sqsubseteq E$, where C and E are concepts. An $\mathcal{EL}^{\bot}(\mathcal{D})$*-ontology* $\mathcal{O}$ or simply *an ontology* $\mathcal{O}$ is a set of axioms. We say that a concept E occurs in a concept C iff E is used as a concept in the construction of C. Moreover, a concept F is said to *positively (negatively) occur* in an axiom $C \sqsubseteq E$ iff it occurs in the concept E (C); we alternatively say that we have a *positive (negative) occurrence* of F.

An interpretation of $\mathcal{EL}^{\bot}(\mathcal{D})$ is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a non-empty set which we call the *domain of the interpretation* and $\cdot^{\mathcal{I}}$ is the *interpretation function*. The interpretation function maps each concept name A to a subset $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$, each role name $R \in N_R$ to a relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and each feature name $F \in N_F$ to a relation $F^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times D$. Note that we do not require the interpretation of features to be functional. In this respect, they correspond to the data properties in OWL 2 [2]. The constructors of $\mathcal{EL}^{\bot}(\mathcal{D})$ are interpreted as indicated in the right column of Table 1. For an axiom α, where $\alpha = C \sqsubseteq D$, we write $\mathcal{I} \models \alpha$ and we say that *an interpretation* $\mathcal{I}$ *satisfies an axiom* α, iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. If $\mathcal{I} \models \alpha$ for every $\alpha \in \mathcal{O}$, then $\mathcal{I}$ *is a model of* $\mathcal{O}$ and we write $\mathcal{I} \models \mathcal{O}$. Additionally, if every model $\mathcal{I}$ of $\mathcal{O}$ satisfies the axiom α then we say that $\mathcal{O}$ *entails* α and we write $\mathcal{O} \models \alpha$. We define the *signature of an ontology* $\mathcal{O}$ as the set $sig(\mathcal{O})$ of concept, role and feature names that occur in $\mathcal{O}$. We say that a concept, role or feature name X is *fresh w.r.t. an ontology* $\mathcal{O}$ iff $X \notin sig(\mathcal{O})$.

One of the most common reasoning tasks w.r.t. an ontology $\mathcal{O}$ is the classification of an ontology $\mathcal{O}$, that is computing all axioms of the form $A \sqsubseteq B$, where A, $B \in N_C^{\top,\bot}$ and $\mathcal{O} \models A \sqsubseteq B$. The set of these axioms is called the *taxonomy* of the ontology $\mathcal{O}$.

We say that an axiom in $\mathcal{EL}^{\bot}(\mathcal{D})$ is in normal form if it has one of the forms NF1-NF6 of the left part of Table 2, where A, A_1, A_2, $B \in N_C^{\top}$, $B' \in N_C^{\top,\bot}$, $R \in N_R$, $F \in N_F$, and r is a $\mathcal{D}$-datatype restriction. Given an $\mathcal{EL}^{\bot}(\mathcal{D})$-ontology, if the normalization rules of the right part of Table 2 are applied, we obtain an

Table 2. Normal form of axioms and normalization rules for $\mathcal{EL}^{\perp}(\mathcal{D})$

Normal forms		Normalization rules
NF1	$A \sqsubseteq B'$	$C \sqcap H \sqsubseteq E \rightarrow \{H \sqsubseteq A_f, C \sqcap A_f \sqsubseteq E\}$
NF2	$A_1 \sqcap A_2 \sqsubseteq B$	$\exists R.G \sqsubseteq D \rightarrow \{G \sqsubseteq A_f, \exists R.A_f \sqsubseteq D\}$
NF3	$A \sqsubseteq \exists R.B$	$G \sqsubseteq H \rightarrow \{G \sqsubseteq A_f, A_f \sqsubseteq H\}$
NF4	$\exists R.B \sqsubseteq A$	$C \sqsubseteq \exists R.H \rightarrow \{C \sqsubseteq \exists R.A_f, A_f \sqsubseteq H\}$
NF5	$A \sqsubseteq \exists F.r$	$B \sqsubseteq C \sqcap D \rightarrow \{B \sqsubseteq C, B \sqsubseteq D\}$
NF6	$\exists F.r \sqsubseteq A$	$\perp \sqsubseteq C \rightarrow \emptyset$

ontology which contains only axioms in normal form [3]. For the rules of Table 2, we have that $B \in N_C^{\top}$, $G, H \notin N_C^{\top}$, $R \in N_R$, C, D, E, G and H are concepts and A_f is a fresh concept name w.r.t. the so far transformed ontology.

3 Numerical Datatypes with Restrictions

In this section we introduce the notion of a Numerical Datatype with Restrictions (NDR) which specifies which datatype relations can be used positively and negatively. We then present a polynomial consequence-based classification procedure for $\mathcal{EL}^{\perp}$ extended with NDRs and prove its soundness. Finally we prove that the procedure is complete provided that the NDR satisfies special safety requirements.

Definition 2 (Numerical Datatype with Restrictions). *A* numerical datatype with restrictions (NDR) *is a triple* $(\mathcal{D}, O_+, O_-)$*, where* $\mathcal{D} \subseteq \mathbb{R}$ *is a numerical domain and* $O_+, O_- \subseteq \{<, \leq, >, \geq, =\}$ *is the set of* positive *and, respectively,* negative relations. *An axiom in* $\mathcal{EL}^{\perp}(\mathcal{D})$ *is an axiom in* $\mathcal{EL}^{\perp}(\mathcal{D}, O_+, O_-)$ *if for every positive (negative) occurrence of a concept* $\exists F.(s, y)$ *in the axiom,* $s \in O_+$ *(*$s \in O_-$*). An* $\mathcal{EL}^{\perp}(\mathcal{D}, O_+, O_-)$-ontology *is a set of axioms in* $\mathcal{EL}^{\perp}(\mathcal{D}, O_+, O_-)$*.*

Subsequently, we describe when a datatype restriction is inconsistent and when one datatype restriction implies another (w.r.t. a domain D). These definitions of inconsistency and implication for datatype restrictions are necessary for the formulation of the inference rules, which we then briefly present.

3.1 The Classification Procedure and Soundness

We are going to describe a classification procedure for $\mathcal{EL}^{\perp}(\mathcal{D}, O_+, O_-)$, which is closely related to the procedure for $\mathcal{EL}^{++}$ [3]. In order to formulate inference rules for datatypes we need to introduce notation for satisfiability of a datatype restriction and implication between datatype restrictions.

Definition 3. *For two* $\mathcal{D}$*-datatype restrictions* r_+ *and* r_-*, we write* $r_+ \rightarrow_{\mathcal{D}} \perp$ *iff there is no* $x \in \mathcal{D}$ *such that* $r_+(x)$ *holds. Otherwise, we write* $r_+ \nrightarrow_{\mathcal{D}} \perp$*. We write that* $r_+ \rightarrow_{\mathcal{D}} r_-$ *iff* $r_+(x)$ *implies* $r_-(x)$*,* $\forall x \in \mathcal{D}$*. Otherwise, we write* $r_+ \nrightarrow_{\mathcal{D}} r_-$*.*

Table 3. Reasoning rules in $\mathcal{EL}^{\bot}(\mathcal{D})$

IR1	$\dfrac{}{A \sqsubseteq A}$	
IR2	$\dfrac{}{A \sqsubseteq \top}$	
CR1	$\dfrac{A \sqsubseteq B}{A \sqsubseteq C'}$	$B \sqsubseteq C' \in \mathcal{O}$
CR2	$\dfrac{A \sqsubseteq B \quad A \sqsubseteq C}{A \sqsubseteq D}$	$B \sqcap C \sqsubseteq D \in \mathcal{O}$
CR3	$\dfrac{A \sqsubseteq B}{A \sqsubseteq \exists R.C}$	$B \sqsubseteq \exists R.C \in \mathcal{O}$
CR4	$\dfrac{A \sqsubseteq \exists R.B \quad B \sqsubseteq C}{A \sqsubseteq D}$	$\exists R.C \sqsubseteq D \in \mathcal{O}$
CR5	$\dfrac{A \sqsubseteq \exists R.B \quad B \sqsubseteq \bot}{A \sqsubseteq \bot}$	
ID1	$\dfrac{}{A \sqsubseteq \bot}$	$A \sqsubseteq \exists F.r_+ \in \mathcal{O}$, $r_+ \to_{\mathcal{D}} \bot$
CD1	$\dfrac{A \sqsubseteq \exists F.r_+}{A \sqsubseteq B}$	$\exists F.r_- \sqsubseteq B \in \mathcal{O}$, $r_+ \to_{\mathcal{D}} r_-$
CD2	$\dfrac{A \sqsubseteq B}{A \sqsubseteq \exists F.r_+}$	$B \sqsubseteq \exists F.r_+ \in \mathcal{O}$

$A, B, C, E \in N_C^{\top}$

$C' \in N_C^{\top,\bot}$

$R \in N_R, F \in N_F$

We assume that deciding whether $r_+ \to_{\mathcal{D}} \bot$ and $r_+ \to_{\mathcal{D}} r_-$ can be done in polynomial time. It is easy to see that this is the case when $\mathcal{D}$ is the set of natural numbers, integers, reals or rationals for the set of relations $\{<, \leq, >, \geq, =\}$.

The classification procedure for $\mathcal{EL}^{\bot}(\mathcal{D})$ takes as an input an $\mathcal{EL}^{\bot}(\mathcal{D})$-ontology $\mathcal{O}$ whose axioms are in normal form and applies the inference rules in Table 3 to derive new axioms of the form NF1, NF3 and NF5 in Table 2. The rules are applied to already derived axioms and use axioms in $\mathcal{O}$ and properties $r_+ \to_{\mathcal{D}} \bot$ and $r_+ \to_{\mathcal{D}} r_-$ as side-conditions. The procedure terminates when no new axiom can be derived. It is easy to see that the procedure runs in polynomial time because there are only polynomially many axioms of the form NF1, NF3 and NF5 possible over $sig(\mathcal{O})$. It can be easily proved that the procedure is sound because the rules derive logical consequences of the axioms.

