Categorical Quantum Circuits

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Introduction

- Quantum circuits are the *de facto* standard tool in quantum information science for representing (unitary) evolutions diagrammatically.
- Since its founding, category theory has become the mathematical formalism of choice for describing composable processes between objects.
- Extending the approach taken by Abramsky and Coecke [LiCS'04, IEEE Computer Science Press (2004)], we reframe the quantum circuit model as a dagger-compact-closed category (introduced in 1995 under a different name by Dolan and Baez [J.Math.Phys. 36 (1995) 6073-6105]) and obtain a diagrammatic language which is a useful superset of standard quantum circuits.
- These diagrams can be manipulated in a very intuitive, visual ways. Objects (boxes etc.) can be slid along wires. The wires themselves can be bent and rearranged. Nodes where several wires meet can be combined and split according to simple rules.

From qubits to qudits

To keep the techniques as general as possible, in our proofs we use generic $\mathbf{d}$-dimensional qudits.

- Instead of the Hadamard gate we use the discrete Fourier transform gate to transform between two mutually unbiased bases:

$$\mathbf{H} := \frac{1}{\sqrt{\mathbf{d}}} \sum_{\mathbf{ab}} e^{i2\pi ab/d} |\mathbf{a}\rangle\langle\mathbf{b}|,$$

- The σ_z and σ_x gates are likewise replaced by the $\mathbf{Z}$ and $\mathbf{X}$ gates:

$$\mathbf{Z} := \sum_{\mathbf{k}} e^{i2\pi k/d} |\mathbf{k}\rangle\langle\mathbf{k}|,$$

$$\mathbf{X} := \sum_{\mathbf{k}} |\mathbf{k} \oplus \mathbf{1}\rangle\langle\mathbf{k}|,$$

which are in fact the same operation in two different bases: $\mathbf{H}\mathbf{X}\mathbf{H}^\dagger = \mathbf{Z}$.

- The CNOT gate is generalized into the negated modular adder gate NADD which, like CNOT, is self-inverse:

$$\mathrm{NADD}_{\mathbf{i},\mathbf{j}} := \sum_{\mathbf{xy}} |\mathbf{x}, \ominus\mathbf{x} \ominus \mathbf{y}\rangle\langle\mathbf{x}, \mathbf{y}|_{\mathbf{i},\mathbf{j}}.$$

