On Prototypes for Winslett’s Semantics of DL-Lite ABox Evolution

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Abstract. Evolution of Knowledge Bases expressed in Description Logics (DLs) proved its importance. Most studies on evolution in DLs have focused on model-based approaches to evolution semantics and in particular on Winslett’s semantics (WS). It was understood that evolution under WS even in tractable DLs, such as DL-Lite, suffers from inexpressibility, i.e., the result of evolution cannot be expressed in the same logics. In this work we show which combination of DL-Lite logical constructs is responsible for the inexpressibility and explain reasons for such a behaviour. We present novel techniques, based on what we called prototypes, to capture Winslett’s evolution in FO for DL-LiteR. We also discuss which fragments of DL-LiteR are closed under evolution.

1 Introduction

Description Logics (DLs) provide excellent mechanisms for representing structured knowledge by means of Knowledge Bases (KBs) $K$ that are composed of two components: TBox (describes intensional or general knowledge about an application domain) and ABox (describes facts about individual objects). DLs constitute the foundations for various dialects of OWL, the Semantic Web ontology language.

Traditionally DLs have been used for modeling static and structural aspects of application domains [1]. Recently, the scope of KBs has broadened, and they are now used also for providing support in the maintenance and evolution phase of information systems. This makes it necessary to study evolution of Knowledge Bases [2], where the goal is to incorporate a new knowledge $N$ into an existing KB $K$ so as to take into account changes that occur in the underlying application domain. In general, $N$ is represented by a set of formulas denoting those properties that should be true after $K$ has evolved, and the result of evolution, denoted $K \circ N$, is also intended to be a set of formulas. In the case where $N$ interacts with $K$ in an undesirable way, e.g., by causing the KB or relevant parts of it to become unsatisfiable, $N$ cannot simply be added to the KB. Instead, suitable changes need to be made in $K$ so as to avoid this undesirable interaction, e.g., by deleting parts of $K$ conflicting with $N$. Different choices for changes are possible, corresponding to different approaches to semantics for KB evolution [3][5].

One approach to evolution semantics that proved its importance is Winslett’s semantics (WS) [6], which is an update semantics in terms of Katsumo and Mendelzon [4], and was originally proposed for propositional theories. Under this semantics the result of evolution $K \circ N$ is a set of models of $N$ that are minimally distanced from models of $K$, where the distance is based on symmetric difference between models (see Section 3 for
Since the result of evolution $\mathcal{K} \diamond \mathcal{N}$ is a set of models, while $\mathcal{K}$ and $\mathcal{N}$ are logical theories, it is desirable to represent $\mathcal{K} \diamond \mathcal{N}$ as a logical theory using the same language as for $\mathcal{K}$ and $\mathcal{N}$. Thus, looking for representations of $\mathcal{K} \diamond \mathcal{N}$ is the main challenge in a study of evolution under WS. When $\mathcal{K}$ and $\mathcal{N}$ are propositional theories, representing $\mathcal{K} \diamond \mathcal{N}$ is well understood [5], while it becomes dramatically more complicated as soon as $\mathcal{K}$ and $\mathcal{N}$ are first-order, e.g., DL KBs [7].

In this work we study how WS can be applied to evolution of KBs under the following two assumptions. First, we assume that both $\mathcal{K}$ and $\mathcal{N}$ are written in a language of the DL-Lite family [8]. The focus on DL-Lite is not surprising since DL-Lite is tightly connected with conceptual data models and it is the basis of OWL 2 QL, a tractable OWL 2 profile. Second, we assume that $\mathcal{N}$ is a new ABox and the TBox of $\mathcal{K}$ should remain the same after the evolution. That is, we study a so-called ABox evolution. ABox evolution is important for areas, e.g., bioinformatics, where the structural knowledge TBox is well crafted and stable, while ABox facts about specific individuals may get changed, or/and new facts can be inserted in the ABox. These ABox changes should be reflected in KBs in a way that the TBox is not affected.