Theorem 1 (Soundness). *Let $\mathcal{O}$ be an $\mathcal{EL}^{\bot}(\mathcal{D})$-ontology consisting of axioms in normal form and $\mathcal{O}'$ consists of all axioms that are derivable using the rules of Table 3 for $\mathcal{O}$. Every model $\mathcal{I}$ of $\mathcal{O}$ is a model of $\mathcal{O}'$ as well.*

Proof. For every axiom $\alpha \in \mathcal{O}'$, we prove that $\mathcal{I} \models \alpha$ by induction on the length of the derivation of α.

Induction base: If α is obtained using rules **IR1** and **IR2** then $\mathcal{I} \models \alpha$ trivially. Suppose that $\alpha = A \sqsubseteq \bot$ is obtained using rule **ID1**. In this case, $A \sqsubseteq \exists F.r_+ \in \mathcal{O}$ and since $\mathcal{I} \models \mathcal{O}$ then $A^{\mathcal{I}} \subseteq (\exists F.r_+)^{\mathcal{I}}$. Since $r_+ \to_{\mathcal{D}} \bot$ we have $(\exists F.r_+)^{\mathcal{I}} = \emptyset$. Therefore, $A^{\mathcal{I}} \subseteq \emptyset$ and so $\mathcal{I} \models A \sqsubseteq \bot$.

Induction step: For the cases when axiom α is obtained using rules **IR1-CR5** (that do not involve datatypes) the proof is identical with the case of $\mathcal{EL}^{++}$ [3]. Suppose that $\alpha = A \sqsubseteq B$ is obtained using **CD1** from $A \sqsubseteq \exists F.r_+$. Then by induction hypothesis, $A^{\mathcal{I}} \subseteq (\exists F.r_+)^{\mathcal{I}}$. Since $\mathcal{I} \models \mathcal{O}$, $(\exists F.r_-)^{\mathcal{I}} \subseteq B^{\mathcal{I}}$ and

$r_+ \rightarrow_\mathcal{D} r_-$, we have that $A^\mathcal{I} \subseteq B^\mathcal{I}$. So, $\mathcal{I} \models A \sqsubseteq B$. Suppose that $\alpha = A \sqsubseteq \exists F.r_+$ is obtained using **CD2** from $A \sqsubseteq B$. Then by induction hypothesis, $A^\mathcal{I} \subseteq B^\mathcal{I}$. Since $\mathcal{I} \models \mathcal{O}$, $B^\mathcal{I} \subseteq (\exists F.r_+)^\mathcal{I}$ and, so, $A^\mathcal{I} \subseteq (\exists F.r_+)^\mathcal{I}$. So, $\mathcal{I} \models A \sqsubseteq \exists F.r_+$.

3.2 Completeness and safe NDRs

The completeness proof is based on the canonical model construction similarly as for $\mathcal{EL}^{++}$ [3]. In order to deal with datatypes in the canonical model we introduce a notion of a *datatype constraint.* Intuitively, a constraint specifies which datatype restrictions should hold in a model and which should not.

Definition 4 (Constraint). *A* constraint *over* $(\mathcal{D}, O_+, O_-)$ *is defined as a pair of sets* (S_+, S_-)*, such that* $S_+ = \{(s_+^1, y_1), \ldots, (s_+^n, y_n)\}$ *with* $s_+^i \in O_+$*,* $S_- = \{(s_-^1, z_1), \ldots, (s_-^m, z_m)\}$ *with* $s_-^j \in O_-$*,* y_i*,* $z_j \in \mathcal{D}$*,* $(s_+^i, y_i) \nrightarrow_\mathcal{D} (s_-^j, z_j)$ *and* $(s_+^i, y_i) \nrightarrow_\mathcal{D} \bot$ *for* $1 \leq i \leq n$*,* $1 \leq j \leq m$ *and* m*,* $n \geq 0$*.*

Definition 5. *A constraint* (S_+, S_-) *over* $(\mathcal{D}, O_+, O_-)$ *is* satisfiable *iff there exists a* solution of (S_+, S_-) *that is a set* $V \subseteq \mathcal{D}$ *such that every* $r_+ \in S_+$ *is satisfied by at least one* $v \in V$ *but no* $r_- \in S_-$ *is satisfied by any* $v \in V$*.*

Our model construction procedure works only for the cases where we can ensure that every constraint over a numerical domain is satisfiable. This leads us to a notion of safety for an NDR.

Definition 6 (NDR Safety). *Let* $(\mathcal{D}, O_+, O_-)$ *be an NDR.* $(\mathcal{D}, O_+, O_-)$ *is* safe *iff every constraint over* $(\mathcal{D}, O_+, O_-)$ *is satisfiable.*

We define strong and weak convexity for NDRs and prove that an NDR is safe iff it is weakly convex.

Definition 7 (Strong and Weak Convexity). *The NDR* $(\mathcal{D}, O_+, O_-)$ *is* strongly convex *when for every* $r_+^i = (s_+^i, y_i)$ *and* $r_-^j = (s_-^j, z_j)$*, with* $s_+^i \in O_+$*,* $s_-^j \in O_-$ *and* y_i*,* $z_j \in \mathcal{D}$ *(*$1 \leq i \leq n$*,* $1 \leq j \leq m$*), if* $\bigwedge_{i=1}^n r_+^i \rightarrow_\mathcal{D} \bigvee_{j=1}^m r_-^j$*, then there exists an* r_-^j *(*$1 \leq j \leq m$*) such that* $\bigwedge_{i=1}^n r_+^i \rightarrow_\mathcal{D} r_-^j$*.* $(\mathcal{D}, O_+, O_-)$ *is* weakly convex *when the implication holds for* $n = 1$*.*

For example the NDR $(\mathbb{Z}, \{<, >\}, \{=\})$ is weakly convex but not strongly convex. It is weakly convex since the implications $((<, y) \rightarrow_\mathbb{Z} \bigvee_{j=1}^m (=, z_j))$ and $((>, y) \rightarrow_\mathbb{Z} \bigvee_{j=1}^m (=, z_j))$ never hold. However, it is not strongly convex: it is $(>, 2) \wedge (<, 5) \rightarrow_\mathbb{Z} (=, 3) \vee (=, 4)$, but also $(>, 2) \wedge (<, 5) \nrightarrow_\mathbb{Z} (=, 3)$ and $(>, 2) \wedge (<, 5) \nrightarrow_\mathbb{Z} (=, 4)$.

Lemma 1. $(\mathcal{D}, O_+, O_-)$ *is safe iff it is weakly convex.*

Proof. We assume that $(\mathcal{D}, O_+, O_-)$ is not weakly convex and we prove that it is non-safe. Since it is not weakly convex we have that for some $r_+ \rightarrow_\mathcal{D} \bigvee_{j=1}^m r_-^j$ there exists no r_-^j such that $r_+ \rightarrow_\mathcal{D} r_-^j$. In order to prove non-safety it is sufficient to define a constraint which is not satisfiable. We define (S_+, S_-), with $S_+ = \{r_+\}$ and $S_- = \{r_-^j\}_{j=1}^m$. (S_+, S_-) is indeed a constraint because $r_+ \not\to_{\mathcal{D}} \bot$ (otherwise $r_+ \to_{\mathcal{D}} r_-^j$ is true for every r_-^j) and for every r_-^j, $r_+ \not\to_{\mathcal{D}} r_-^j$ (otherwise $r_+ \to_{\mathcal{D}} r_-^j$ is true for at least one r_-^j). Additionally, it is not satisfiable, because from $r_+ \to_{\mathcal{D}} \bigvee_{j=1}^m r_-^j$ there can be found no x such that $r_+(x)$ and $\bigwedge_{j=1}^m \neg r_-^j(x)$.

We prove that if $(\mathcal{D}, O_+, O_-)$ is not safe, then it is not weakly convex. Since it is not safe then there exists a non-satisfiable constraint (S_+, S_-), where $S_+ = \{r_+^i\}_{i=1}^n$ and $S_- = \{r_-^j\}_{j=1}^m$. If $S_- = \emptyset$, then since $r_+^i \not\to_{\mathcal{D}} \bot$ for $1 \le i \le n$, there is a solution $V = \{x_i \mid 1 \le i \le n\}$ for (S_+, S_-). Thus, $S_- \neq \emptyset$. If $S_+ = \emptyset$ then there is the solution $V = \emptyset$ for (S_+, S_-). Thus, $S_+ \neq \emptyset$. Since (S_+, S_-) is a constraint, then $r_+^i \not\to_{\mathcal{D}} r_-^j$ for $1 \le i \le n$ and $1 \le j \le m$. Since (S_+, S_-) is not satisfiable for every $1 \le i \le n$ there exists no x such that $r_+^i(x)$ and $\bigwedge_{j=1}^m \neg r_-^j(x)$, that is if $r_+^i(x)$ then $r_-^j(x)$ holds for at least one r_-^j or, otherwise written, $r_+^i \to_{\mathcal{D}} \bigvee_{j=1}^m r_-^j$. From this and $r_+^i \not\to_{\mathcal{D}} r_-^j$ for every r_-^j, $(\mathcal{D}, O_+, O_-)$ is not weakly convex. □

Theorem 2 (Completeness). *Let $(\mathcal{D}, O_+, O_-)$ be a safe NDR, let $\mathcal{O}$ be an $\mathcal{EL}^\bot(\mathcal{D}, O_+, O_-)$-ontology containing axioms in normal form and let $\mathcal{O}'$ be the saturation of $\mathcal{O}$ under the rules of Table 3. For every A, $B \in (N_C^\top \cap sig(\mathcal{O}))$, if $\mathcal{O} \models A \sqsubseteq B$, then $A \sqsubseteq B \in \mathcal{O}'$ or $A \sqsubseteq \bot \in \mathcal{O}'$.*

