Computing Laboratory

PHASE GROUPS AND LOCAL HIDDEN VARIABLES

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Abstract

This report extends previous work [3] in which quantum mechanics and a quantum-like theory (Spekkens's toy bit theory [5]) were compared within the framework of symmetric monoidal categories. In this context, each quantum-like theory is naturally associated with an Abelian group, termed its *phase group*. Quantum mechanics and the toy bit theory exhibit different phase groups, Z_4 and $Z_2 \times Z_2$ respectively, and it was shown that this difference exactly underlies the fact that while the predictions of the toy theory can be modelled by local hidden variables, those of the stabiliser theory cannot. In this report we extend this work to more general groups: given a quantum-like theory with some phase group we derive a group theoretic criterion which determines whether a local hidden variable interpretation is impossible for the theory. This result is essentially a generalisation of Mermin's famous no-go theorem [4] employing the GHZ state.

1 Introduction

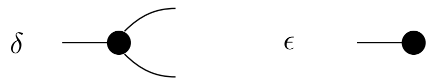
Basis structures (commutative isometric dagger Frobenius comonoids) arise in the categories associated with several quantum-like theories, where they provide the abstract counterparts of orthonormal bases, and are thus associated with the measurement of observables. Every basis structure has a corresponding Abelian group, termed its *phase group*. Previous work [3] investigated the categories **Stab** and **Spek** which correspond respectively to qubit stabiliser quantum mechanics, and the toy bit theory proposed by Rob Spekkens [5]. The two categories exhibit different phase groups, Z_4 and $Z_2 \times Z_2$ respectively. It was shown that this difference exactly underlies the fact that while the predictions of the toy theory can be modelled by local hidden variables, those of the stabiliser theory cannot.

This work attempts to extend this result to more general phase groups. Whilst it does not succeed in encompassing all possible phase groups, it does extend the result to a large class, many of which might be expected to occur in the categories corresponding to theories of interest. We begin in section 1 by reviewing the key categorical structures which will be used in the paper, and then move on to explicitly characterise which phase groups are covered by the new result, in section 2. Section 3 shows how the issue of local hidden variable theories is treated in the abstract categorical setting. Sections 4 and 5 lay the groundwork for the main result which is described in section 6. This result is essentially a generalisation of Mermin's famous no-go theorem [4] employing the GHZ state. Section 7 makes a link between the main result of the paper and the subject of *group extensions*. Appendix A illustrates the results of the paper using a concrete example. Consideration of this example may aid comprehension of the proof of the general result.

2 Review of basis structures and phase groups

Here we briefly review the key categorical structures used in the paper. For full details the reader is directed to [2].

Definition 2.1 *In a †-SMC a* basis structure Δ *on an object A is a commutative isometric dagger Frobenius comonoid* $(A, \delta : A \to A \otimes A, \epsilon : A \to I)$*. For more details on this definition* *see section 4 of [2] where basis structures are referred to as 'observable structures'. We represent the morphisms δ and ϵ graphically as:*



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We will frequently use the notation $\mathcal{B}_A$ to denote the set of basis structures on the object A.

Definition 2.2 *A morphism $x : I \to A$ is an* eigenstate *of a basis structure $\Delta = (A, \delta, \epsilon)$ iff it satisfies the following conditions:*

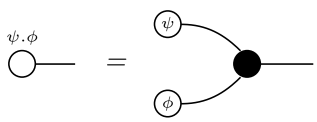
$$\delta \circ x = x \otimes x \quad x = x_* \quad \epsilon \circ x = 1_I \tag{2}$$

Essentially eigenstates are 'copied' by the δ morphism. We will frequently use the notation C_Δ to denote the set of eigenstates of the basis structure Δ.

Definition 2.3 *Given a basis structure $\Delta = (A, \delta, \epsilon)$ in a †-SMC $\mathcal{C}$, the* basis structure multiplication *is a map:*

$$-.- : \mathcal{C}(I, A) \times \mathcal{C}(I, A) \to \mathcal{C}(I, A) \quad \textit{where} \quad \psi.\phi = \delta^\dagger \circ (\psi \otimes \phi) \tag{3}$$

or diagrammatically:



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It can be shown [2] from the defining properties of a basis structure that $(\mathcal{C}(I, A), -.-, \epsilon^\dagger)$ is a commutative monoid. We refer to this as the basis structure monoid *corresponding to Δ.*

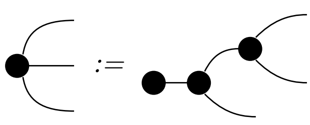
Definition 2.4 *Given a basis structure $\Delta = (A, \delta, \epsilon)$ a state $\psi : I \to A$ is* unbiased *with respect to Δ iff. $\psi.\psi_* = \epsilon^\dagger$. We will frequently use the notation U_Δ to denote the set of unbiased states of the basis structure Δ.*

Definition 2.5 *It can be shown [2] that $(U_\Delta, -.-, \epsilon^\dagger, (-)_*)$ is an Abelian sub-group of the basis structure monoid. We refer to this group as the* phase group *of Δ.*

Definition 2.6 *In a †-SMC, the* GHZ state *$\Psi_\Delta : I \to A \otimes A \otimes A$ corresponding to the basis structure $\Delta = (A, \delta, \epsilon)$ is the composition:*

$$\Psi_\Delta := (\delta \otimes 1_A) \circ \delta \circ \epsilon^\dagger \tag{5}$$

or graphically:



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3 Domain of applicability

By the *process category* of a theory we mean the category whose objects correspond to the systems of the theory, and whose morphisms correspond to the processes which can be undergone by these systems. Such a category is naturally symmetric monoidal. In this paper we will require additional structure on our process category.

1. The process category must have a dagger functor and basis structures on all of its objects. It should also have zero morphisms.
2. These formal mathematical features of the process category must relate to the theory in specific ways. Basis structures correspond to observables in the theory. The eigenstates e_i of a basis structure Δ are in bijection with the outcomes of a measurement of this observable. Given a system prepared in a state $\psi : I \to A$, if $e_i^\dagger \circ \psi = 0_{I,I}$, then the corresponding measurement outcome has zero probability. e_i is described as a *forbidden outcome with respect to* ψ.
3. Given e_i and e_j, *distinct* eigenstates of the same basis structure, we require that $e_i^\dagger \circ e_j = 0_{I,I}$. One can show that this follows from requiring that there are only two idempotent scalars in the category, $1_{I,I}$ and $0_{I,I}$.