There are several works on WS for both DL-Lite and more expressive DLs. Liu, Lutz, Milicic, and Wolter studied Winslett’s evolution in expressive DLs [7], for KBs with empty TBoxes. Most of DLs they considered are not closed under WS and in order to close these logics they used “@” operator. Poggi, Lembo, De Giacomo, Lenzerini, and Rosati applied WS to DL-Lite [9] and proposed an algorithm to compute the result of evolution. It turned out that their algorithm is wrong, i.e. it is neither sound, nor complete [10]. Actually, such an algorithm cannot exist since Calvanese, Kharlamov, Nutt, and Zheleznyakov showed that, e.g., DL-Lite$_{FR}$ is not closed under WS of evolution [11], that is, there are $\mathcal{K}$ and $\mathcal{N}$ such that $\mathcal{K} \diamond \mathcal{N}$ is not axiomatizable in this family. Recently [12] we introduced prototypes, which are in a way generalization of the notion of canonical model, and proposed a way to capture some fragments of DL-Lite in FO[2], a fragment of first-order logic that uses two variables only.

Current work extends the preliminary results of [12]. Our goals here are
(i) to clarify our prototype-based techniques which was only sketched in [12],
(ii) to extend the techniques to wider DL-Lite fragments,
(iii) to gain a better understanding on which fragments of DL-Lite are closed under WS and how to approximate evolution results in DL-Lite.

We would also like to promote prototypes since we believe they are an useful tool to study evolution of ontologies and might be not only of DL-Lite ones.

In Sections 2 and 3 we define DL-Lite$_{R}$ and ABox evolution under WS. In Section 4 we give an intuition of our approach to capture WS of evolution for DL-Lite$_{R}$ KBs using prototypes and FO[2] theories. In Sections 5 and 6 we formalize the approach. Finally, we discuss properties and approximation of these theories.

## 2 DL-Lite$_{R}$

We introduce some basic notions of DLs (see [11] for more details). We consider a logic DL-Lite$_{R}$ of DL-Lite family of DLs [8,13]. DL-Lite$_{R}$ has the following constructs for (complex) concepts and roles: (i) $B ::= A \mid \exists R, (ii) C ::= B \mid \neg B, (iii) R ::= P \mid P^{-}$, where $A$ and $P$ stand for an atomic concept and role, respectively, which are just
names. A DL knowledge base (KB) $K = (T, A)$ is compound of two sets of assertions: TBox $T$, and ABox $A$. DL-Lite TBox assertions are concept inclusion assertions of the form $B \sqsubseteq C$ and role inclusion assertions $R_1 \sqsubseteq R_2$, while ABox assertions are membership assertions of the form $A(a)$, $\neg A(a)$, and $R(a, b)$. The active domain of $K$, denoted $\text{adom}(K)$, is the set of all constants occurring in $K$. The DL-Lite family has nice computational properties, for example, KB satisfiability has polynomial-time complexity in the size of the TBox and logarithmic-space in the size of the ABox $[14][15]$.

The semantics of DL-Lite KBs is given in the standard way: using first order interpretations $I$, all over the same countable domain $\Delta$. We assume that $\Delta$ contains the constants and $c^2 = c$, i.e., we adopt standard names. Alternatively, we view interpretations as sets of atoms: $A(a) \in I$ iff $a \in A^I$ and $P(a, b) \in I$ iff $(a, b) \in P^I$.

Definitions of $I$ being a model of an ABox or a TBox assertion $F$, denoted $I \models F$, and a KB $K$, denoted $I \models K$, are standard, as well as the notion of satisfiability. We use $\text{Mod}(K)$ to denote the set of all models of $K$. We use entailment on KBs $K \models K'$ in the standard sense. An ABox $T$-entails an ABox $A'$, denoted $A \models_T A'$, if $T \cup A = A'$, and $A$ is $T$-equivalent to $A'$, denoted $A \equiv_T A'$, if $A \models_T A'$ and $A' \models_T A$.