Proof. The proof is analogous to the completeness proof of the subsumption algorithm for $\mathcal{EL}^{++}$ [3]; we build a canonical model $\mathcal{I}$ for $\mathcal{O}$ using $\mathcal{O}'$ and show that if $A \not\sqsubseteq B \in \mathcal{O}'$ and $A \not\sqsubseteq \bot \in \mathcal{O}'$ then $\mathcal{I} \not\models A \sqsubseteq B$.
For every $A \in N_C$, $F \in N_F$, define $S_+(A, F)$ and $S_-(A, F)$, as follows:

$$S_+(A, F) = \{r_+ \mid A \sqsubseteq \exists F.r_+ \in \mathcal{O}', A \sqsubseteq \bot \notin \mathcal{O}'\} \quad (3)$$
$$S_-(A, F) = \{r_- \mid \exists F.r_- \sqsubseteq B \in \mathcal{O}, A \sqsubseteq B \notin \mathcal{O}'\} \quad (4)$$

We now show that $(S_+(A, F), S_-(A, F))$ is a constraint w.r.t $(\mathcal{D}, O_+, O_-)$. First we prove that $r_+ \not\to_{\mathcal{D}} \bot$, $\forall r_+ \in S_+(A, F)$, which is true because otherwise due to rule **ID1** it would be $A \sqsubseteq \bot \in \mathcal{O}'$, in contradiction to the definition of $S_+(A, F)$. Additionally, there is no $r_+ \in S_+(A, F)$ and $r_- \in S_-(A, F)$ such that $r_+ \to_{\mathcal{D}} r_-$, otherwise from $A \sqsubseteq \exists F.r_+ \in \mathcal{O}'$, $\exists F.r_- \sqsubseteq B \in \mathcal{O}$ and **CD1** it would be $A \sqsubseteq B \in \mathcal{O}'$ which contradicts the definition of $S_-(A, F)$. Since $(S_+(A, F), S_-(A, F))$ is a constraint over $(\mathcal{D}, O_+, O_-)$ and $(\mathcal{D}, O_+, O_-)$ is safe, there exists a solution $V(A, F) \subseteq \mathcal{D}$ of $(S_+(A, F), S_-(A, F))$. We now construct the canonical model $\mathcal{I}$:

$$\Delta^{\mathcal{I}} = \{x_A \mid A \in N_C^\top \cap sig(\mathcal{O}), A \sqsubseteq \bot \notin \mathcal{O}'\} \quad (5)$$
$$B^{\mathcal{I}} = \{x_A \mid x_A \in \Delta^{\mathcal{I}}, A \sqsubseteq B \in \mathcal{O}'\} \quad (6)$$
$$R^{\mathcal{I}} = \{(x_A, x_B) \mid A \sqsubseteq \exists R.B \in \mathcal{O}', x_A, x_B \in \Delta^{\mathcal{I}}\} \quad (7)$$
$$F^{\mathcal{I}} = \{(x_A, v) \mid v \in V(A, F)\} \quad (8)$$

We prove that $\mathcal{I} \models \mathcal{O}$ by showing that $\mathcal{I} \models \alpha$, when α takes one of the NF1-NF6.

NF1 $A \sqsubseteq B$: We need to prove $A^{\mathcal{I}} \subseteq B^{\mathcal{I}}$. Take an $x \in A^{\mathcal{I}}$. By (6), $x = x_C$ such that $C \sqsubseteq A \in \mathcal{O}'$. From $A \sqsubseteq B \in \mathcal{O}$ and since $\mathcal{O}'$ is closed under **CR1**, we have $C \sqsubseteq B \in \mathcal{O}'$. Hence $x = x_C \in B^{\mathcal{I}}$ by (6).

If $B = \bot$, then we need to show that $A^{\mathcal{I}} = \emptyset$. If there exists $x \in A^{\mathcal{I}}$, then by (6) $x = x_C$ such that $C \sqsubseteq A \in \mathcal{O}'$. Since $\mathcal{O}'$ is closed under **CR1** and $A \sqsubseteq \bot \in \mathcal{O}'$, we have $C \sqsubseteq \bot \in \mathcal{O}'$. Thus, $x = x_C \notin \Delta^{\mathcal{I}}$ by (5), which contradicts our assumption that $x \in A^{\mathcal{I}}$.

We examine separately the case when $A = \top$. We have that $x_A \in \Delta^{\mathcal{I}}$ and we need to show that $x_A \in B^{\mathcal{I}}$. From rule **IR2**, we have that $A \sqsubseteq \top \in \mathcal{O}'$. From rule **CR1**, $A \sqsubseteq B \in \mathcal{O}'$; since $x_A \in \Delta^{\mathcal{I}}$ and $A \sqsubseteq B \in \mathcal{O}'$ we get $x_A \in B^{\mathcal{I}}$ by (6).

NF2 $A_1 \sqcap A_2 \sqsubseteq B$: We prove $(A_1 \sqcap A_2)^{\mathcal{I}} \subseteq B^{\mathcal{I}}$. Take an $x \in (A_1 \sqcap A_2)^{\mathcal{I}}$; then, $x \in A_1^{\mathcal{I}}$, $x \in A_2^{\mathcal{I}}$ and by (6) $x = x_A$ for some concept name A such that $A \sqsubseteq A_1 \in \mathcal{O}'$ and $A \sqsubseteq A_2 \in \mathcal{O}'$. Since $A \sqsubseteq A_1 \in \mathcal{O}'$, $A \sqsubseteq A_2 \in \mathcal{O}'$ and $A_1 \sqcap A_2 \sqsubseteq B \in \mathcal{O}$ closure under rule **CR2** gives $A \sqsubseteq B \in \mathcal{O}'$ and, therefore, $x \in B^{\mathcal{I}}$, by (6).

NF3 $A \sqsubseteq \exists R.B$: We show $A^{\mathcal{I}} \subseteq (\exists R.B)^{\mathcal{I}}$; take an $x \in A^{\mathcal{I}}$. By (6), $x = x_C$ where $C \sqsubseteq A \in \mathcal{O}'$. Since $A \sqsubseteq \exists R.B \in \mathcal{O}$ and $\mathcal{O}'$ is closed under **CR3**, we have $C \sqsubseteq \exists R.B \in \mathcal{O}'$. Since $x_C \in \Delta^{\mathcal{I}}$, we have $C \sqsubseteq \bot \notin \mathcal{O}'$ and, hence, $B \sqsubseteq \bot \notin \mathcal{O}'$ by **CR5**. Thus, $x_B \in \Delta^{\mathcal{I}}$ and $(x_C, x_B) \in R^{\mathcal{I}}$ by (7). Since $B \sqsubseteq B \in \mathcal{O}'$ by **IR1**, we have $x_B \in B^{\mathcal{I}}$ by (6). Thus, $x = x_C \in (\exists R.B)^{\mathcal{I}}$.

NF4 $\exists R.B \sqsubseteq A$: We prove $(\exists R.B)^{\mathcal{I}} \subseteq A^{\mathcal{I}}$; take an $x \in (\exists R.B)^{\mathcal{I}}$. Then, there exists $y \in \Delta^{\mathcal{I}}$ such that $(x, y) \in R^{\mathcal{I}}$ and $y \in B^{\mathcal{I}}$. By (7) and (6) $x = x_C$ and $y = x_D$ such that $C \sqsubseteq \exists R.D \in \mathcal{O}'$ and $D \sqsubseteq B \in \mathcal{O}'$ respectively. Since $\exists R.B \sqsubseteq A \in \mathcal{O}$ and $\mathcal{O}'$ is closed under **CR4**, we have $C \sqsubseteq A \in \mathcal{O}'$. Hence, $x = x_C \in A^{\mathcal{I}}$ by (6).

NF5 $A \sqsubseteq \exists F.r_+$: We show that $A^{\mathcal{I}} \subseteq (\exists F.r_+)^{\mathcal{I}}$; take an $x \in A^{\mathcal{I}}$. By (6), there exists a concept name C such that $x = x_C$ and $C \sqsubseteq A \in \mathcal{O}'$. Since $A \sqsubseteq \exists F.r_+ \in \mathcal{O}$ and $\mathcal{O}'$ is closed under **CD2**, we have $C \sqsubseteq \exists F.r_+ \in \mathcal{O}'$. We use (3) and (4) to build $(S_+(C, F), S_-(C, F))$; we have $r_+ \in S_+(C, F)$. By (8) we have $(x_C, v) \in F^{\mathcal{I}}$ for every $v \in V(C, F)$. Since $r_+ \in S_+(C, F)$, there exists $v \in V(C, F)$ such that v satisfies r_+ and, hence, $x = x_C \in (\exists F.r_+)^{\mathcal{I}}$.

NF6 $\exists F.r_- \sqsubseteq B$: We prove that $(\exists F.r_-)^{\mathcal{I}} \subseteq B^{\mathcal{I}}$; take an $x \in (\exists F.r_-)^{\mathcal{I}}$. By (5), there exists a concept name C such that $x = x_C$. We use (3) and (4) and construct $(S_+(C, F), S_-(C, F))$. Since $x_C \in (\exists F.r_-)^{\mathcal{I}}$, by (8), there exists $v \in V(C, F)$, such that $r_-(v)$ and $V(C, F)$ is a solution for $(S_+(C, F), S_-(C, F))$. Hence, $r_- \notin S_-(C, F)$, and so, $C \sqsubseteq B \in \mathcal{O}'$ by (4). Now by (6) and $C \sqsubseteq B \in \mathcal{O}'$, we have that $x_C \in B^{\mathcal{I}}$.