One can motivate these features to some extent by physical considerations, but we will not discuss that here. It should be noted that not all theories of interest (perhaps not even most) will have process categories with these features. An example is *boxworld*, a theory proposed in [1], which exhibits a greater degree of non-locality than quantum mechanics; its process category has no dagger functor or basis structures. That said, there are theories of interest which do satisfy these conditions, for example quantum mechanics, and the toy bit theory due to Spekkens [5].

Furthermore, our analysis applies only to phase groups satisfying two conditions (1) the *observable-coset* condition and (2) the *quotient-sub-periood (QSP)* condition.

Definition 3.1 *A phase group U_Δ satisfies the* observable-coset condition *if:*

1. *It has a sub-group consisting of all the eigenstates of some other basis structure Δ'. This is termed the* observable sub-group*, and will be denoted by C_0.*
2. *The cosets of the observable sub-group each themselves consist of all of the eigenstates of some basis structure. These are termed* observable cosets*, and will be denoted by C_i.*

Definition 3.2 *The* observable quotient group *is the group U_Δ / C_0. Its elements are the observable cosets.*

The observable-coset condition is clearly a categorical property of the process category, rather than a group theoretic property. For example it is impossible to say whether the group Z_4 has the observable coset property. However, we can say that the Z_4 phase group appearing in the elementary object basis structure in **Stab** does have the observable coset property.

Definition 3.3 *A phase group U_Δ which satisfies the observable-coset condition will additionally satisfy the* quotient-sub-period *condition, or* QSP *condition, if for any $a \in C_0$, $a^{|C_Q|} = e$.*

The observable-coset condition is key, in that many of the definitions that follow are only well-defined when it holds (indeed the QSP condition just stated only makes sense when the observable-coset condition holds). Most of the definitions which follow in the next few sections still make sense for a phase group which does not satisfy the QSP condition. However, our main result will only apply to those which do satisfy it. Throughout the following sections we will assume that we are dealing with a phase group U_Δ, which satisfies the observable-coset property, and corresponds to a basis structure $\Delta = (A, \delta, \epsilon)$. We will explicitly note when we require a phase group to satisfy the QSP condition.

4 Abstract local hidden variables

We aim to address the issue of local hidden variables in the abstract categorical arena. To pursue this we will need to develop an abstract notion of hidden variable, and equally we will need some idea of what it means for hidden variables to be local. Before going any further we first need to introduce the concept of *observable structures.* Whilst in the theories we are considering every basis structure corresponds to a unique observable, the converse is not true in general: usually there will be multiple basis structures corresponding to the same observable. This is because it is possible for different basis structures to have the same eigenstates, and it is really these which identify a basis structure with a given observable. For this reason, we partition $\mathcal{B}_A$ (the set of basis structures on A) into classes whose members all have the same eigenstates. Such classes are termed *observable structures.* They are conventionally represented by a capital omega Ω, differentiated by subscripts if necessary, and we denote the set of observable structures on an object A by $\mathcal{O}_A$.

We can now proceed to give an abstract account of hidden variables.

Definition 4.1 *The* hidden state space *of an object A is the set $\Xi_A = \prod_{\Omega_i \in \mathcal{O}_A} C_{\Omega_i}$.*

Each element $h \in \Xi_A$ is termed a *hidden state.* Essentially a hidden state is a list of one outcome for all measurements which can be made on the system - it should be interpreted as representing a definite set of values for each of the system's observables. Note that every object in a process category with basis structures has a hidden state space, regardless of whether or not the accompanying theory has a hidden variable interpretation.

We take a rather abstract view of locality, and simply say that in a local theory we should be able to distinguish distinct systems, and that the choice of what observable to measure on one system should make no difference to the outcome of measurements on another system. We include no reference to spatio-temporal concepts. The monoidal product will naturally represent a compound system. However, we need to make the assumption that any object A in the process category has a unique monoidal decomposition $A = A_1 \otimes \cdots \otimes A_n$ into objects $A_1, \ldots, A_n$ which represent elementary systems; this will not necessarily be reflected in the categorical structure. Those objects for which $n > 1$ will be described as *composite*, those for which $n = 1$ as *elementary.*

Definition 4.2 *The* local hidden state space (LHSS) *of a composite object* $A = A_1 \otimes \cdots \otimes A_n$ *is the set* $\Lambda_A = \prod_i \Xi_{A_i}$*. The elements of* Λ_A *are termed* hidden states*.*

The hidden states of an LHSS are tuples of values. However it will be convenient to index the components of the tuple with two labels: h^i_j. This represents the value of observable Ω_j on the constituent system A_i.

Definition 4.3 *Each* $h \in \Lambda_A$ *induces a* value function *on each constituent object* A_i*:*

$$v^i_h : \mathcal{O}_{A_i} \to \mathcal{R}_{A_i} :: \Omega_j \mapsto h^i_j \tag{7}$$

Definition 4.4 *An* observable n-tuple *on a composite object* $A = A_1 \otimes \cdots \otimes A_n$ *is a tuple of observable structures* $(\Omega_1, \ldots, \Omega_n)$ *where* $\Omega_i \in \mathcal{O}_{A_i}$*. An* outcome n-tuple *corresponding to this observable n-tuple is a tuple of eigenstates* $(x_1, x_2, \ldots, x_n)$ *where* $x_1 \in C_{\Omega_i}$*.*

Evidently the hidden states of a local hidden state space map observable n-tuples into outcome n-tuples, via the value functions.

Definition 4.5 *A* local hidden state distribution (LHSD) *over a local hidden state space* Λ *is a* σ*-additive measure* $\mu : \mathcal{B}(\Lambda) \to \mathbb{R}$*, such that* $\mu(\Lambda) = 1$*.*

A LHSD μ allows us to calculate the probabilities of the different outcomes for measurement of any tuple of observables, via the following prescription.

$$\mathrm{prob}_\mu(x_1, \ldots, x_n) = \mu(\{h \in \Lambda | v^1_h(\Omega_1) = x_1, \ldots v^n_h(\Omega_n) = x_n\}) \tag{8}$$

Now, any theory will have some algorithm for calculating the probability of the measurement outcome corresponding to the outcome tuple $(x_1, \ldots, x_n)$, given that the system is prepared in state ψ (in all theories examined so far, the probability is some function of the scalar $(x_1 \otimes \cdots \otimes x_n)^\dagger \circ \psi$). We term this the theory's *probability rule.*