The deductive closure of a TBox $T$, denoted $cl(T)$, is the set of all TBox assertions $F$ such that $T \models F$. For satisfiable KBs $K = (T, A)$, a full closure of $A$ (wrt $T$), $\text{fcl}_T(A)$, is the set of all membership assertions $f$ (both positive and negative) over $\text{adom}(K)$ such that $A \models_T f$. Clearly, in DL-Lite both $cl(T)$ and $\text{fcl}_T(A)$ are computable in time quadratic in, respectively, $|T|$, i.e., the number of assertions of $T$, and $|T \cup A|$. For the ease of exhibition and wlg we assume that all TBoxes and ABoxes are closed.

A homomorphism $h$ from a model $I$ to a model $J$ is a structure-preserving mapping from $\Delta$ to $\Delta$ satisfying: (i) $h(a) = a$ for every constant $a$; (ii) if $a \in A^I$ (resp., $(\alpha, \beta) \in P^I$), then $h(\alpha) \in A^J$ (resp., $(h(\alpha), h(\beta)) \in P^J$) for every $A$ (resp., $P$). We write $I \hookrightarrow J$ if there is a homomorphism from $I$ to $J$. A canonical model $I$ of $K$, denoted as $I^\text{can}_K$ or just $I^\text{can}$ when $K$ is clear from the context, is a model of $K$ which can be homomorphically embedded in every model of $K$ $[3]$.

3 Winslett’s Semantics for Evolution of Knowledge Bases

We start with ABox evolution of single models under Winslett’s semantics. Let $K = (T, A)$ be a DL-Lite KB, $T$ a model of $K$, and $N$ a new ABox satisfiable with $T$. Evolution of a model $I$ of $K$ is based on the symmetric difference $\ominus$: $S_1 \ominus S_2 = (S_1 \setminus S_2) \cup (S_2 \setminus S_1)$, and defined as follows. The (result of) evolution of $I$ with $N$ under Winslett’s semantics (WS) $[2]$, denoted $I \circ N$, is the set of models $J$ such that: (i) $J \in \text{Mod}(T \cup N)$, and (ii) there is no model $J' \in \text{Mod}(T \cup N)$ satisfying $I \ominus J' \subset I \ominus J$.

Note that in Case (i) we have $\text{Mod}$ of both $T$ and $N$, which means that the evolution preserves both the old TBox and the new knowledge. Case (ii) guarantees the principle of minimal change $[5]$. We extend the definition to KBs.

The result of evolution of $K$ with $N$ under WS, denoted $K \circ N$, is the following set of models:

$$K \circ N = \bigcup_{I \in \text{Mod}(K)} I \circ N.$$ 

In terms of $[10]$, WS corresponds to $L^\text{a}_\sqsubseteq$ semantics, i.e., local model-based semantics based on atoms and set inclusion.
The input for evolution is two finite syntactic objects: a KB $K = (T, A)$ and a new information $N$, while the output $K \circ N$ is a set of models, which is an infinite object for DL-Lite$_R$. Indeed, $K \circ N$ is in general infinite. One can easily come up with examples where $K \circ N$ has an infinite number of infinite models. These observations imply that storing $K \circ N$ is infeasible and in practice one would like to represent the evolution as a KB $K'$. Moreover, one would like to stay within the same formalism and express $K'$ in DL-Lite$_R$.

Example 1. Consider the following DL-Lite KB $K_1 = (T_1, A_1)$ and $N_1 = \{C(b)\}$:

$$T_1 = \{A \sqsubseteq \exists R, \exists R^+ \sqsubseteq \neg C\}; \quad A_1 = \{A(a), C(e), C(d), R(a, b)\}.$$  

Consider the following model $I$ of $K_1$:

$$I: \quad A^I = \{a, x\}, \quad C^I = \{d, e\}, \quad R^I = \{(a, b), (x, b)\},$$

where $x \in \Delta \setminus \text{adm}$. The following models belong to $I \circ N_1$:

- $J_0$: $A^I = \emptyset$, $C^I = \{d, e, b\}$, $R^I = \emptyset$,
- $J_1$: $A^I = \{x\}$, $C^I = \{e, b\}$, $R^I = \{(x, d)\}$,
- $J_2$: $A^I = \{x\}$, $C^I = \{d, b\}$, $R^I = \{(x, e)\}$.