We now show that if $A \sqsubseteq B \notin \mathcal{O}'$ and $A \sqsubseteq \bot \notin \mathcal{O}'$, then $\mathcal{O} \not\models A \sqsubseteq B$ by proving $\mathcal{I} \not\models A \sqsubseteq B$ (since $\mathcal{I} \models \mathcal{O}$). $A^{\mathcal{I}} \nsubseteq B^{\mathcal{I}}$ holds, because $x_A \in \Delta^{\mathcal{I}}$ (from $A \sqsubseteq \bot \notin \mathcal{O}'$ and (5)), $x_A \in A^{\mathcal{I}}$ (from $A \sqsubseteq A \in \mathcal{O}'$)using rule **IR1** and by (6) and $x_A \notin B^{\mathcal{I}}$ (from $A \sqsubseteq B \notin \mathcal{O}'$ and (6)). □

Table 4. Maximal safe NDRs for $\mathbb{N}$

NDR	O_+	O_-
NDR_1	$\{=\}$	$\{<, \leq, >, \geq, =\}$
NDR_2	$\{<, \leq, >, \geq, =\}$	$\{<, \leq\}$
NDR_3	$\{<, \leq, >, \geq, =\}$	$\{>, \geq\}$
NDR_4	$\{>, \geq, =\}$	$\{<, \leq, =\}$

Table 5. Transformations $C_1 \Rightarrow C_2$ preserving constraints and their satisfiability for $\mathbb{N}$, where S_-, S_+ and S are sets of datatype restrictions and $y_1 \leq y_2$, $z_1 \leq z_2$

$C_1 = (S \cup S_+^1, S_-)$, $C_2 = (S \cup S_+^2, S_-)$		$C_1 = (S_+, S \cup S_-^1)$, $C_2 = (S_+, S \cup S_-^2)$	
S_+^1	S_+^2	S_-^1	S_-^2
$\{(<, y)\}$	$\{(\leq, y-1)\}$	$\{(<, z)\}$	$\{(\leq, z-1)\}$
$\{(>, y)\}$	$\{(\geq, y+1)\}$	$\{(>, z)\}$	$\{(\geq, z+1)\}$
$\{(\leq, y_1), (\leq, y_2)\}$	$\{(\leq, y_1)\}$	$\{(\leq, z_1), (\leq, z_2)\}$	$\{(\leq, z_2)\}$
$\{(\geq, y_1), (\geq, y_2)\}$	$\{(\geq, y_2)\}$	$\{(\geq, z_1), (\geq, z_2)\}$	$\{(\geq, z_1)\}$
$\{(=, y_1), (\leq, y_2)\}$	$\{(=, y_1)\}$	$\{(=, z_1), (\leq, z_2)\}$	$\{(\leq, z_2)\}$
$\{(\geq, y_1), (=, y_2)\}$	$\{(=, y_2)\}$	$\{(\geq, z_1), (=, z_2)\}$	$\{(\geq, z_1)\}$
		$\{(<, 0)\}$	$\emptyset$

4 Maximal Safe NDRs for $\mathbb{N}$

In this section we present a full classification of safe NDRs for natural numbers; for the current section we assume that every constraint is over the domain $\mathbb{N}$ ($0 \in \mathbb{N}$). Table 4 lists all maximal safe NDRs for $\mathbb{N}$. We prove that: (i) every NDR in Table 4 is safe, (ii) extending any of these NDRs with a new relation leads to non-safety and (iii) every safe NDR is contained in some NDR in Table 4.

Table 5 presents some basic transformations that preserve satisfiability of constraints.

Lemma 2. *Let C_1 and C_2 be as defined in Table 5 and $(\mathbb{N}, O_+, O_-)$ be an NDR. Then (i) C_1 is a constraint over $(\mathbb{N}, O_+, O_-)$ iff C_2 is a constraint over $(\mathbb{N}, O_+, O_-)$ and (ii) if C_1 and C_2 are both constraints over $(\mathbb{N}, O_+, O_-)$, then C_1 is satisfiable iff C_2 is satisfiable.*

Corollary 1. *Let $\mathsf{NDR}_i = (\mathbb{N}, O_+^i, O_-^i)$, with $1 \leq i \leq 4$. For every (S_+^1, S_-^1) over NDR_i there exists a constraint (S_+^2, S_-^2) over NDR_i, $y_1, \ldots, y_n \in \mathbb{N}$ and $z_1, \ldots, z_m \in \mathbb{N}$ such that:*

$$S_+^2 \subseteq \{(\leq, y_1), (=, y_2), \ldots, (=, y_{n-1}), (\geq, y_n)\}$$
$$S_-^2 \subseteq \{(\leq, z_1), (=, z_2), \ldots, (=, z_{m-1}), (\geq, z_m)\}$$

where $y_1 < \ldots < y_n$, $z_1 < \ldots < z_m$, $z_1 < y_1$, $z_m > y_n$, $y_i \neq z_j$ ($2 \leq i \leq n-1$, $2 \leq j \leq m-1$, m, $n \geq 0$) and (S_+^1, S_-^1) over NDR_i is satisfiable iff (S_+^2, S_-^2) over NDR_i is satisfiable.

The proof of Lemma 2 and Corollary 1 is trivial by a routine check of all cases.

Table 6. Examples of non-safe NDRs for $\mathbb{N}$ where $(s_+, y) \rightarrow_{\mathbb{N}} (s_-^1, z_1) \vee (s_-^2, z_2)$, $(s_+, y) \not\rightarrow_{\mathbb{N}} (s_-^1, z_1)$ and $(s_+, y) \not\rightarrow_{\mathbb{N}} (s_-^2, z_2)$

$\{s_+\}$	$\{s_-^1, s_-^2\}$	y	z_1	z_2
$\{<\}, \{\leq\}$	$\{<, \geq\}, \{\leq, >\}, \{\leq, \geq\}$	3	1	1
$\{<\}, \{\leq\}$	$\{<, >\}$	3	2	1
$\{>\}, \{\geq\}$	$\{<, \geq\}, \{\leq, >\}, \{\leq, \geq\}$	1	3	3
$\{>\}, \{\geq\}$	$\{<, >\}$	1	3	2
$\{>\}$	$\{=, \geq\}$	1	2	3
$\{>\}$	$\{=, >\}$	1	2	2
$\{\geq\}$	$\{=, \geq\}$	1	1	2
$\{\geq\}$	$\{=, >\}$	1	1	1
$\{<\}$	$\{=\}$	3	1	2
$\{\leq\}$	$\{=\}$	2	1	2

Lemma 3. *Every NDR in Table 4 is safe.*

Proof. We prove Lemma 3 by building a solution V for every constraint over NDRs in Table 4. By Corollary 1 we can assume w.l.o.g. the following restrictions for (S_+, S_-) and construct the corresponding solution V:

NDR_1: For S_+ we have that $S_+ \subseteq \{(=, y_1), \ldots, (=, y_n)\}$ and for S_- that $S_- \subseteq \{(\leq, z_1), (=, z_2), \ldots, (=, z_{m-1}), (\geq, z_m)\}$ with $z_1 < y_1 < \ldots < y_n < z_m$, $z_1 < \ldots < z_m$ and $y_i \neq z_j$ $(1 \leq i \leq n,\ 2 \leq j \leq m-1)$. $V = \{y_1, \ldots, y_n\}$.

NDR_2: For S_+ we have that $S_+ \subseteq \{(\leq, y_1), (=, y_2), \ldots, (=, y_{n-1}), (\geq, y_n)\}$ and for S_- that $S_- \subseteq \{(\leq, z_1)\}$ with $z_1 < y_1 < \ldots < y_n$. $V = \{y_1, \ldots, y_n\}$.

NDR_3: For S_+ we have that $S_+ \subseteq \{(\leq, y_1), (=, y_2), \ldots, (=, y_{n-1}), (\geq, y_n)\}$ and for S_- that $S_- \subseteq \{(\geq, z_1)\}$ with $y_1 < \ldots < y_n < z_1$. $V = \{y_1, \ldots, y_n\}$.

NDR_4: For S_+ we have that $S_+ \subseteq \{(=, y_1), \ldots, (=, y_{n-1}), (\geq, y_n)\}$ and for S_- that $S_- \subseteq \{(\leq, z_1), (=, z_2), \ldots, (=, z_m)\}$ with $y_1 < \ldots < y_n$, $z_1 < \ldots < z_m$, $z_1 < y_1$ and $y_i \neq z_j$ $(1 \leq i \leq n-1,\ 2 \leq j \leq m)$. $V = \{y_1, \ldots, y_{n-1}, y'_n\}$, where $y'_n = \max(y_n, z_m) + 1$. □

Lemma 4. *Let* $(\mathbb{N}, O_+, O_-)$ *be an NDR. Then:*

(a) If $O_+ \cap \{<, \leq, >, \geq\} \neq \emptyset$, $O_- \cap \{<, \leq\} \neq \emptyset$ *and* $O_- \cap \{>, \geq\} \neq \emptyset$, *then* $(\mathbb{N}, O_+, O_-)$ *is non-safe.*
(b) If $O_+ \cap \{>, \geq\} \neq \emptyset$, $O_- \cap \{>, \geq\} \neq \emptyset$ *and* $\{=\} \subseteq O_-$, *then* $(\mathbb{N}, O_+, O_-)$ *is non-safe.*
(c) If $O_+ \cap \{<, \leq\} \neq \emptyset$ *and* $\{=\} \subseteq O_-$, *then* $(\mathbb{N}, O_+, O_-)$ *is non-safe.*

Proof. In order to prove that the NDR is non-safe it suffices, from Lemma 1 to prove that it is not weakly convex. We provide restrictions (s_+, y), (s_-^1, z_1) and (s_-^2, z_2), such that $s_+ \in O_+$, s_-^1, $s_-^2 \in O_-$ and $(s_+, y) \rightarrow_{\mathbb{N}} (s_-^1, z_1) \vee (s_-^2, z_2)$, $(s_+, y) \not\rightarrow_{\mathbb{N}} (s_-^1, z_1)$, $(s_+, y) \not\rightarrow_{\mathbb{N}} (s_-^2, z_2)$ that consist a violation of the weak convexity condition. Table 6 provides the counterexamples; the first four, next four and last two lines refer to Lemma 4(a), 4(b) and 4(c) respectively. □

Lemma 5. *Every NDR in Table 4 is maximal safe, that is if any relation is added to O_+ or O_- it becomes non-safe.*

Proof. We examine all cases of adding a new relation to NDRs in Table 4:
NDR_1: If any of the $<$, $\leq$, $>$, $\geq$ is added to O_+, then NDR_1 becomes non-safe due to Lemma 4(a).
NDR_2: If $>$ or $\geq$ is added to O_-, then non-safety is due to Lemma 4(a). When $=$ is added to O_- then NDR_2 becomes non-safe due to Lemma 4(c).
NDR_3: If $<$ or $\leq$ is added to O_-, then non-safety is due to Lemma 4(a). When $=$ is added to O_- then NDR_3 becomes non-safe due to Lemma 4(c).
NDR_4: If $>$ or $\geq$ is added to O_-, then non-safety is due to Lemma 4(b). For adding $<$ or $\leq$ to O_+, non-safety is due to Lemma 4(c). □

It remains to be proved that every safe NDR is contained in some NDR in Table 4. In the following, we assume that O^i_+ and O^i_- are defined such that $\mathsf{NDR}_i = (\mathbb{N}, O^i_+, O^i_-)$ with $1 \leq i \leq 4$.