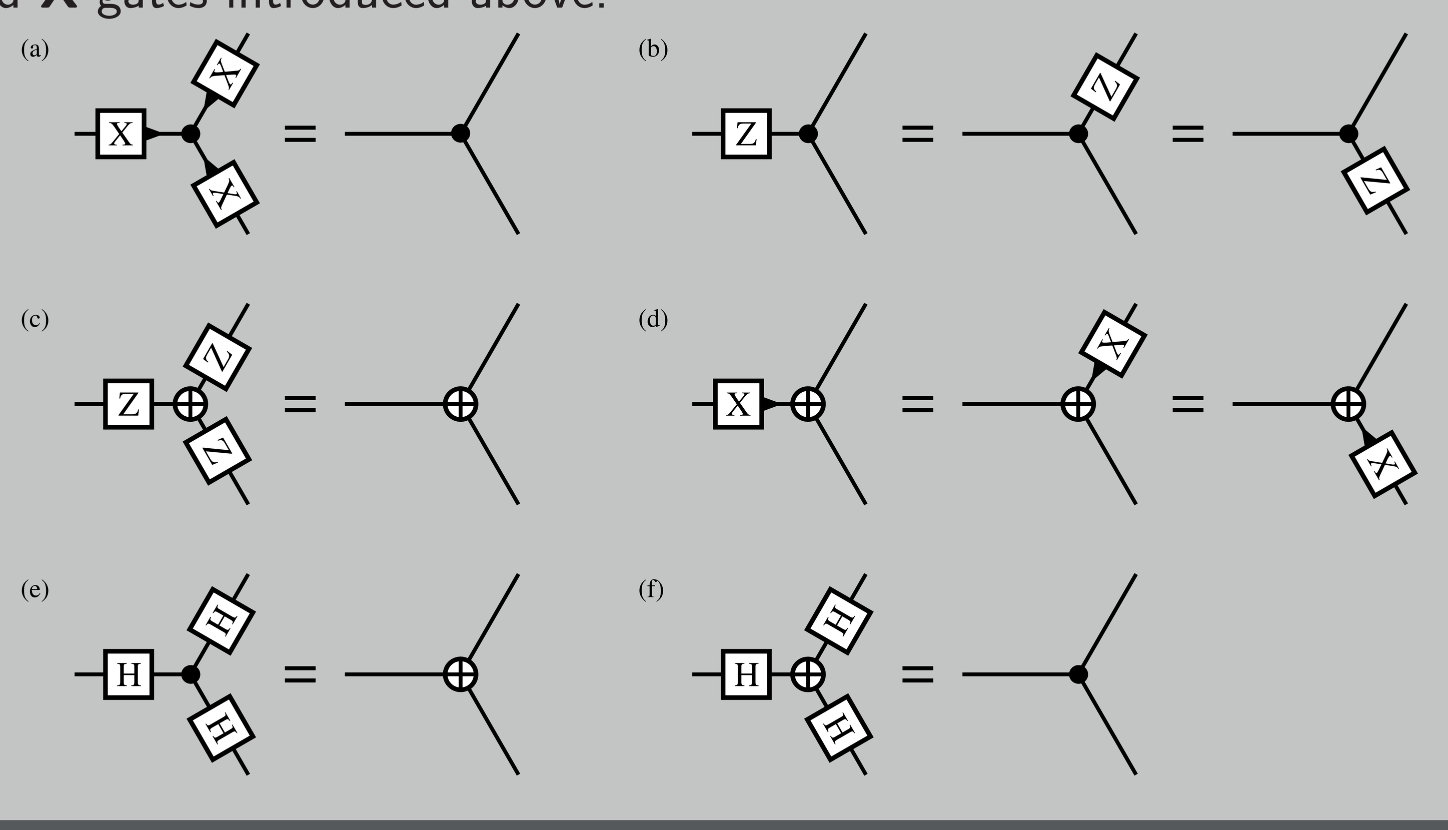
Dots

- We build a class of highly symmetrical linear operators called dots, which can be used e.g. to decompose multiqudit logic gates into simpler, more symmetric structures.
- The NADD gate, for example, can be decomposed into a copy dot ($\bullet$) and a plus dot ($\oplus$), defined as

$$\mathrm{COPY}_z^{m\to n} := \sum_{\mathbf{k}} |\underbrace{\mathbf{k}\cdots\mathbf{k}}_{\mathbf{n}}\rangle\langle\underbrace{\mathbf{k}\cdots\mathbf{k}}_{\mathbf{m}}|,$$

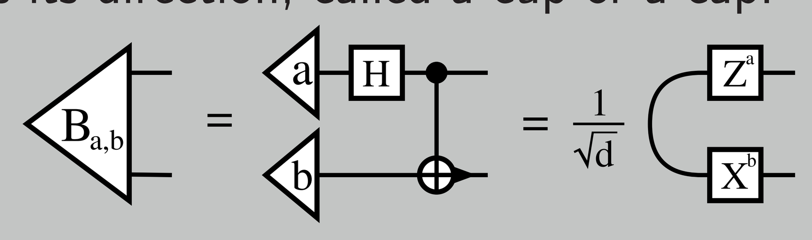
$$\mathrm{PLUS}_z^{m\to n} := \frac{1}{\mathbf{d}^{(m+n-2)/2}} \sum_{\substack{\mathbf{r}_1\cdots\mathbf{r}_m \\ \mathbf{s}_1\cdots\mathbf{s}_n}} \delta_{(\sum_i r_i \oplus \sum_j s_j),0} |\mathbf{s}_1\cdots\mathbf{s}_n\rangle\langle\mathbf{r}_1\cdots\mathbf{r}_m|.$$

- These dots have reasonably simple commutation relations with the $\mathbf{Z}$ and $\mathbf{X}$ gates introduced above:



Cups and caps

- The *unit* and *counit* morphisms in the category correspond to maximally entangled two-qudit states, and are represented diagrammatically by a wire that reverses its direction, called a cup or a cap.