Indeed, all the models satisfy $N_1$ and $T_1$. To see that they are in $I \circ N_1$ observe that every model $J(I) \in (I \circ N_1)$ can be obtained from $I$ by making modifications that guarantee that $J(I) \models (N_1 \cup T_1)$ and that the distance between $I$ and $J(I)$ is minimal. What are these modifications? Since in every $J(I)$ the new assertion $C(b)$ holds and $(C \sqsubseteq \neg \exists R^+)$ in $T_1$, there should be no $R$-atoms with $b$-fillers at the second coordinate in $J(I)$. Hence, the necessary modifications of $I$ are either to drop (some of) the $R$-atoms $R(a, b)$ and $R(x, b)$, or to modify (some of) them, by substituting the $b$-fillers with another ones, while keeping the elements $a$ and $x$ on the first position. The model $J_0$ corresponds to the case when both $R$-atoms are dropped, while in $J_1$ and $J_2$ only $R(a, b)$ is dropped and $R(x, b)$ is modified to $R(x, d)$ and $R(x, e)$, respectively. Note that the modification in $R(x, b)$ leads to a further change in the interpretation of $C$ in both $J_1$ and $J_2$, namely, $C(d)$ and $C(e)$ should be dropped, respectively.

4 Prototypes for Winslett’s Semantics

We first present a general discussion on issues with capturing WS in DL-Lite, then give an intuition of our approach for capturing it in FO[2], and finally give an example of how the approach works. In the next section we formalize the approach.

ABox Evolution of a DL-Lite KB $K$ with an ABox $N$ is the set of models $K \circ N$ that may not have a canonical one [12]. This immediately yields that $K \circ N$ cannot be described (aka axiomatized) in any language of the DL-Lite family.

Example 2. We now illustrate the lack of canonical models in $K_1 \circ N_1$ from Example 1. One can verify that any model $J_{can}$ that can be homomorphically embedded into $J_0$, $J_1$, and $J_2$ is such that $A^{J_{can}} = R^{J_{can}} = \emptyset$, and $c, d \notin C^{J_{can}}$. It is easy to check that such a model does not belong to $K_1 \circ N_1$. Hence, there is no canonical model in $K \circ N$ and it is inexpressible in DL-Lite.
A closer look at sets $K \diamond N$ for different $K$ and $N$ gave a surprising outcome: all of them satisfy the following property.

**Theorem 3.** $K \diamond N$ can be divided (but in general not partitioned) into finitely many subsets $S_0, \ldots, S_n$ of models, where each $S_i$ has a canonical model $J_i$. Each of these canonical models is a minimal element in $K \diamond N$ wrt homomorphisms.

We called these $J_i$s prototypes [12]. Thus, capturing $K \diamond N$ in some logics boils down to

(i) capturing each $S_i$ with some theory $K_{S_i}$ and

(ii) taking the disjunction across all $K_{S_i}$.

This will give the desired theory $K' = K_{S_1} \lor \cdots \lor K_{S_n}$.

Unfortunately, some of $K_{S_i}$ are not DL-Lite theories (while they are FO[2] theories, see Section 5 for details).

We construct $K'$ in two steps. First, we construct DL-Lite KBs $K(J_i)$ for each $J_i$ such that $K(J_i)$ is a sound approximation of $S_i$s, that is, $S_i \subseteq \text{Mod}(K(J_i))$. Second, based on $K$ and $N$, we construct an FO[2] formula $\Psi$, which cancels out all the models in $\text{Mod}(K(J_i)) \setminus S_i$, that is, $K_{S_0} \lor \cdots \lor K_{S_n} = \Psi \land (K(J_0) \lor \cdots \lor K(J_n))$.