Definition 4.6 *Consider a state* $\psi : I \to A$*. The object* A *will have a local hidden state space* Λ_A*. Now consider a LHSD* μ *on* Λ_A*: if the probability predictions due to* μ *match those derived for* ψ *from the theory's probability rule, then we say that* μ *provides a* local hidden variable interpretation (LHVI) *for* ψ*.*

Definition 4.7 *Consider a state* $\Psi : I \to A_1 \otimes \cdots \otimes A_n$ *in a process category* $\mathcal{C}$*. An* n*-tuple of values* $(x_1, x_2, \ldots, x_n)$*, where* $x_i \in C_\Omega$*,* $\Omega \in \mathcal{O}_{A_i}$*, is termed a* forbidden outcome n-tuple with respect to Ψ *if:*

$$(x_1 \otimes x_2 \otimes \cdots \otimes x_n)^\dagger \circ \Psi = 0_{I,I} \tag{9}$$

n-tuples which are not forbidden with respect to Ψ *are* allowed *with respect to it.*

The following result will be important later on when we develop a no-go proof to rule out hidden variable interpretations for certain theories.

Lemma 4.8 *If the value functions of a hidden state h map an observable n-tuple into an outcome n-tuple which is forbidden with respect to a state Ψ, then for an LHSD μ to constitute a LHVI for Ψ, it must assign a probability of zero to h.*

Proof: Suppose $(x_1, \ldots, x_n)$ is a forbidden tuple with respect to a state $\Psi : I \to A$. The corresponding observable n-tuple is $(\Omega_1, \ldots, \Omega_n)$. An LHSD μ on the local hidden state space of A can only constitute a LHVI for Ψ if:

$$\mu(\{h \in \Lambda | v_h^1(\Omega_1) = x_1, \ldots v_h^n(\Omega_n) = x_n\}) = 0 \tag{10}$$

Clearly we can conclude that all hidden states $h \in \Lambda$ for which $v_h^j(\Omega_j) = x_j$ must be assigned a generalised probability of zero. $\square$

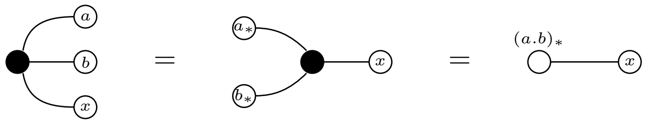
Throughout the remainder of this chapter we will only be interested in GHZ states, and if we refer to an outcome n-tuple as forbidden, we will mean that it is forbidden with respect to whichever GHZ state is under consideration. Since the GHZ states are states on a 3-composite object, we will be referring throughout to *forbidden outcome triples*, or, for short, *forbidden triples*.

5 Phase group and forbidden triples

Given a basis structure $\Delta = \{A, \delta, \epsilon\}$, the corresponding monoidal product on two points $a, b : I \to A$ is defined by $a.b = \delta^\dagger \circ (a \otimes b)$. We will now show that this monoid catalogues the forbidden triples of the GHZ state corresponding to Δ.

Lemma 5.1 *Given $a, b : I \to A$, and a basis structure Δ on A with corresponding monoidal product $-.-$, suppose $\exists \Delta' \in \mathcal{B}_A$ (not necessarily equal to the original Δ) such that $(a.b)_* \in C_{\Delta'}$. Then $(a, b, (a.b)_*)$ is an allowed triple with respect to the GHZ state corresponding to Δ. Furthermore $\forall x \in C_{\Delta'}, x \neq (a.b)_*$, we have that (a, b, x) is a forbidden triple with respect to this GHZ state.*

Proof: First note that:



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Then if $x = (a.b)_*$ then the rightmost diagram equals 1_I, and $(a, b, (a.b)_*)$ is an allowed triple. And if $x \neq (a.b)_*$ the rightmost diagram equals $0_{I,I}$, and (a, b, x) is a forbidden triple. $\square$

Hence, every pair of points a and b of A for which $a.b$ is an eigenstate of some basis structure has a set of associated forbidden triples. Obviously there is no overlap between these forbidden triples, since a and b are different in each case.

The phase group of a basis structure is a sub-group of the monoid, so we naturally expect the phase group to have some bearing on the allowed triples of GHZ states. If a phase group satisfies the observable-coset condition then it pins down the allowed and forbidden triples very precisely for certain observable triples, as we now see.

Definition 5.2 *In an phase group U_Δ with the observable-coset property, a triple of observable structures $(\Omega_1, \Omega_2, \Omega_3)$ is said to be a* forbidden-outcome observable triple *or* FO-observable triple *if C_{Ω_1}, C_{Ω_2} and C_{Ω_3} are observable cosets and $C_{\Omega_3} = (C_{\Omega_1}.C_{\Omega_2})^{-1}$ where $-.-$ and $(-)^{-1}$ denote group multiplication and inverse with respect to the observable quotient group.*

Proposition 5.3 *Given an FO-observable triple $(\Omega_1, \Omega_2, \Omega_3)$, any element of $C_{\Omega_1} \times C_{\Omega_2} \times C_{\Omega_3}$ which is not of the form $(a, b, (a.b)_*)$ is a forbidden triple.*

Proof: Any element of $C_{\Omega_1} \times C_{\Omega_2} \times C_{\Omega_3}$ which is not of the form above takes the form (a, b, x) where: (i) $x, a.b \in C_3$; and (ii) $x \neq a.b$. From lemma 5.1 this implies (a, b, x) is forbidden. □

In such a situation then, the phase group gives complete information on the forbidden and allowed outcome triples for the FO-observable triples.

6 Generalised Mermin table and generalised parities

Our analysis of locality is essentially a generalisation of the famous no-go proof employing GHZ states proposed by Mermin [4]. In this section we show how to generalise the key ingredients of that proof, ready for our generalised version of the proof in the next section.

Throughout this section we will denote the observable sub-group by C_0, and the observable-cosets by $C_1, C_2, \ldots$ etc. The corresponding observable structures will be denoted by $\Omega_0, \Omega_1, \ldots$ etc. $C_0, C_1, \ldots$ are the elements of $\mathcal{C}_Q$. We will denote the group multiplication by $-.-$ and the inverse operation by $(-)^{-1}$. For the remainder of these sections we will use n to denote the order of the observable quotient group i.e. $n = |\mathcal{C}_Q|$.