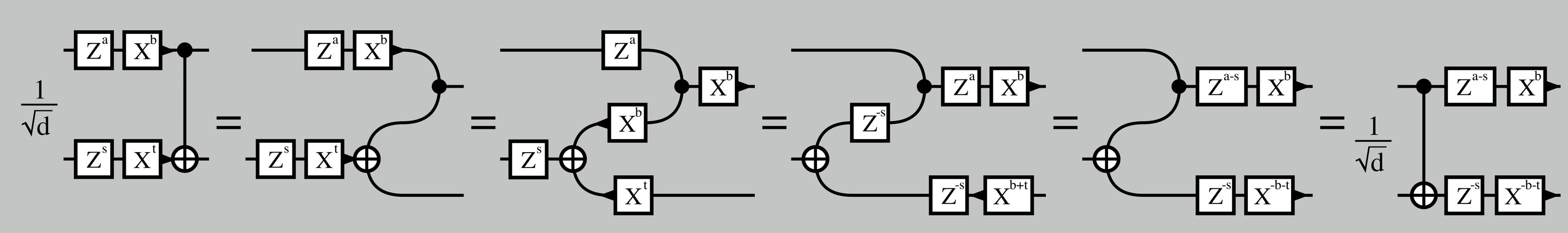


- Together with the $\mathbf{Z}$ and $\mathbf{X}$ gates they can be used to represent any of the $\mathbf{d}^2$ orthonormal generalized Bell states:

$$|\mathbf{B}_{\mathbf{a},\mathbf{b}}\rangle_{\mathbf{i},\mathbf{j}} := \frac{1}{\sqrt{\mathbf{d}}} \sum_{\mathbf{k}} e^{i2\pi ak/d} |\mathbf{k}, \mathbf{k} \oplus \mathbf{b}\rangle = \mathrm{ADD}_{\mathbf{i},\mathbf{j}} \mathbf{H}_{\mathbf{i}} |\mathbf{a}, \mathbf{b}\rangle_{\mathbf{i},\mathbf{j}}.$$

Example: Commuting local gates through a NADD gate

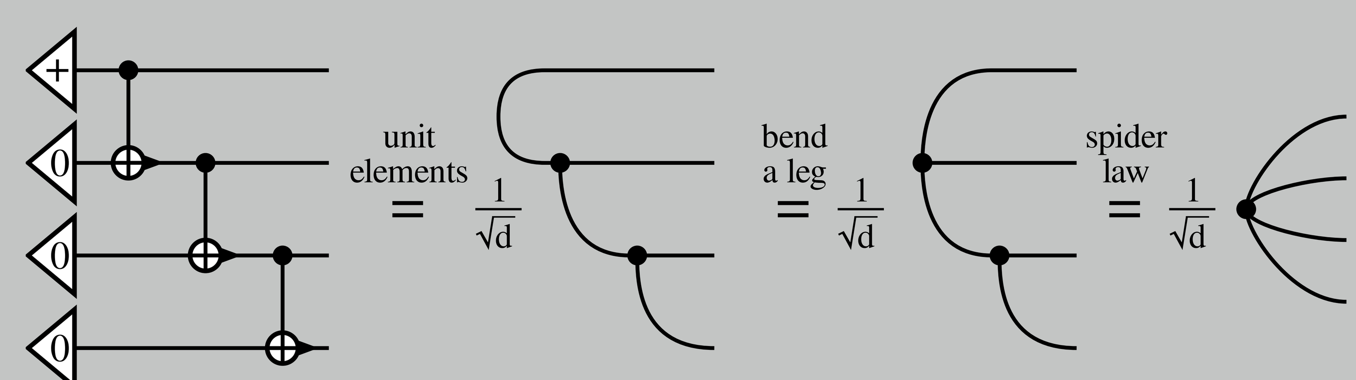
- NADD is decomposed into a copy dot and a plus dot.
- $\mathbf{Z}$ and $\mathbf{X}$ gates are commuted through the dots.
- The NADD gate is reassembled, thus recovering the generalization of the familiar commutation rule for CNOT, σ_z and σ_x gates.