Lemma 6. *If $(\mathbb{N}, O_+, O_-)$ is a safe NDR, then $O_+ \subseteq O^i_+$ and $O_- \subseteq O^i_-$ for some i $(1 \leq i \leq 4)$.*

Proof. The proof is by case analysis of possible relations in O_+ and O_-.
Case 1: $O_+ \cap \{<, \leq, >, \geq\} = \emptyset$. In this case, $O_+ \subseteq O^1_+$ and $O_- \subseteq O^1_-$.
Case 2: $O_+ \cap \{<, \leq, >, \geq\} \neq \emptyset$. If $O_- \cap \{<, \leq\} \neq \emptyset$ and $O_- \cap \{>, \geq\} \neq \emptyset$ at the same time, then from Lemma 4(a), the NDR is non-safe. Therefore, we examine two cases: either $O_- \subseteq \{>, \geq, =\}$ or $O_- \subseteq \{<, \leq, =\}$.
Case 2.1: $O_- \subseteq \{>, \geq, =\}$. We further distinguish whether $O_- \subseteq \{>, \geq\}$ or $\{=\} \subseteq O_-$.
Case 2.1.1: $O_- \subseteq \{>, \geq\} = O^3_-$ and $O_+ \subseteq O^3_+$.
Case 2.1.2: $\{=\} \subseteq O_-$. By Lemma 4(c) it should be $O_+ \subseteq \{>, \geq, =\} = O^4_+$ otherwise the NDR is non-safe. If $O_- \cap \{>, \geq\} \neq \emptyset$ then the NDR is non-safe by Lemma 4(b); otherwise $O_- = \{=\} \subseteq O^4_-$.
Case 2.2: $O_- \subseteq \{<, \leq, =\} = O^4_-$. If $O_+ \subseteq \{>, \geq, =\}$, then $O_+ \subseteq O^4_+$. Otherwise, $O_+ \cap \{<, \leq\} \neq \emptyset$ and we distinguish cases whether $O_- \subseteq \{<, \leq\}$ or $\{=\} \in O_-$.
Case 2.2.1: $O_- \subseteq \{<, \leq\} = O^2_-$ and $O_+ \subseteq O^2_+$.
Case 2.2.2: $\{=\} \in O_-$. In this case, S is non-safe by Lemma 4(c). □

5 Maximal Safe NDRs for $\mathbb{Z}$

We now identify the maximal safe NDRs for the domain of integers.

Lemma 7. *Let C_1 and C_2 be as defined in Table 5 and $(\mathbb{Z}, O_+, O_-)$ be an NDR. Then (i) C_1 is a constraint over $(\mathbb{Z}, O_+, O_-)$ iff C_2 is a constraint over $(\mathbb{Z}, O_+, O_-)$ and (ii) if C_1 and C_2 are both constraints over $(\mathbb{Z}, O_+, O_-)$, then C_1 is satisfiable iff C_2 is satisfiable.*

Table 7. Transformations $C_1 \Rightarrow C_2$ preserving constraints and their satisfiability for $\mathbb{Z}$, where S_-, S_+ and S are sets of datatype restrictions and $y_1 \leq y_2$, $z_1 \leq z_2$

$C_1 = (S \cup S^1_+, S_-)$, $C_2 = (S \cup S^2_+, S_-)$		$C_1 = (S_+, S \cup S^1_-)$, $C_2 = (S_+, S \cup S^2_-)$	
S^1_+	S^2_+	S^1_-	S^2_-
$\{(<, y)\}$	$\{(\leq, y-1)\}$	$\{(<, z)\}$	$\{(\leq, z-1)\}$
$\{(>, y)\}$	$\{(\geq, y+1)\}$	$\{(>, z)\}$	$\{(\geq, z+1)\}$
$\{(\leq, y_1), (\leq, y_2)\}$	$\{(\leq, y_1)\}$	$\{(\leq, z_1), (\leq, z_2)\}$	$\{(\leq, z_2)\}$
$\{(\geq, y_1), (\geq, y_2)\}$	$\{(\geq, y_2)\}$	$\{(\geq, z_1), (\geq, z_2)\}$	$\{(\geq, z_1)\}$
$\{(=, y_1), (\leq, y_2)\}$	$\{(=, y_1)\}$	$\{(=, z_1), (\leq, z_2)\}$	$\{(\leq, z_2)\}$
$\{(\geq, y_1), (=, y_2)\}$	$\{(=, y_2)\}$	$\{(\geq, z_1), (=, z_2)\}$	$\{(\geq, z_1)\}$

Table 8. Maximal safe NDRs for $\mathbb{Z}$

NDR	O_+	O_-
NDR_1	$\{=\}$	$\{<, \leq, >, \geq, =\}$
NDR_2	$\{<, \leq, >, \geq, =\}$	$\{=\}$
NDR_3	$\{<, \leq, >, \geq, =\}$	$\{<, \leq\}$
NDR_4	$\{<, \leq, >, \geq, =\}$	$\{>, \geq\}$
NDR_5	$\{>, \geq, =\}$	$\{<, \leq, =\}$
NDR_6	$\{<, \leq, =\}$	$\{>, \geq, =\}$

Corollary 2. *Let* $\mathsf{NDR}_i = (\mathbb{Z}, O^i_+, O^i_-)$*, with* $1 \leq i \leq 6$*. For every* (S^1_+, S^1_-) *over* NDR_i *there exists a constraint* (S^2_+, S^2_-) *over* NDR_i*,* $y_1, \ldots, y_n \in \mathbb{Z}$ *and* $z_1, \ldots, z_m \in \mathbb{Z}$ *such that:*

$$S^2_+ \subseteq \{(\leq, y_1), (=, y_2), \ldots, (=, y_{n-1}), (\geq, y_n)\}$$
$$S^2_- \subseteq \{(\leq, z_1), (=, z_2), \ldots, (=, z_{m-1}), (\geq, z_m)\}$$

where $y_1 < \ldots < y_n$*,* $z_1 < \ldots < z_m$*,* $z_1 < y_1$*,* $z_m > y_n$ *,* $y_i \neq z_j$ *(*$2 \leq i \leq n-1$*,* $2 \leq j \leq m-1$*,* $m, n \geq 0$*) and* (S^1_+, S^1_-) *over* NDR_i *is satisfiable iff* (S^2_+, S^2_-) *over* NDR_i *is satisfiable.*

Table 8 provides the safe NDRs for integers. When we compare the results with Table 4 we notice two new maximal safe NDRs, namely NDR_2 and NDR_6. The reason is that integers do not have a minimal element such as 0 in the case of naturals. In particular positive occurrences of $<$ (or $\leq$) and negative occurrence of $=$ are no longer dangerous (e.g. $(\leq, 1) \rightarrow_{\mathbb{Z}} (=, 1) \vee (=, 0)$ does not hold anymore).

Lemma 8. *Every NDR in Table 8 is safe.*

Proof. We prove Lemma 8 by building a solution V for every constraint over NDRs in Table 8. By Corollary 2 we can assume w.l.o.g. the following restrictions for (S_+, S_-) and construct the corresponding solution V:

NDR_1: For S_+ we have that $S_+ \subseteq \{(=, y_1), \ldots, (=, y_n)\}$ and for S_- that $S_- \subseteq \{(\leq, z_1), (=, z_2), \ldots, (=, z_{m-1}), (\geq, z_m)\}$ with $y_1 < \ldots < y_n$, $z_1 < \ldots < z_m$ and $y_i \neq z_j$ $(1 \leq i \leq n$, $2 \leq j \leq m-1)$. $V = \{y_1, \ldots, y_n\}$.

NDR_2: For S_+ we have that $S_+ \subseteq \{(\leq, y_1), (=, y_2), \ldots, (=, y_{n-1}), (\geq, y_n)\}$ and for S_- that $S_- \subseteq \{(=, z_1), \ldots, (=, z_m)\}$ with $y_1 < \ldots < y_n$, $z_1 < \ldots < z_m$ and $y_i \neq z_j$ $(1 \leq j \leq m,\ 2 \leq i \leq n-1)$. If we set $y'_1 = \min(y_1, z_1) - 1$ and $y'_n = \max(y_n, z_m) + 1$, we have $V = \{y'_1, y_2, \ldots, y_{n-1}, y'_n\}$.

NDR_3: For S_+ we have that $S_+ \subseteq \{(\leq, y_1), (=, y_2), \ldots, (=, y_{n-1}), (\geq, y_n)\}$ and for S_- that $S_- \subseteq \{(\leq, z_1)\}$ with $z_1 < y_1 < \ldots < y_n$. $V = \{y_1, \ldots, y_n\}$.

NDR_4: For S_+ we have that $S_+ \subseteq \{(\leq, y_1), (=, y_2), \ldots, (=, y_{n-1}), (\geq, y_n)\}$ and for S_- that $S_- \subseteq \{(\geq, z_1)\}$ with $y_1 < \ldots < y_n < z_1$. $V = \{y_1, \ldots, y_n\}$.