Definition 6.1 *The* generalised Mermin table *of a phase group U_Δ satisfying the observable-coset condition is an array of observable structures of n^2 rows and three columns, with each row being of the form:*

$$\Omega_i \quad \Omega_j \quad \Omega_k$$

where $i, j = 0, \ldots, n-1$ and the corresponding observable-cosets satisfy $C_k = (C_i.C_j)^{-1}$, i.e. each row contains the elements of a FO-observable triple. Clearly the rows are indexed by i and j so we will refer, for example to the $(i, j)^{th}$-row.

We will denote the elements of the observable sub-group by $C_0 = \{a_0, \ldots, a_{m-1}\}$ where a_0 denotes the group identity element. Throughout the remainder of these sections we will use m to denote the order of the observable sub-group.

Definition 6.2 *A* labelling of observable coset elements *is a function:*

$$\mathcal{L} : \mathcal{C}_Q \backslash \{C_0\} \to U_\Delta :: C_i \mapsto c^i \tag{12}$$

such that $c^i \in C_i$, i.e. it consists of selecting a representative element from each observable coset. Each different set of choices yields a different labelling - thus there are m^{n-1} labellings.

Clearly, having chosen a labelling of observable coset elements we can write the observable coset C_i as $\{c^i.a_0, \ldots, c^i.a_{m-1}\}$.

Definition 6.3 *Relative to a given labelling of observable coset elements, the* label *of an element $c^i.a_j$ is the observable sub-group element a_j. The label of an element of the observable sub-group is simply the element itself.*

Definition 6.4 *Given a hidden state $h \in \Lambda_{A\otimes A\otimes A}$, the h-*realisation *of the generalised Mermin table of U_Δ is obtained via the following procedure. Beginning with the generalised Mermin table, with three columns, and rows of the form:*

$$\Omega_i \quad \Omega_j \quad \Omega_k$$

take the value function of each observable structure:

$$v_h^1(\Omega_i) \quad v_h^2(\Omega_j) \quad v_h^3(\Omega_k)$$

Choosing a specific labelling $\mathcal{L}$, we can write this row as:

$$c^i.a_p \quad c^j.a_q \quad c^k.a_r$$

Remark 6.5 *An h-realisation of a generalised Mermin table can more succinctly be described as a re-writing of the original table where every appearance of Ω_i in a given column is replaced by* the same element *of C_i, which we write as $c^i.a_{i'}$. The label $a_{i'}$ will be different in each column, and will depend on h.*

Definition 6.6 *The* generalised parity*, with respect to a labelling $\mathcal{L}$, of a row or column in an h-*realisation *of a generalised Mermin table is the product of all the labels of the elements appearing in that row or column. Thus, the generalised parity is an element of the observable sub-group.*

For example, the generalised parity of the final example row in definition 6.4 is $a_p.a_q.a_r$.

7 A no-go proof

Lemma 7.1 *Consider a phase group U_Δ which satisfies both the observable-coset and QSP conditions. Given any labelling $\mathcal{L}$,* all *h-realisations of the generalised Mermin table of U_Δ will have, for all three columns, a generalised parity equal to the identity.*

Proof: From the definition of a generalised Mermin table (definition 6.1), each of the n observable structures $\{\Omega_i\}_{i=0,\ldots,n-1}$ (corresponding to the observable sub-group C_0 and observable cosets $\{C_i\}_{i=1,\ldots,n-1}$) appear in the first and second columns of the table n times. Simple group theory tells us that $\forall\, C_i, C_k \in \mathcal{C}_Q$ there exists a unique $C_j \in \mathcal{C}_Q$ such that

$$(C_i.C_j)^{-1} = C_k \tag{13}$$

So, in the third column, Ω_k appears in the same row as Ω_i, for each $i = 0, \ldots, n-1$, exactly once. Thus we conclude that each of the n observable structures $\{\Omega_i\}_{i=0,\ldots,n-1}$ appear n times in the third column of the table as well.

Now focus on a particular column, for definiteness the first. The argument will apply equally to the second and third columns. From the argument above, and noting remark 6.5, we see, in the first column of any h-realisation of the table, there are n occurrences of some element $c^i.a_{i'}$ for each $i = 0, \ldots n-1$ (there is no need for the different i' to be distinct). The generalised parity of the first column of this h-realisation will be $(\prod_{i=0}^{n} a_{i'})^n$. Now, $\prod_{i'=0}^{n} a_{i'}$ is some element of C_0. From the QSP condition $(\prod_{i=0}^{n} a_{i'})^n = a_0$. □

Lemma 7.2 *Consider a phase group U_Δ which satisfies the observable-coset property. Given any labelling $\mathcal{L}$, all h-realisations which map the observable triple in the top row of the generalised Mermin table into an* allowed *outcome triple have a generalised parity for this row equal to the identity.*

Proof: The top row of a generalised Mermin table is:

$$\Omega_0 \quad \Omega_0 \quad \Omega_0$$

From proposition 5.3 we know that all allowed outcome triples for this triple of observables are of the form $(a_i, a_j, (a_i.a_j)_*)$. Recall that the lower star operation gives the phase group inverse, by definition. In any h-realisation with such an outcome triple as its top row:

$$a_i \quad a_j \quad (a_i.a_j)_*$$

the generalised parity of the first row is clearly a_0, the identity element. □

Lemma 7.3 *Consider a phase group U_Δ which satisfies the observable-coset property. Given any labelling $\mathcal{L}$, all h-realisations which map the observable triple in a given row of the generalised Mermin table into an* allowed *outcome triple have the same generalised parity for this row. This value of this parity will, in general, depend on $\mathcal{L}$.*

Proof: Consider a general row of the generalised Mermin table:

$$\Omega_i \quad \Omega_j \quad \Omega_k$$

Recall that the value of k is determined by the observable quotient group via $C_k = (C_i.C_j)^{-1}$. Again, from proposition 5.3, we know that all allowed outcome triples for this triple of observables are of the form $(c^i.a_p, c^j.a_q, ((c^i.a_p).(c^j.a_q))_*)$ where we have chosen a specific labelling. Note that the final outcome in the triple can be re-written:

$$((c^i.a_p).(c^j.a_q))_* = (c^i.c^j)_*.(a_p.a_q)_* = d^k.(a_p.a_q)_* \tag{14}$$

where $d^k = (c^i.c^j)_* \in C_k$ but in general $d^k \neq c^k$, i.e. d^k is not the representative element of C_k picked out by the labelling function. This last point is important because it implies that if this outcome triple is a row in an h-realisation:

$$c^i.a_p \quad c^j.a_q \quad d^k.(a_p.a_q)_*$$

the $(a_p.a_q)_*$ appearing in the third column is *not* the label of that element, and hence is not what we need to use to calculate the generalised parity of this row. However, since $d^k \in C_k$ we know that there exists $a(i,j) \in C_0$ such that $d^k = c^k.a(i,j)$. We can then re-write the row above as:

$$c^i.a_p \quad c^j.a_q \quad c^k.(a(i,j).(a_p.a_q)_*)$$

Clearly now $a(i,j).(a_p.a_q)_*$ *is* the label for the element in the third column. We can use it to calculate the generalised parity for this row, which clearly equals $a(i,j)$. □