To get a better intuition on our approach, consider Figure 1, where the result of evolution $K \diamond N$ is depicted as the figure with solid-line borders (each point within the figure is a model in $K \diamond N$). Assume that $K \diamond N$ can be divided in four subsets $S_0, \ldots, S_3$.

To emphasize this fact, $K \diamond N$ looks similar to a hand with four fingers, where each finger represents an $S_i$. Consider the left part of Figure 1. Each of $S_i$s has a canonical model depicted as a star. Using DL-Lite$_R$, we can provide KBs $K(J_0), \ldots, K(J_3)$ that are sound approximations of corresponding $S_i$s. We depict the models $\text{Mod}(K(J_i))$ as ovals with dashed-line boarders. Consider the right part of Figure 1. In this figure we depict in grey the models $\text{Mod}(K(J_i)) \setminus S_i$ that are cut off by $\Psi$.

Before proceeding to the next section where we formalize our approach, we introduce prototypes formally.

**Definition 4.** Let $K$ be a DL-Lite$_R$ KB and $N$ be an ABox. A prototypical set for $K \diamond N$ is a minimal subset $\mathcal{J} = \{J_0, \ldots, J_n\}$ of $K \diamond N$ satisfying the property:

for every $J \in K \diamond N$ there is $J_i \in \mathcal{J}$ such that $J_i \hookrightarrow J$.

We call every $J_i \in \mathcal{J}$ a prototype for $K \diamond N$. Note that prototypes generalize canonical models in the sense that every set of models with a canonical one, say $\text{Mod}(K)$ for a DL-Lite$_R$ KB $K$, has a prototype, which is exactly the canonical model.

## 5 Computing Winslett’s Semantics When No Roles Interact

We first discuss some of the reasons of WS inexpressibility in our examples and DL-Lite$_R$. 

![Fig. 1. Graphical representation of our approach to capture the result of evolution under WS.](image-url)
Dual-Affection of Roles. As we discussed in the previous section and illustrated in Example 1, sets of models \( K \circ N \) that result from Winslett’s evolution do not have canonical models. We now give an intuition why in \( K \circ N \) canonical models are missing. Observe that in Example 1 the role \( R \) is affected by the old TBox \( T_1 \) as follows:

(i) \( T_1 \) places (i.e., enforces the existence of) \( R \)-atoms in the evolution result, and on one of coordinates of these \( R \)-atoms, there are constants from specific sets, e.g., \( A \subseteq R \) of \( T_1 \) enforces \( R \)-atoms with constants from \( A \) on the first coordinate, and

(ii) \( T_1 \) forbids \( R \)-atoms in \( K_1 \circ N_1 \) with specific constants on the other coordinate, e.g., \( \exists R^{-} \subseteq \neg C \) forbids \( R \)-atoms with \( C \)-constants on the second coordinate.

Due to this dual-affection (both positive and negative) of the role \( R \) in \( T_1 \), we were able to provide an ABox \( A_1 \) and \( N_1 \), which together triggered the case analyses of modifications on the model \( I \), that is, \( A_1 \) and \( N_1 \) were triggers for \( R \). Existence of such an affected \( R \) and triggers \( A_1 \) and \( N_1 \) made \( K_1 \circ N_1 \) inexpressible in DL-Lite\(_R\). Therefore, we now learn how to detect dually-affected roles in TBoxes and how to understand whether these roles are triggered by an ABox and a new (ABox) information.

Formally, let \( T \) be a TBox, a role \( R \) is dually-affected in \( T \) if for some concepts \( A \) and \( B \) it holds that \( T \models A \subseteq R \) and \( T \models \exists R^{-} \subseteq \neg B \). Let \( N \) be an ABox satisfiable with \( T \), then a dually-affected role \( R \) is triggered by \( N \) if there is a concept \( B \) such that \( T \models \exists R^{-} \subseteq \neg B \) and \( N \models \top B(b) \) for some constant \( b \). The set \( TR(T,N) \) (or simply \( TR \)) is the set of all roles (dually-affected in \( T \)) that are triggered by \( N \).