NDR_5: For S_+ we have that $S_+ \subseteq \{(=, y_1), \ldots, (=, y_{n-1}), (\geq, y_n)\}$ and for S_- that $S_- \subseteq \{(\leq, z_1), (=, z_2), (=, z_3), \ldots, (=, z_m)\}$ with $y_1 < \ldots < y_n$, $z_1 < y_1$, $z_1 < \ldots < z_m$ and $y_i \neq z_j$ $(1 \leq i \leq n-1, 2 \leq j \leq m)$. $V = \{y_1, y_2, \ldots, y_{n-1}, y'_n\}$, where $y'_n = \max(y_n, z_m) + 1$.

NDR_6: For S_+ we have that $S_+ \subseteq \{(\leq, y_1), (=, y_2), \ldots, (=, y_n)\}$ and for S_- that $S_- \subseteq \{(=, z_1) \ldots, (=, z_{m-1}), (\geq, z_m)\}$ with $y_1 < \ldots < y_n$, $z_1 < \ldots < z_m$ and $y_n < z_m$, $y_i \neq z_j$ $(2 \leq i \leq n,\ 1 \leq j \leq m-1)$. $V = \{y'_1, y_2, \ldots, y_{n-1}, y_n\}$, where $y'_1 = \min(y_1, z_1) - 1$. □

Lemma 9. *Let $(\mathbb{Z}, O_+, O_-)$ be an NDR. Then:*

(a) If $O_+ \cap \{<, \leq, >, \geq\} \neq \emptyset$, $O_- \cap \{<, \leq\} \neq \emptyset$ and $O_- \cap \{>, \geq\} \neq \emptyset$, then $(\mathbb{Z}, O_+, O_-)$ is non-safe.

(b) If $O_+ \cap \{>, \geq\} \neq \emptyset$, $O_- \cap \{>, \geq\} \neq \emptyset$ and $\{=\} \subseteq O_-$, then $(\mathbb{Z}, O_+, O_-)$ is non-safe.

(c) If $O_+ \cap \{<, \leq\} \neq \emptyset$, $O_- \cap \{<, \leq\} \neq \emptyset$ and $\{=\} \subseteq O_-$, then $(\mathbb{Z}, O_+, O_-)$ is non-safe.

Proof. In order to prove that the NDR is non-safe it suffices, by Lemma 1 to prove that it is not weakly convex. We provide restrictions (s_+, y), (s^1_-, z_1) and (s^2_-, z_2), such that $s_+ \in O_+$, s^1_-, $s^2_- \in O_-$ and $(s_+, y) \rightarrow_{\mathbb{Z}} (s^1_-, z_1) \vee (s^2_-, z_2)$, $(s_+, y) \nrightarrow_{\mathbb{Z}} (s^1_-, z_1)$, $(s_+, y) \nrightarrow_{\mathbb{Z}} (s^2_-, z_2)$ that consist a violation of the weak convexity condition. Table 9 provides the counterexamples; the first four, following four and last four lines refer to Lemma 9(a), 9(b) and 9(c) respectively. □

Lemma 10. *Every NDR in Table 8 is maximal safe, that is if any relation is added to O_+ or O_- it becomes non-safe.*

Proof. We examine all cases of adding a new relation to NDRs in Table 8:
$\underline{\mathsf{NDR}_1}$: If any of the $<$, $\leq$, $>$, $\geq$ is added to O_+, then NDR_1 becomes non-safe due to Lemma 9(a).
$\underline{\mathsf{NDR}_2}$: If $>$ or $\geq$ is added to O_-, then non-safety is due to Lemma 9(b). When $<$ or $\leq$ is added to O_- then NDR_2 becomes non-safe due to Lemma 9(c).
$\underline{\mathsf{NDR}_3}$: If $>$ or $\geq$ is added to O_-, then non-safety is due to Lemma 9(a). When $=$ is added to O_- then NDR_3 becomes non-safe due to Lemma 9(c).
$\underline{\mathsf{NDR}_4}$: If $<$ or $\leq$ is added to O_-, then non-safety is due to Lemma 9(a). For adding $=$ to O_-, non-safety is due to Lemma 9(b).
$\underline{\mathsf{NDR}_5}$: If $>$ or $\geq$ is added to O_-, then non-safety is due to Lemma 9(a). When $<$ or $\leq$ is added to O_+ then NDR_5 becomes non-safe due to Lemma 9(c).

Table 9. Examples of non-safe NDRs for $\mathbb{Z}$ where $(s_+, y) \rightarrow_{\mathbb{Z}} (s_-^1, z_1) \vee (s_-^2, z_2)$, $(s_+, y) \not\rightarrow_{\mathbb{Z}} (s_-^1, z_1)$ and $(s_+, y) \not\rightarrow_{\mathbb{Z}} (s_-^2, z_2)$

$\{s_+\}$	$\{s_-^1, s_-^2\}$	y	z_1	z_2
$\{<\}$, $\{\leq\}$	$\{<, \geq\}$, $\{\leq, >\}$, $\{\leq, \geq\}$	3	1	1
$\{<\}$, $\{\leq\}$	$\{<, >\}$	3	2	1
$\{>\}$, $\{\geq\}$	$\{<, \geq\}$, $\{\leq, >\}$, $\{\leq, \geq\}$	1	3	3
$\{>\}$, $\{\geq\}$	$\{<, >\}$	1	3	2
$\{>\}$	$\{=, \geq\}$	1	2	3
$\{>\}$	$\{=, >\}$	1	2	2
$\{\geq\}$	$\{=, \geq\}$	1	1	2
$\{\geq\}$	$\{=, >\}$	1	1	1
$\{<\}$	$\{=, \leq\}$	3	2	1
$\{<\}$	$\{=, <\}$	3	2	2
$\{\leq\}$	$\{=, \leq\}$	2	2	1
$\{\leq\}$	$\{=, <\}$	2	2	2

NDR_6: If $<$ or $\leq$ is added to O_-, then non-safety is due to Lemma 9(a). For adding $>$ or $\geq$ to O_+, non-safety is due to Lemma 9(b). □

In the following, we assume that O_+^i and O_-^i are defined such that $\mathsf{NDR}_i = (\mathbb{N}, O_+^i, O_-^i)$ with $1 \leq i \leq 6$.

Lemma 11. *If $(\mathbb{N}, O_+, O_-)$ is a safe NDR, then $O_+ \subseteq O_+^i$ and $O_- \subseteq O_-^i$ for some i ($1 \leq i \leq 6$).*

Proof. The proof is by case analysis of possible relations in O_+ and O_-.

Case 1: $O_+ \cap \{<, \leq, >, \geq\} = \emptyset$. In this case, $O_+ \subseteq O_+^1$ and $O_- \subseteq O_-^1$.

Case 2: $O_+ \cap \{<, \leq, >, \geq\} \neq \emptyset$. If $O_- \cap \{<, \leq\} \neq \emptyset$ and $O_- \cap \{>, \geq\} \neq \emptyset$ at the same time, then from Lemma 9(a), the NDR is non-safe. Therefore, only two cases are possible: either $O_- \subseteq \{>, \geq, =\}$ or $O_- \subseteq \{<, \leq, =\}$.

Case 2.1: $O_- \subseteq \{>, \geq, =\}$. We further distinguish on whether $O_- \subseteq \{>, \geq\}$ or $\{=\} \subseteq O_-$.

Case 2.1.1: $O_- \subseteq \{>, \geq\} = O_-^4$ and $O_+ \subseteq O_+^4$.

Case 2.1.2: $\{=\} \subseteq O_-$. If $O_- = \{=\}$, then $O_- \subseteq O_-^2$ and $O_+ \subseteq O_+^2$. Otherwise, $O_- \cap \{>, \geq\} \neq \emptyset$. We examine two cases: either $O_+ \cap \{>, \geq\} \neq \emptyset$ or $O_+ \subseteq \{<, \leq, =\}$.

Case 2.1.2.1: $O_+ \cap \{>, \geq\} \neq \emptyset$. In this case by 9(b) the NDR is non-safe.

Case 2.1.2.2: $O_+ \subseteq \{<, \leq, =\} = O_+^6$ and $O_- \subseteq O_-^6$.

Case 2.2: $O_- \subseteq \{<, \leq, =\}$. We further distinguish on whether $O_- \subseteq \{<, \leq\}$ or $\{=\} \subseteq O_-$.

Case 2.2.1: $O_- \subseteq \{<, \leq\} = O_-^3$ and $O_+ \subseteq O_+^3$.

Case 2.2.2: $\{=\} \subseteq O_-$. If $O_- = \{=\}$, then $O_- \subseteq O_-^2$ and $O_+ \subseteq O_+^2$. Otherwise, $O_- \cap \{<, \leq\} \neq \emptyset$. We examine two cases: either $O_+ \cap \{<, \leq\} \neq \emptyset$ or $O_+ \subseteq \{>, \geq, =\}$.

Case 2.2.2.1: $O_+ \cap \{<, \leq\} \neq \emptyset$. In this case by 9(c) the NDR is non-safe.

Case 2.2.2.2: $O_+ \subseteq \{>, \geq, =\} = O_+^5$ and $O_- \subseteq O_-^5$.