Definition 7.4 *With respect to a labelling $\mathcal{L}$, the* allowed parity *for the $(i,j)^{th}$ row of a generalised Mermin table is $a(i,j) \in C_0$, defined by:*

$$a(i,j) = (c^i.c^j.c^k)_* \tag{15}$$

where $C_k = (C_i.C_j)^{-1}$ and $c^i = \mathcal{L}(C_i) \dots$ etc. Taking into account lemma 7.2 we define $a(0,0) = a_0$.

Corollary 7.5 *Any $h \in \Lambda_{A\otimes A\otimes A}$ whose h-realisation has a generalised parity for the $(i,j)^{th}$ row which is not equal to the allowed parity $a(i,j)$, maps the observable triple in the $(i,j)^{th}$ row of the generalised Mermin table into a forbidden outcome triple. Consequently, for a LHVI to exist for the GHZ state, the corresponding LHSD must assign h a generalised probability of zero.*

Proposition 7.6 *For a phase group satisfying the observable-coset and QSP conditions the product of the allowed parities of all rows $\prod_{i,j=0}^{n} a(i,j)$ is independent of the labelling. We term this product the* Mermin parameter *of the phase group.*

Proof: Consider a re-labelling which changes the representative element of just one of the cosets. For the coset C_m, instead of c^m we choose d^m. Note that $\exists\ a^* \in C_0$ such that $d^m = c^m.a^*$. Note that in all labellings $c^0 = a_0$, so we can assume that $m \neq 0$. With respect to this new labelling we get a new set of allowed parities for the rows of the Mermin table, $a'(i,j)$. We need to determine how they relate to the previous allowed parities $a(i,j)$, which were defined by the relation $c^i.c^j = c^k.a(i,j)$. There are several distinct situations to consider.

- $i,j,k \neq m$: We simply have $a'(i,j) = a(i,j)$.
- $i = m,\ j = 0$ or $i = 0,\ j = m$: $a(i,0)$ and $a(0,j)$ both equal a_0 in all labellings, so again we have $a'(i,j) = a(i,j)$. There are two such cases.
- $i = m,\ j \neq 0, m$ or $i \neq 0, m,\ j = m$: In the first instance we have $d^m.c^j = c^k.a'(m,j)$, in the second we have $c^i.d^m = c^k.a'(i,m)$, from which we conclude that in either instance $a'(i,j) = a^*.a(i,j)$. There are $2n-4$ such cases.

- $i = j = m$: Here we have $a'(i,j) = (a^*)^2.a(i,j)$. There is one such case.
- $k = m$: Here we have $c^i.c^j = d^m.a'(i,j)$, and can thus conclude that $a'(i,j) = (a^*)^{-1}.a(i,j)$. There are n such cases, however two of them coincide with the second situation in this list. There is no overlap with the other situations in the list.

Overall then, we conclude that:

$$\prod_{i,j=0}^{n} a'(i,j) = (a^*)^{2n-2}.(a^*)^{-(n-2)}.[\prod_{i,j=0}^{n} a(i,j)] = (a^*)^n.[\prod_{i,j=0}^{n} a(i,j)] \tag{16}$$

If the QSP condition holds then $(a^*)^n = a_0$ and we have $\prod_{i,j=0}^{n} a'(i,j) = \prod_{i,j=0}^{n} a(i,j)$. Since we can move between any two labellings via a sequence where we only change the representative element of one coset, we have shown that the Mermin parameter is independent of labelling. □

Theorem 7.7 *Given a phase group U_Δ which satisfies the observable-coset and QSP conditions, for which the Mermin parameter does not equal a_0, the corresponding GHZ state Ψ_Δ does not have a LHVI.*

Proof: We will define the *table parity* of an h-realisation of a generalised Mermin table as the product of the labels of all elements appearing in the h-realisation. Clearly the table parity can be calculated either by taking the product of the generalised parities of all three columns, or by taking the product of the generalised parities of all n^2 rows. Using the column method, from lemma 7.1, any h-realisation must have a table parity of a_0. Using the row method, from corollary 7.5, any h-realisation in which every row is an allowed triple must have a table parity equal to the Mermin parameter. If the Mermin parameter is not equal to a_0, then there does not exist an h-realisation in which every row is an allowed triple i.e. every h-realisation has at least one row which is a forbidden triple for the corresponding observable triple in the generalised Mermin table. From lemma 4.8 we then conclude that any LHSD which was an LHVI for the GHZ state would have to assign a probability of zero to all hidden states h. But by its definition a LHSD must assign a non-zero probability to some states. Thus we have a contradiction, and conclude that no LHVI exists. □

8 Connection to group extensions

In group theory the *group extension problem* is the following: given an Abelian group G_1 and some other group G_2, find all groups G with a normal sub-group isomorphic to G_1, such that $G/G_1 \cong G_2$. We will concentrate on the special case where all three groups are Abelian.

Let us suggestively denote the elements of G_1 by $\{a_0, \ldots, a_{m-1}\}$ with a_0 the identity, and those of $G/G_1 (\cong G_2)$ by $\{C_0, \ldots, C_{n-1}\}$ with C_0 the identity. Now choose a representative element c^i from each C_i. Clearly now the elements of G are $\{c^i.a_j\}_{i=0,\ldots,n-1;j=0,\ldots,m-1}$. To fully specify G, it remains to determine the product of two arbitrary elements $(c^i.a_p).(c^j.a_q)$. Note first that:

$$(c^i.a_p).(c^j.a_q) = (c^i.c^j).(a_p.a_q) \tag{17}$$

Now $(a_p.a_q)$ is fully specified by G_1. It remains to determine $(c^i.c^j)$. We know that $(c^i.c^j) \in C_k = C_i.C_j$ where k is determined by G_2. Whilst, in general, $(c^i.c^j) \neq c^k$, we do know that $\exists\, \tilde{a}(i,j) \in C_0$ such that $(c^i.c^j) = c^k.\tilde{a}(i,j)$, so that we can write the product of two arbitrary elements in G as:

$$(c^i.a_p).(c^j.a_q) = c^k.\tilde{a}(i,j).(a_p.a_q) \tag{18}$$

Clearly the choices for $\tilde{a}(i,j)$ constitute the only degrees of freedom not pre-determined by G_1 or G_2, and thus different choices for these parameters will give us the different possible group extensions G.