Description Logics DL-Lite\(_R^I\). We now show a restriction of DL-Lite\(_R\) for which we later present an algorithm to capture WS using prototypical set. DL-Lite\(_R^I\) (where \( I \) stands for (mutual) independence of roles) is a restriction of DL-Lite\(_R\) in which TBoxes \( T \) satisfy: for any two roles \( R \) and \( R' \), \( T \not\models \exists R \subseteq \exists R' \) and \( T \not\models \exists R \subseteq \neg \exists R' \). That is, we forbid direct role interaction (subsumption and disjointness) between role projections. Some interaction is still possible: role projections may contain the same concept. This restriction allows us to analyze evolution affecting roles independently for every role.

Components for Computation. We now introduce several notions and notations that we further use in the description of our algorithm. An alignment of a model \( I \) with \( N \), denoted \( \text{Align}(I,N) \), is the interpretation:

\[
\text{Align}(I,N) = \{ f \mid f \in I \text{ and } f \text{ is satisfiable with } N \}.
\]

An auxiliary set of atoms \( AA \) (Auxiliary Atoms) that, due to evolution, should be deleted from the original KB and have some extra condition on the first coordinate is:

\[
\text{AA}(T,A,N) = \{ R(a,b) \in fcl_T(A) \mid T \models A \subseteq R, A \models \top A(a), N \models \top \neg \exists R^{-}(b) \}.
\]

For the set \( TR \) we define the set of forbidden atoms \( FA(T,A,N)(R_i) \) of the original ABox as:

\[
\{ D(c) \in fcl_T(A) \mid \exists R^{-}(c) \land D(c) \models \top \bot, N \not\models \top D(c), \text{ and } N \not\models \top \neg D(c) \}.
\]
Consequently, the set of forbidden atoms for the entire KB \((\mathcal{T}, \mathcal{A})\) and \(\mathcal{N}\) is

\[\text{FA}(\mathcal{T}, \mathcal{A}, \mathcal{N}) = \cup_{R_i \in \text{TR}} \text{FA}(\mathcal{T}, \mathcal{A}, \mathcal{N})(R_i).\]

In the following we omit the arguments \((\mathcal{T}, \mathcal{A}, \mathcal{N})\) whenever they are clear from the context. For a role \(R\), the set \(SC(R)\), where \(SC\) stands for sub-concepts, is a set of those concepts which are immediately under \(\exists R\) in the concept hierarchy generated by \(\mathcal{T}\):

\[SC(R) = \{A \mid \mathcal{T} \models A \sqsubseteq R\text{ and there is no } A' \text{ s.t. } \mathcal{T} \models A \sqsubseteq A' \text{ and } \mathcal{T} \models A' \sqsubseteq R\}.\]

If \(f\) is an ABox assertion, then \(\text{root}^f_{\mathcal{T}}(f)\) is a set of all the atoms that \(\mathcal{T}\)-entail \(f\). For example, \(A(x) \in \text{root}^f_{\mathcal{T}}(\exists R(x))\) if \(\mathcal{T} \models A \sqsubseteq \exists R\).

We are ready to proceed to construction of prototypes.

**Constructing Zero-Prototype.** The procedure \(BZP(\mathcal{K}, \mathcal{N})\) (Build Zero Prototype) in Figure 2 constructs the main prototype \(\mathcal{J}_0\) for \(\mathcal{K}\) and \(\mathcal{N}\) from \(\text{DL-Lite}^{\mathcal{T}}\), which we call zero-prototype. Based on \(\mathcal{J}_0\) we will construct all the other prototypes. To build \(\mathcal{J}_0\) one has to align the canonical model of \(\mathcal{K}\) with \(\mathcal{N}\), and then delete from the resulting set of atoms all the auxiliary atoms \(R(a, b)\) (from \(AA(\mathcal{K}, \mathcal{N})\)). In the case when no \(R(a, \beta)\) for some constant \(\beta\) such that \(R(a, \beta) \in AA(\mathcal{K}, \mathcal{N})\) is in the canonical model, we also delete atoms \(\text{root}^f_{\mathcal{T}}(\exists R(a))\), since their presence in the model and the absence of \(R\)-atoms with \(a\) at the first coordinate would contradict the TBox.