□

Table 10. Transformations $C_1 \Rightarrow C_2$ preserving constraints and their satisfiability for $\mathbb{R}$, where S_-, S_+ and S are sets of datatype restrictions, $y_1 \leq y_2$, $z_1 \leq z_2$, $C_1 \cup C_2 = \{(s_i, x_i)\}_{i=1}^k$ and $\epsilon = \min\{|c-d| \mid c \neq d, c, d \in \{x_i\}_{i=1}^k\}/2$

$C_1 = (S \cup S_+^1, S_-)$, $C_2 = (S \cup S_+^2, S_-)$		$C_1 = (S_+, S \cup S_-^1)$, $C_2 = (S_+, S \cup S_-^2)$	
S_+^1	S_+^2	S_-^1	S_-^2
$\{(<, y)\}$	$\{(\leq, y-\epsilon)\}$	$\{(<, z)\}$	$\{(\leq, z-\epsilon)\}$
$\{(>, y)\}$	$\{(\geq, y+\epsilon)\}$	$\{(>, z)\}$	$\{(\geq, z+\epsilon)\}$
$\{(\leq, y_1), (\leq, y_2)\}$	$\{(\leq, y_1)\}$	$\{(\leq, z_1), (\leq, z_2)\}$	$\{(\leq, z_2)\}$
$\{(\geq, y_1), (\geq, y_2)\}$	$\{(\geq, y_2)\}$	$\{(\geq, z_1), (\geq, z_2)\}$	$\{(\geq, z_1)\}$
$\{(=, y_1), (\leq, y_2)\}$	$\{(=, y_1)\}$	$\{(=, z_1), (\leq, z_2)\}$	$\{(\leq, z_2)\}$
$\{(\geq, y_1), (=, y_2)\}$	$\{(=, y_2)\}$	$\{(\geq, z_1), (=, z_2)\}$	$\{(\geq, z_1)\}$

Table 11. Maximal safe NDRs for $\mathbb{R}$ and $\mathbb{Q}$

NDR	O_+	O_-
NDR_1	$\{=\}$	$\{<, \leq, >, \geq, =\}$
NDR_2	$\{<, \leq, >, \geq, =\}$	$\{\leq, =\}$
NDR_3	$\{<, \leq, >, \geq, =\}$	$\{\geq, =\}$
NDR_4	$\{<, \leq, >, \geq, =\}$	$\{<, \leq\}$
NDR_5	$\{<, \leq, >, \geq, =\}$	$\{>, \geq\}$
NDR_6	$\{<, >, \geq, =\}$	$\{<, \leq, =\}$
NDR_7	$\{<, \leq, >, =\}$	$\{>, \geq, =\}$

6 Maximal Safe NDRs for $\mathbb{R}$ and $\mathbb{Q}$

We continue with the domain of real numbers ($\mathbb{R}$) which does not differ from the set of natural numbers ($\mathbb{Q}$).

Lemma 12. *Let C_1 and C_2 be as defined in Table 5 and $(\mathbb{R}, O_+, O_-)$ be an NDR. Then (i) C_1 is a constraint over $(\mathbb{R}, O_+, O_-)$ iff C_2 is a constraint over $(\mathbb{R}, O_+, O_-)$ and (ii) if C_1 and C_2 are both constraints over $(\mathbb{R}, O_+, O_-)$, then C_1 is satisfiable iff C_2 is satisfiable.*

Corollary 3. *Let $\mathsf{NDR}_i = (\mathbb{R}, O_+^i, O_-^i)$, with $1 \leq i \leq 7$. For every (S_+^1, S_-^1) over NDR_i there exists a constraint (S_+^2, S_-^2) over NDR_i, $y_1, \ldots, y_n \in \mathbb{R}$ and $z_1, \ldots, z_m \in \mathbb{R}$ such that:*

$$S_+^2 \subseteq \{(\leq, y_1), (=, y_2), \ldots, (=, y_{n-1}), (\geq, y_n)\}$$
$$S_-^2 \subseteq \{(\leq, z_1), (=, z_2), \ldots, (=, z_{m-1}), (\geq, z_m)\}$$

where $y_1 < \ldots < y_n$, $z_1 < \ldots < z_m$, $z_1 < y_1$, $z_m > y_n$, $y_i \neq z_j$ ($2 \leq i \leq n-1$, $2 \leq j \leq m-1$, m, $n \geq 0$) and (S_+^1, S_-^1) over NDR_i is satisfiable iff (S_+^2, S_-^2) over NDR_i is satisfiable.

Table 11 presents the maximal safe NDRs for reals, which are the same for rationals. Reals and rationals are examples of dense domains: between every two different numbers there always exists a third one. This property is responsible for new safe NDRs. Specifically, either $\leq$ or $\geq$ can be added to O_-

Table 12. Examples of non-safe NDRs for $\mathbb{R}$ where $(s_+, y) \rightarrow_{\mathbb{R}} (s_-^1, z_1) \vee (s_-^2, z_2)$, $(s_+, y) \not\rightarrow_{\mathbb{R}} (s_-^1, z_1)$ and $(s_+, y) \not\rightarrow_{\mathbb{R}} (s_-^2, z_2)$

$\{s_+\}$	$\{s_-^1, s_-^2\}$	y	z_1	z_2
$\{<\}$, $\{\leq\}$	$\{<, \geq\}$, $\{\leq, >\}$, $\{\leq, \geq\}$	3	1	1
$\{<\}$, $\{\leq\}$	$\{<, >\}$	3	2	1
$\{>\}$, $\{\geq\}$	$\{<, \geq\}$, $\{\leq, >\}$, $\{\leq, \geq\}$	1	3	3
$\{>\}$, $\{\geq\}$	$\{<, >\}$	1	3	2
$\{\geq\}$	$\{=, >\}$	1	1	1
$\{\leq\}$	$\{=, <\}$	1	1	1

of NDR_2 from Table 8 because it does not violate the weak convexity property (e.g. $(\leq, 5) \not\rightarrow_{\mathbb{R}} (=, 5) \vee (\leq, 4)$). For the same reason, O_+ of NDR_5 and NDR_6 from Table 8 can be extended with $<$ and $>$ respectively because the weak convexity property which did not apply for $\mathbb{Z}$ now applies for $\mathbb{R}$ (e.g. $(<, 5) \not\rightarrow_{\mathbb{R}} (=, 4) \vee (\leq, 3)$).

Lemma 13. *Every NDR in Table 11 is safe.*

Proof. We prove Lemma 13 by building a solution V for every constraint over NDRs in Table 11. By Corollary 3 we can assume w.l.o.g. the following restrictions for (S_+, S_-) and construct the corresponding solution V:

NDR_1: For S_+ we have that $S_+ \subseteq \{(=, y_1), \ldots, (=, y_n)\}$ and for S_- that $S_- \subseteq \{(\leq, z_1), (=, z_2), \ldots, (=, z_{m-1}), (\geq, z_m)\}$ with $y_1 < \ldots < y_n$, $z_1 < \ldots < z_m$ and $y_i \neq z_j$ $(1 \leq i \leq n,\ 2 \leq j \leq m-1)$. $V = \{y_1, \ldots, y_n\}$.

NDR_2: For S_+ we have that $S_+ \subseteq \{(\leq, y_1), (=, y_2), \ldots, (=, y_{n-1}), (\geq, y_n)\}$ and for S_- that $S_- \subseteq \{(\leq, z_1), (=, z_2), \ldots, (=, z_m)\}$ with $y_1 < \ldots < y_n$, $z_1 < y_1$, $z_1 < \ldots < z_m$ and $y_i \neq z_j$ $(2 \leq j \leq m,\ 2 \leq i \leq n-1)$. $V = \{y_1, \ldots, y_{n-1}, y_n'\}$, where $y_1' = y_1 - \epsilon/2$ and $y_n' = \max(y_n, z_1) + \epsilon$.

NDR_3: For S_+ we have that $S_+ \subseteq \{(\leq, y_1), (=, y_2), \ldots, (=, y_{n-1}), (\geq, y_n)\}$ and for S_- that $S_- \subseteq \{(=, z_1), \ldots, (=, z_{m-1}), (\geq, z_m)\}$ with $y_1 < \ldots < y_n$, $y_n < z_m$ $z_1 < \ldots < z_m$ and $y_i \neq z_j$ $(1 \leq j \leq m-1,\ 2 \leq i \leq n-1)$. If we set $y_1' = \min(y_1, z_m) - \epsilon$ and $y_n' = y_n + \epsilon/2$, we have $V = \{y_1', y_2, \ldots, y_{n-1}, y_n'\}$.

NDR_4: For S_+ we have that $S_+ \subseteq \{(\leq, y_1), (=, y_2), \ldots, (=, y_{n-1}), (\geq, y_n)\}$ and for S_- that $S_- \subseteq \{(\leq, z_1)\}$ with $z_1 < y_1 < \ldots < y_n$. $V = \{y_1, \ldots, y_n\}$.

NDR_5: For S_+ we have that $S_+ \subseteq \{(\leq, y_1), (=, y_2), \ldots, (=, y_{n-1}), (\geq, y_n)\}$ and for S_- that $S_- \subseteq \{(\geq, z_1)\}$ with $y_1 < \ldots < y_n < z_1$. $V = \{y_1, \ldots, y_n\}$.

NDR_6: For S_+ we have that $S_+ \subseteq \{(\leq y_1), (=, y_2), \ldots, (=, y_{n-1}), (\geq, y_n)\}$ and for S_- that $S_- \subseteq \{(\leq, z_1), (=, z_2), (=, z_3), \ldots, (=, z_m)\}$ with $y_1 < \ldots < y_n$, $z_1 < y_1$, $z_1 < \ldots < z_m$ and $y_i \neq z_j$ $(2 \leq i \leq n-1,\ 2 \leq j \leq m)$. We set $y_n' = \max(y_n, z_m) + \epsilon$ and $V = \{y_1, \ldots, y_{n-1}, y_n'\}$.