The two sets of parameters $\tilde{a}(i,j)$ (which determine which group extension is realised) and $a(i,j)$ (which determine locality properties) are not identical, but are closely related.

Lemma 8.1 *In a phase group satisfying the observable-coset and QSP conditions, the product of the group extension parameters $\tilde{a}(i,j)$ is equal to the inverse of the Mermin parameter.*

Proof: Let us assume that we have $C_i.C_j = C_k$ and $(C_k)^{-1} = C_l$. Then the defining property of the $a(i,j)$ is $(c^i.c^j)_* = c^l.a(i,j)$ whilst that of the $\tilde{a}(i,j)$ is $c^i.c^j = c^k.\tilde{a}(i,j)$.

First note that $(c^k)_* = d^l$, with $e^l \in C_l$. Now define a new parameter $a(k) \in C_0$, such that $d^l = c^l.a(k)$, so that we have $(c^k)_* = c^l.a(k)$. From $(c^i.c^j)_* = c^k.a(i,j)$ we deduce $c^i.c^j = (c^k.a(i,j))_* = c^k_*.(a(i,j))_* = c^l.a(k).(a(i,j))_*$. We thus conclude that $\tilde{a}(i,j) = a(k).(a(i,j))_*$.

The product of the group extension parameters is $\prod_{i,j=0}^n \tilde{a}(i,j) = \prod_{i,j=0}^n (a(k).(a(i,j))_*) = (\prod_{i,j=0}^n a(k)).M_*$, where M denotes the Mermin parameter. Now note that there will be precisely n combinations of i,j for which $C_i.C_j = C_k$. Thus for each value of k, $a(k)$ will appear in the product n times. From the QSP condition we know that $a(k)^n = a_0$, thus we can conclude that $\prod_{i,j=0}^n a(k) = a_0$ and $\prod_{i,j=0}^n \tilde{a}(i,j) = M_*$. □

The most straightforward example of a group extension for G_1 and G_2 is the *direct product* $G_1 \times G_2$. In fact we can immediately show that a direct product phase group won't exhibit Mermin-style non-locality:

Lemma 8.2 *Given a phase group satisfying the observable-coset and QSP conditions, which can be written as $G_1 \times G_2$ where G_1 is the observable subgroup, the Mermin parameter is equal to the identity element.*

Proof: The elements of $G_1 \times G_2$ can be written as (a_i, C_j). The elements of the form (a_i, C_0) form the subgroup isomorphic to G_1. Elements of the form (a_i, C_j), for constant j form a coset to this subgroup. Now recall our earlier discussion of *labellings* of the elements of cosets. Suppose we pick a particular labelling for $G_1 \times G_2$ such that $c^i = (a_0, C_i)$ for all cosets C_i. In this case we get $c^i.c^j = (a_0, C_i).(a_0, C_j) = (a_0, C_k) = c^k$, for all i,j. Recall that the *allowed parities* $a(i,j)$ are defined by $c^i.c^j = c^k.a(i,j)$. From this we conclude that in this labelling, $a(i,j) = a_0$ for all i,j, and thus that the Mermin parameter $M = \bigodot_{i,j=1}^n a(i,j)$ is also equal to a_0, for all labellings. □

The converse however, is not true: a Z_9 phase group with Z_3 observable subgroup provides a counter-example, as shown in the appendix.

9 Conclusions

We have extended the result of [3] to a much wider class of phase groups. Certainly not all phase groups are included, but many of those which arise in theories of interest do seem to be. For example, stabiliser theory can be extended to systems of higher dimension than two. For dimensions of prime power at least, we expect the corresponding phase groups to satisfy the observable coset and QSP conditions.

So far only a small number of cases have been examined: Z_4 and $Z_2 \times Z_2$, both with a Z_2 observable sub-group, and Z_9 and $Z_3 \times Z_3$, both with a Z_3 observable sub-group. Of these only the Z_4 case has a non-identity Mermin parameter. Clearly it would be beneficial to work through other examples. Given the discussion of the relation to the group extension problem in section 7, one might imagine that examples with a non-identity Mermin parameter could be constructed via a judicious choice of the parameters $\tilde{a}(i,j)$. This is not a straightforward task however: we do not have total freedom to choose any set of $\tilde{a}(i,j)$ - only some will result in a valid group extension. A better understanding of the restrictions on the $\tilde{a}(i,j)$, and indeed of group extensions in general is clearly desirable. Relating some kind of classification of group extensions to the existence of local hidden variable interpretations would obviously be a very nice result.

A Example: Z_9 phase group with Z_3 observable sub-group

To illustrate the main result of this paper we now consider a concrete example.

- We have an object in a process category with (at least) four basis structures, Δ_Z, Δ_A, Δ_B and Δ_C. Each of these basis structures has three eigenstates, all of which are distinct from each other.
- The eigenstates of Δ_A, Δ_B and Δ_C together constitute the unbiased states of Δ_Z. The phase group of Δ_Z thus has nine elements, and in this case we choose it to be the cyclic group Z_9.
- The eigenstates of Δ_A constitute the Z_3 sub-group of the phase group, while the eigenstates of Δ_B and Δ_C respectively constitute the two cosets of this sub-group. Thus this phase group satisfies the observable-coset condition (definition 3.1) by design.

So in our previous terminology we have $U_{\Delta_Z} = Z_9$ and $C_0 = Z_3$. From this we conclude that the observable quotient group C_Q is equal to $Z_9/Z_3 \cong Z_3$. It is now straightforward to see that this phase group also satisfies the QSP condition (definition 3.3): $\forall\ a \in C_0 \cong Z_3$ we have $a^{|C_Q|} = a^3 = a_0$ (where a_0 is the identity element of C_0).