**Constructing Other Prototypes.** The procedure \(BP(\mathcal{K}, \mathcal{N}, \mathcal{J}_0)\) (Build Prototypes) of constructing \(\mathcal{J}\) for the case of \(\text{DL-Lite}^{\mathcal{T}}\), takes \(\mathcal{J}_0\) and manipulates with it by first dropping atoms from \(\text{FA}\) and then adding atoms in order to compensate the dropped ones so that the result is an evolved model under WS. It can be found in Figure 3.

We conclude the discussion on the algorithms with a theorem:

**Theorem 5.** Let \(\mathcal{K} = (\mathcal{T}, \mathcal{A})\) be a \(\text{DL-Lite}^{\mathcal{T}}\) KB, and \(\mathcal{N}\) a \(\text{DL-Lite}^{\mathcal{T}}\) ABox consistent with \(\mathcal{T}\). Then the set \(BP(\mathcal{K}, \mathcal{N}, \mathcal{J}_0)\) is a prototypal set for \(\mathcal{K} \circ \mathcal{N}\).

Continuing with Example 1 it is easy to check that the prototypical set for \(\mathcal{K}_1\) and \(\mathcal{N}_1\) is \(\{\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\}\), where \(\mathcal{J}_0, \mathcal{J}_1, \) and \(\mathcal{J}_2\) are described in the example and

\[\mathcal{J}_3:\ A^I = \{x, y\}, \ C^I = \{b\}, \ R^I = \{(x, d), (y, e)\} .\]

We proceed to correctness of \(BP\) in capturing evolution in \(\text{DL-Lite}^{\mathcal{T}}\), where we use the following set \(\text{FC}[\mathcal{T}, \mathcal{A}, \mathcal{N}](R_i) = \{c \mid D(c) \in \text{FA}[\mathcal{T}, \mathcal{A}, \mathcal{N}](R_i)\}\), that collects all the constants that participate in the forbidden atoms.
Theorem 6. Let $\mathcal{K} = (T, A)$ be a DL-Lite $^\Phi$ KB, $\mathcal{N}$ a DL-Lite $^R$ ABox consistent with $T$, and BP($\mathcal{K}, \mathcal{N}$; BZP($\mathcal{K}, \mathcal{N}$)) = $\{J_0, \ldots, J_n\}$ is a prototypical set for $\mathcal{K} \circ \mathcal{N}$. Then

$$\mathcal{K} \circ \mathcal{N} = \text{Mod}(T) \cap \text{Mod}(A_0 \lor \ldots \lor A_n) \cap \text{Mod}(\Phi \land \Psi),$$

where $A_i$ is a DL-Lite $^R$ ABox such that $J_i$ is a canonical model for $(T, A_i)$, and

$$\Phi = \bigwedge_{R_i \in TR} \left( \bigwedge_{c_j \in \mathcal{P}(R_i)} \forall x. \left[ (R_i(x, c_j) \rightarrow (\text{root}_{\mathcal{T}}(\exists R_i(x)) \neq \emptyset) \right] \land \forall y. (R_i(x, c_j) \land R_i(x, y) \rightarrow y = c_j) \right),$$

$$\Psi = \bigwedge_{R(a,b) \in S_{at}} \exists R(a) \rightarrow \text{root}_{\mathcal{T}}(\exists R(a)) \cap \text{fcl}_{\mathcal{T}}(A).$$

The $A_i$ mentioned in Theorem 6 can be constructed in the similar way that the corresponding prototypes $J_i$, taking the original ABox $A$ instead of $I^{\text{can}}$. Note that an ABox may include a negative literals, like $\neg B(c)$. Those should be treated in the same way that the positive literal (atoms) are. We will denote such an ABox as $A[J_i]$.