NDR_7: For S_+ we have that $S_+ \subseteq \{(\leq, y_1), (=, y_2), \ldots, (=, y_{n-1}), (\geq, y_n)\}$ and for S_- that $S_- \subseteq \{(=, z_1), \ldots, (=, z_{m-1}), (\geq, z_m)\}$ with $y_1 < \ldots < y_n$, $z_1 < \ldots < z_m$ and $y_n < z_m$, $y_i \neq z_j$ $(2 \leq i \leq n-1,\ 1 \leq j \leq m-1)$. $V = \{y_1', y_2, \ldots, y_n\}$, where $y_1' = \min(y_1, z_1) - \epsilon$. □

Lemma 14. *Let* $(\mathbb{R}, O_+, O_-)$ *be an NDR. Then:*

(a) If $O_+ \cap \{<, \leq, >, \geq\} \neq \emptyset$*,* $O_- \cap \{<, \leq\} \neq \emptyset$ *and* $O_- \cap \{>, \geq\} \neq \emptyset$*, then* $(\mathbb{R}, O_+, O_-)$ *is non-safe.*
(b) If $\{\geq\} \in O_+$ *and* $O_- \cap \{>, =\} \neq \emptyset$*, then* $(\mathbb{R}, O_+, O_-)$ *is non-safe.*
(c) If $\{\leq\} \in O_+$ *and* $O_- \cap \{<, =\} \neq \emptyset$*, then* $(\mathbb{R}, O_+, O_-)$ *is non-safe.*

Proof. In order to prove that the NDR is non-safe it suffices, from Lemma 1 to prove that it is not weakly convex. We provide restrictions (s_+, y), (s_-^1, z_1) and (s_-^2, z_2), such that $s_+ \in O_+$, s_-^1, $s_-^2 \in O_-$ and $(s_+, y) \rightarrow_{\mathbb{R}} (s_-^1, z_1) \vee (s_-^2, z_2)$, $(s_+, y) \not\rightarrow_{\mathbb{R}} (s_-^1, z_1)$, $(s_+, y) \not\rightarrow_{\mathbb{R}} (s_-^2, z_2)$ that consist a violation of the weak convexity condition. Table 12 provides the counterexamples; the first four, fifth and sixth line(s) refer to Lemma 14(a), 14(b) and 14(c) respectively. □

Lemma 15. *Every NDR in Table 11 is maximal safe, that is if any relation is added to* O_+ *or* O_- *it becomes non-safe.*

Proof. We examine all cases of adding a new relation to NDRs in Table 11:
$\underline{\mathsf{NDR}_1}$: If any of the $<$, $\leq$, $>$, $\geq$ is added to O_+, then NDR_1 becomes non-safe due to Lemma 14(a).
$\underline{\mathsf{NDR}_2}$: If $\geq$ is added to O_-, then non-safety is due to Lemma 14(a). When $>$ is added to O_- then non-safety is due to Lemma 14(b). Finally, if $<$ is added to O_- NDR_2 becomes non-safe due to Lemma 14(c).
$\underline{\mathsf{NDR}_3}$: If $\leq$ is added to O_-, then non-safety is due to Lemma 14(a). When $>$ is added to O_- then non-safety is due to Lemma 14(b). Finally, if $<$ is added to O_- then NDR_3 becomes non-safe due to Lemma 14(c).
$\underline{\mathsf{NDR}_4}$: If $>$ or $\geq$ is added to O_-, then non-safety is due to Lemma 14(a). For adding $=$ to O_-, non-safety is due to Lemma 14(c).
$\underline{\mathsf{NDR}_5}$: If $<$ or $\leq$ is added to O_-, then non-safety is due to Lemma 14(a). When $=$ is added to O_- then NDR_5 becomes non-safe due to Lemma 14(b).
$\underline{\mathsf{NDR}_6}$: If $>$ or $\geq$ is added to O_-, then non-safety is due to Lemma 14(a). For adding $\leq$ to O_+, non-safety is due to Lemma 14(c).
$\underline{\mathsf{NDR}_7}$: If $<$ or $\leq$ is added to O_-, then non-safety is due to Lemma 14(a). For adding $\geq$ to O_+, non-safety is due to Lemma 14(b).

□

In the following, we assume that O_+^i and O_-^i are defined such that $\mathsf{NDR}_i = (\mathbb{N}, O_+^i, O_-^i)$ with $1 \leq i \leq 7$.

Lemma 16. *If* $(\mathbb{R}, O_+, O_-)$ *is a safe NDR, then* $O_+ \subseteq O_+^i$ *and* $O_- \subseteq O_-^i$ *for some* i *(*$1 \leq i \leq 7$*).*

Proof. The proof is by case analysis of possible relations in O_+ and O_-.

<u>Case 1:</u> $O_+ \cap \{<, \leq, >, \geq\} = \emptyset$. In this case, $O_+ \subseteq O_+^1$ and $O_- \subseteq O_-^1$.

<u>Case 2:</u> $O_+ \cap \{<, \leq, >, \geq\} \neq \emptyset$. If $O_- \cap \{<, \leq\} \neq \emptyset$ and $O_- \cap \{>, \geq\} \neq \emptyset$ at the same time, then from Lemma 14(a), the NDR is non-safe. Therefore, we examine two cases: either $O_- \subseteq \{>, \geq, =\}$ or $O_- \subseteq \{<, \leq, =\}$.

Case 2.1: $O_- \subseteq \{>, \geq, =\}$. We further examine whether $O_- \subseteq \{>, \geq\}$ or $\{=\} \subseteq O_-$.

Case 2.1.1: $O_- \subseteq \{>, \geq\} = O_-^5$ and $O_+ \subseteq O_+^5$.

Case 2.1.2: $\{=\} \subseteq O_-$. We distinguish two cases: either $O_- \subseteq \{\geq, =\}$ or $O_- \cap \{>\} \neq \emptyset$.

Case 2.1.2.1: $O_- \subseteq \{\geq, =\}$. In this case $O_+ \subseteq O_+^3$ and $O_- \subseteq O_-^3$.

Case 2.1.2.2: $O_- \cap \{>, =\} \neq \emptyset$. If $O_+ \cap \{\geq\} \neq \emptyset$, then the NDR is non-safe from 14(b). Therefore, $O_+ \subseteq \{<, >, \geq, =\} = O_+^7$ and $O_- \subseteq \{>, \geq, =\} = O_-^7$.

Case 2.2: $O_- \subseteq \{<, \leq, =\}$. We further examine whether $O_- \subseteq \{<, \leq\}$ or $\{=\} \subseteq O_-$.

Case 2.2.1: $O_- \subseteq \{<, \leq\} = O_-^4$ and $O_+ \subseteq O_+^4$.

Case 2.2.2: $\{=\} \subseteq O_-$. We distinguish two cases: either $O_- \subseteq \{\leq, =\}$ or $O_- \cap \{<\} \neq \emptyset$.

Case 2.2.2.1: $O_- \subseteq \{\leq, =\}$. In this case $O_+ \subseteq O_+^2$ and $O_- \subseteq O_-^2$.

Case 2.2.2.2: $O_- \cap \{<\} \neq \emptyset$. If $O_+ \cap \{\leq\} \neq \emptyset$, then the NDR is non-safe from 14(c). Therefore, $O_+ \subseteq \{<, >, \geq, =\} = O_+^6$ and $O_- \subseteq \{<, \leq, =\} = O_-^6$. □

7 Related Work

Datatypes have been extensively studied in the context of DLs [3, 7, 8]. Extensions of expressive DLs with datatypes have been examined in depth [7] with the main focus on decidability. Baader, Brandt and Lutz [3] formulated tractable extensions of $\mathcal{EL}$ with datatypes using a p-admissibility restriction for datatypes. A datatype $\mathcal{D}$ is p-admissible if (i) satisfiability and implication of conjunctions of datatype restrictions can be decided in polynomial time, and (ii) $\mathcal{D}$ is convex: if a conjunction of datatype restrictions implies a disjunction of datatype restrictions then it also implies on of its disjuncts [3]. In our case instead of condition (i) we require that implication and satisfiability of just datatype restrictions (not conjunctions since we do not consider functional features) is decidable in polynomial time. Condition (ii) is relaxed to the requirement of safety for NDRs since we take into account not only the domain of the datatypes and the types of restrictions but also the polarity of their occurrences. The relaxed restrictions allow for more expressive usage of datatypes in tractable languages, as demonstrated by the example given in the introduction. Furthermore, Baader, Brandt and Lutz did not provide a classification of datatypes that are p-admissible; in our case we provide such a classification for natural numbers, integers, rationals and reals. The EL Profile of OWL 2 [2] is inspired by $\mathcal{EL}^{++}$ and restricts all OWL 2 datatypes to satisfy p-admissibility. In particular, only equality can be used in datatype restrictions. Our result can allow for a significant extension of datatypes in the OWL 2 EL Profile, where in addition inequalities can be used negatively.

Our work is not the only one where the convexity property is relaxed without losing tractability. It has been shown [8] that the convexity requirement is not necessary provided that (i) the ontology contains only concept definitions of the form $A \equiv C$, where A is a concept name, and (ii) every concept name occurs at most once in the left-hand side of the definition. In some applications this requirement can be too restrictive since it disallows the usage of general concept inclusion axioms (GCIs), such as the axiom (2) given in the introduction, which do not cause any problem in our case.

8 Conclusions and Future Work

In this work we made a fine-grained analysis of extensions of $\mathcal{EL}$ with numerical datatypes, focusing not only on the types of relations but also on the polarities of their occurrences in axioms. We made a full classification of cases where these restrictions result in a tractable extension for natural numbers, integers and reals. One practically relevant case for these datatypes is when positive occurrences of datatype expressions can only use equality and negative occurrences can use any of the numerical relations considered. This case was motivated by an example of a pharmacy-related ontology, and can be proposed as a candidate for a future extension of the OWL 2 EL Profile. For the cases where the extension is tractable, we provided a polynomial sound and complete consequence-based reasoning procedure, which can be seen as an extension of the completion-based procedure for $\mathcal{EL}$. We think that the procedure can be straightforwardly extended to accommodate other constructors in $\mathcal{EL}^{++}$ such as (complex) role inclusions, nominals, domain and range restrictions and assertions, since these constructors do not interact with datatypes [9]. We hope to investigate these extensions in future works.

In future work we also plan to consider other OWL datatypes, such as strings, binary data or date and time, functional features, and to try to extend the consequence-based procedure for Horn $\mathcal{SHIQ}$ [10] with our rules for datatypes. For example, to extend the procedure with functional features, we probably need a notion of "functional safety" for an NDR that corresponds to the strong convexity property (see Definition 7). In order to achieve even higher expressivity for datatypes we shall study how to combine different restrictions on the datatypes occurring in an ontology so that tractability is preserved. For example, using two safe NDRs in a single ontology may result in intractability, as is the case for NDR_1 and NDR_2 for integers (see Table 8). One possible solution to this problem is to specify explicitly which features can be used with which NDRs in order to separate their usage in ontologies.

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