Example: Simplifying a GHZ circuit

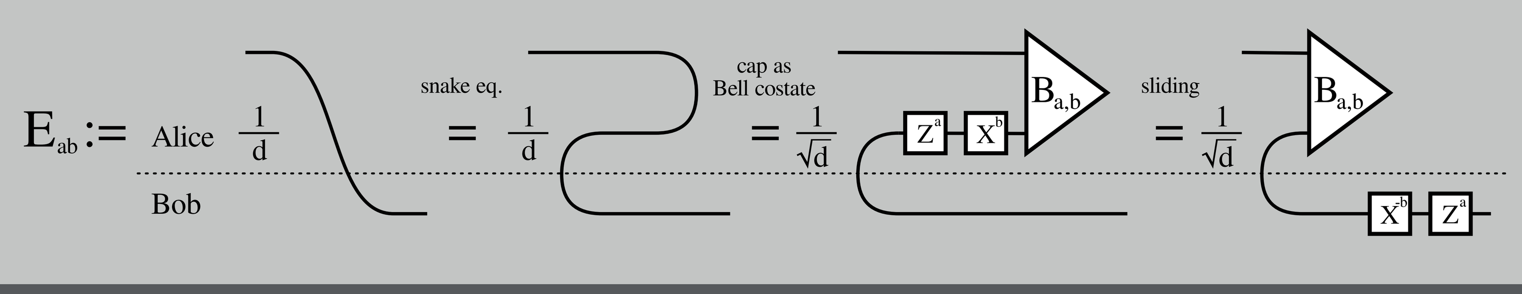
- Using various copy and plus dot identities, one can reduce the following circuit (producing a four-qudit Greenberger-Horne-Zeilinger state) into a single scaled copy dot which is equivalent to the GHZ state as expected:

$$\frac{1}{\sqrt{\mathbf{d}}} \mathrm{COPY}_z^{0\to 4} = \frac{1}{\sqrt{\mathbf{d}}} \sum_{\mathbf{k}} |\mathbf{kkkk}\rangle = |\mathrm{GHZ}_4\rangle.$$



Example: Teleportation

- Using the properties of the cups and caps we obtain a causal diagram that represents the $(\mathbf{a}, \mathbf{b})$ outcome of a Bell measurement by Alice, followed by local corrections dependent on the measurement result by Bob, corresponding to the Kraus operator $\mathbf{E}_{\mathbf{ab}} = \frac{1}{\mathbf{d}}\mathbb{1}$ for all $\mathbf{a}, \mathbf{b}$.
- Together these diagrams represent a physical operation, $\rho \mapsto \sum_{\mathbf{ab}} \mathbf{E}_{\mathbf{ab}} \rho \mathbf{E}_{\mathbf{ab}}^\dagger = \rho$, which faithfully transports any quantum state ρ from Alice to Bob.



Conclusions

- We have provided an explicit representation of dagger-compact-closed categories in terms of the quantum circuit model.
- The corresponding string diagram calculus extends the language of quantum circuit diagrams, enabling powerful new ways of manipulating circuits of systems of arbitrary dimension while remaining within a familiar, useful framework.
- Our approach has further applications in tensor network theory.

http://arxiv.org/abs/1010.4840 ville.bergholm@iki.fi