We now turn to deriving the generalised Mermin table for this phase group. From here on we will switch to a slightly different notation for the cosets. The observable sub-group C_0 consists of the eigenstates of Δ_A, while the cosets C_1 and C_2 consist of the eigenstates of Δ_B and Δ_C respectively. For simplicity we will simply replace C_0, C_1 and C_2 with C_A, C_B and C_C. The rows of the generalised Mermin table consist of triples of observables $\Omega_i, \Omega_j, \Omega_k$ such that the corresponding cosets satisfy $C_i.C_j = C_k^{-1}$ (quotient group product and inverse). In the case of our Z_3 quotient group this leads to the following nine-row table:

$$\begin{array}{ccc} A & A & A \\ A & B & C \\ A & C & B \\ B & A & C \\ B & B & B \\ B & C & A \\ C & A & B \\ C & B & A \\ C & C & C \end{array} \tag{19}$$

The elements of a Z_9 group would be conventionally written as $e, g, g^2, \ldots, g^8$ with e denoting the identity element. In this notation the Z_3 sub-group is $C_A = \{e, g^3, g^6\}$ and its two cosets are $C_B = \{g, g^4, g^7\}$ and $C_C = \{g^2, g^5, g^8\}$. In our approach however, we write each group element as the product of a representative element from its coset and an element of the observable sub-group. Choosing a representative element from each coset amounts to choosing a *labelling* (definition 6.2). Again we adopt slightly different notation - in the full proof we used c^i to denote the representative element from the coset C_i. Here we will use b to denote the representative element from C_B and c to denote that from C_C. In all labellings the representative element from the observable sub-group (here C_A) is chosen to be the identity. Thus, in this case choosing a labelling means assigning one of $\{g, g^4, g^7\}$ as b and one of $\{g^2, g^5, g^8\}$ as c. It is clear that we have nine possible labellings. We will choose two (more or less at random) to serve as examples. Labelling 1 (L1) is the assignment $b = g, c = g^2$; and labelling 2 (L2) is the assignment $b = g, c = g^5$. In L1 for example, the element g^4 is written as $b.a_1$, and thus the *label* (definition 6.3) of this element is the subgroup element a_1.

With the labellings chosen it is now straightforward to calculate the allowed parities $a(i, j)$ (definition 7.4). For example, consider calculating $a(B, B)$ in L2. Using equation 15, we find $a(B, B) = (b.c.a)_* = (g.g^5.e)^{-1} = (g^6)^{-1} = g^3 = a_1$. We can do such a calculation for all of the $a(i, j)$, and they are listed below in table 21. However, it is perhaps enlightening to look at this in a slightly different way. The phase group tables for our two labellings are shown in figure 1. These tables describe the multiplication of exactly the same group, Z_9. The rows and columns are labelled identically, but because these labels correspond to different group elements in each case (e.g. in L1 $c.a_0 = g^2$, whilst in L2 $c.a_0 = g^5$), the two tables look different. Recalling that the allowed triples for FO-observable triples take the form $(x, y, (x.y)_*)$ (proposition 5.3) we can now present the *allowed triple tables* for L1 and L2: these are obtained from the group tables by taking the inverse of every entry in the table. The tables are displayed in figure 2.

The allowed triples are given by the elements labelling a row, a column, and the entry in the table where the row and column intersect (for example $(c.a_0, b.a_1, a_0)$ is an allowed triple in L2). Each of the nine blocks within the table contains the allowed triples for one of the nine FO-observable triples i.e. for one of the nine rows of the Mermin table. Consider a particular assignment of outcomes to the observable triples in the generalised Mermin table (an h-realisation of the Mermin table - definition 6.4). Each row of the table is assigned a triple of outcomes and multiplying together the labels of these outcomes gives us the generalised parity

L1	a_0	a_1	a_2	$b.a_0$	$b.a_1$	$b.a_2$	$c.a_0$	$c.a_1$	$c.a_2$
a_0	a_0	a_1	a_2	$b.a_0$	$b.a_1$	$b.a_2$	$c.a_0$	$c.a_1$	$c.a_2$
a_1	a_1	a_2	a_0	$b.a_1$	$b.a_2$	$b.a_0$	$c.a_1$	$c.a_2$	$c.a_0$
a_2	a_2	a_0	a_1	$b.a_2$	$b.a_0$	$b.a_1$	$c.a_2$	$c.a_0$	$c.a_1$
$b.a_0$	$b.a_0$	$b.a_1$	$b.a_2$	$c.a_0$	$c.a_1$	$c.a_2$	a_0	a_1	a_2
$b.a_1$	$b.a_1$	$b.a_2$	$b.a_0$	$c.a_1$	$c.a_2$	$c.a_0$	a_1	a_2	a_0
$b.a_2$	$b.a_2$	$b.a_0$	$b.a_1$	$c.a_2$	$c.a_0$	$c.a_1$	a_2	a_0	a_1
$c.a_0$	$c.a_0$	$c.a_1$	$c.a_2$	a_0	a_1	a_2	$b.a_1$	$b.a_2$	$b.a_0$
$c.a_1$	$c.a_1$	$c.a_2$	$c.a_0$	a_1	a_2	a_0	$b.a_2$	$b.a_0$	$b.a_1$
$c.a_2$	$c.a_2$	$c.a_0$	$c.a_1$	a_2	a_0	a_1	$b.a_0$	$b.a_1$	$b.a_2$

L2	a_0	a_1	a_2	$b.a_0$	$b.a_1$	$b.a_2$	$c.a_0$	$c.a_1$	$c.a_2$
a_0	a_0	a_1	a_2	$b.a_0$	$b.a_1$	$b.a_2$	$c.a_0$	$c.a_1$	$c.a_2$
a_1	a_1	a_2	a_0	$b.a_1$	$b.a_2$	$b.a_0$	$c.a_1$	$c.a_2$	$c.a_0$
a_2	a_2	a_0	a_1	$b.a_2$	$b.a_0$	$b.a_1$	$c.a_2$	$c.a_0$	$c.a_1$
$b.a_0$	$b.a_0$	$b.a_1$	$b.a_2$	$c.a_0$	$c.a_1$	$c.a_2$	a_2	a_0	a_1
$b.a_1$	$b.a_1$	$b.a_2$	$b.a_0$	$c.a_1$	$c.a_2$	$c.a_0$	a_0	a_1	a_2
$b.a_2$	$b.a_2$	$b.a_0$	$b.a_1$	$c.a_2$	$c.a_0$	$c.a_1$	a_1	a_2	a_0
$c.a_0$	$c.a_0$	$c.a_1$	$c.a_2$	a_2	a_0	a_1	$b.a_0$	$b.a_1$	$b.a_2$
$c.a_1$	$c.a_1$	$c.a_2$	$c.a_0$	a_0	a_1	a_2	$b.a_1$	$b.a_2$	$b.a_0$
$c.a_2$	$c.a_2$	$c.a_0$	$c.a_1$	a_1	a_2	a_0	$b.a_2$	$b.a_0$	$b.a_1$

Figure 1: Z_9 group tables for two labellings

L1	a_0	a_1	a_2	$b.a_0$	$b.a_1$	$b.a_2$	$c.a_0$	$c.a_1$	$c.a_2$
a_0	a_0	a_2	a_1	$c.a_2$	$c.a_1$	$c.a_0$	$b.a_2$	$b.a_1$	$b.a_0$
a_1	a_2 A	a_1 A	a_0 A	$c.a_1$ B	$c.a_0$ A	$c.a_2$ C	$b.a_1$ C	$b.a_0$ A	$b.a_2$ B
a_2	a_1	a_0	a_2	$c.a_0$	$c.a_2$	$c.a_1$	$b.a_0$	$b.a_2$	$b.a_1$
$b.a_0$	$c.a_2$	$c.a_1$	$c.a_0$	$b.a_2$	$b.a_1$	$b.a_0$	a_2	a_1	a_0
$b.a_1$	$c.a_1$ A	$c.a_0$ B	$c.a_2$ C	$b.a_1$ B	$b.a_0$ B	$b.a_2$ B	a_1 C	a_0 B	a_2 A
$b.a_2$	$c.a_0$	$c.a_2$	$c.a_1$	$b.a_0$	$b.a_2$	$b.a_1$	a_0	a_2	a_1
$c.a_0$	$b.a_2$	$b.a_1$	$b.a_0$	a_2	a_1	a_0	$c.a_1$	$c.a_0$	$c.a_2$
$c.a_1$	$b.a_1$ A	$b.a_0$ C	$b.a_2$ B	a_1 B	a_0 C	a_2 A	$c.a_0$ C	$c.a_2$ C	$c.a_1$ C
$c.a_2$	$b.a_0$	$b.a_2$	$b.a_1$	a_0	a_2	a_1	$c.a_2$	$c.a_1$	$c.a_0$

L2	a_0	a_1	a_2	$b.a_0$	$b.a_1$	$b.a_2$	$c.a_0$	$c.a_1$	$c.a_2$
a_0	a_0	a_2	a_1	$c.a_1$	$c.a_0$	$c.a_2$	$b.a_1$	$b.a_0$	$b.a_2$
a_1	a_2 A	a_1 A	a_0 A	$c.a_0$ B	$c.a_2$ A	$c.a_1$ C	$b.a_0$ C	$b.a_2$ A	$b.a_1$ B
a_2	a_1	a_0	a_2	$c.a_2$	$c.a_1$	$c.a_0$	$b.a_2$	$b.a_1$	$b.a_0$
$b.a_0$	$c.a_1$	$c.a_0$	$c.a_2$	$b.a_2$	$b.a_1$	$b.a_0$	a_1	a_0	a_2
$b.a_1$	$c.a_0$ A	$c.a_2$ B	$c.a_1$ C	$b.a_1$ B	$b.a_0$ B	$b.a_2$ B	a_0 C	a_2 B	a_1 A
$b.a_2$	$c.a_2$	$c.a_1$	$c.a_0$	$b.a_0$	$b.a_2$	$b.a_1$	a_2	a_1	a_0
$c.a_0$	$b.a_1$	$b.a_0$	$b.a_2$	a_1	a_0	a_2	$c.a_1$	$c.a_0$	$c.a_2$
$c.a_1$	$b.a_0$ A	$b.a_2$ C	$b.a_1$ B	a_0 B	a_2 C	a_1 A	$c.a_0$ C	$c.a_2$ C	$c.a_1$ C
$c.a_2$	$b.a_2$	$b.a_1$	$b.a_0$	a_2	a_1	a_0	$c.a_2$	$c.a_1$	$c.a_0$

Figure 2: Allowed triple tables for two labellings

of that row. One of the key results of the paper was that for all the allowed triples for a given row, the generalised parity is the same. In terms of the allowed triple tables this translates as saying that in any block, multiplying together the labels of the elements corresponding to any row, any column, and the entry where they intersect must always give the same answer. Clearly this is the case for the top left block, since the labels here are exactly the entries, and thus have the form x, y and $(x.y)_*$, multiplying together to give the identity.

Next consider, for example, the bottom right block, corresponding to the observable triple CCC. For calculating parities we are only interested in the labels of the elements, so let's strip away the pre-factors of c:

$$\begin{array}{|c|c|c|}\hline c.a_1 & c.a_0 & c.a_2 \\ \hline c.a_0 & c.a_2 & c.a_1 \\ \hline c.a_2 & c.a_1 & c.a_0 \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|c|}\hline a_1 & a_0 & a_2 \\ \hline a_0 & a_2 & a_1 \\ \hline a_2 & a_1 & a_0 \\ \hline \end{array} \quad = \quad a_1 . \begin{array}{|c|c|c|}\hline a_0 & a_2 & a_1 \\ \hline a_2 & a_1 & a_0 \\ \hline a_1 & a_0 & a_2 \\ \hline \end{array} \tag{20}$$

It's straightforward then to see that the table with the pre-factors removed is simply a permutation of the top left (AAA) block, obtained by multiplying all entries of that block by a_1. (This a_1 arises from the multiplication of the pre-factors of the elements labelling the rows and columns, in this case $(c.c)_* = c.a_1$). Combining this with our earlier analysis of the top left block, we can see that the generalised parity of any allowed triple for CCC must be a_1.

Whatever way we calculate them, we can now fill in the parities for each row in the Mermin table, and calculate the Mermin parameter:

$$\begin{array}{ccccc} & & & \text{L1} & \text{L2} \\ A & A & A & a_0 & a_0 \\ A & B & C & a_2 & a_1 \\ A & C & B & a_2 & a_1 \\ B & A & C & a_2 & a_1 \\ B & B & B & a_2 & a_2 \\ B & C & A & a_2 & a_1 \\ C & A & B & a_2 & a_1 \\ C & B & A & a_2 & a_1 \\ C & C & C & a_1 & a_1 \\ & & & M = a_0 & M = a_0 \end{array} \tag{21}$$

As expected (proposition 7.6) this is the same for both labellings. Interestingly the Mermin parameter is equal to the identity, and thus we cannot immediately conclude that the GHZ state corresponding to this phase group would not have a local hidden variable interpretation.

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