Theorem 7. A prototype $J_i$ is a canonical model of the KB $(T, A[J_i])$.

Continuing with Example 1 the ABoxes $A[J_0]$ and $A[J_1]$ are as follows:

$$A[J_0] = \{ C(d), C(e), C(b) \}; \quad A[J_1] = \{ A(x), C(e), C(b), R(x, d) \}.$$ 

$A[J_2]$ and $A[J_3]$ can be built in the similar way. Note that only $A[J_0]$ is in DL-Lite$^R$, while writing $A[J_1], \ldots, A[J_3]$ requires variables in ABoxes. Variables, also known as soft constants, are not allowed in DL-Lite$^R$ ABoxes, while present in DL-Lite$^R_S$ ABoxes. Soft constants $x$ are constants not constrained by the Unique Name Assumption: it is not necessary that $x^T = x$. Since DL-Lite$^R_S$ is tractable and FO rewritable $^{[13]}$, expressing $A[J_i]$ in DL-Lite$^R_S$ instead of DL-Lite$^R$ does not affect tractability.

6 Computing Winslett’s Semantics with Roles Interaction

The algorithm BP for constructing prototypical set works only when roles do not interact. The following example illustrates that it does not work in a general case.

Example 8. Consider a KB $\mathcal{K}_2 = (T_2, A_2)$ and a new ABox $\mathcal{N}_2 = \{ C(b) \}$:

- TBox $T_2$: $\exists R^0 \subseteq \neg \exists P^-$, $\exists R^0 \subseteq \neg C$, $A \subseteq \exists R$, $B \subseteq \exists P$;
- ABox $A_2$: $R(a,b)$, $A(a)$, $R(f,g)$, $A(f)$, $P(c,d)$, $B(c)$, $C(e)$.

One can check that the following model $J'$ is in $\mathcal{K}_2 \circ \mathcal{N}_2$:

$$A^{J'} = \{ y \}, \quad B^{J'} = \{ z \}, \quad C^{J'} = \{ b, e \}, \quad R^{J'} = \{ (y, d) \}, \quad P^{J'} = \{ (z, g) \}.$$ 

At the same time, BP over $\mathcal{K}_2$ and $\mathcal{N}_2$ returns the following four prototypes only:

<table>
<thead>
<tr>
<th>$A^{J_i}$</th>
<th>$B^{J_i}$</th>
<th>$C^{J_i}$</th>
<th>$R^{J_i}$</th>
<th>$P^{J_i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 0$</td>
<td>${ f }$</td>
<td>${ e }$</td>
<td>${ b, c }$</td>
<td>${ (f, g) }$</td>
</tr>
<tr>
<td>$i = 1$</td>
<td>${ f, x }$</td>
<td>${ e }$</td>
<td>${ b }$</td>
<td>${ (f, g), (x, e) }$</td>
</tr>
<tr>
<td>$i = 2$</td>
<td>${ f, y }$</td>
<td>$\emptyset$</td>
<td>${ b, e }$</td>
<td>${ (f, g), (y, d) }$</td>
</tr>
<tr>
<td>$i = 3$</td>
<td>${ f, x, y }$</td>
<td>$\emptyset$</td>
<td>${ b }$</td>
<td>${ (f, g), (x, e), (y, d) }$</td>
</tr>
</tbody>
</table>

where $x$ and $y$ are fresh constants. It is easy to see that none of $J_i$s is homomorphically embeddable in $J'$. Thus, BP does not capture $J'$ and it is incomplete. $\blacksquare$
The reason is: when, while constructing prototypes with BP, we delete a forbidden atom (an atom from FA), it may trigger another dually-affected role and such triggering may require further modifications, which are not accounted by BP. In order to compute all prototypes we should run BP recursively: considering the prototypes obtained at the previous step as zero ones. We present a recursive algorithm \( BP_{rec} \) for building prototypes for general DL-Lite KBs in Figure 4. The following theorem shows the correctness of the algorithm.

**Theorem 9.** Let \( \mathcal{K} = (\mathcal{T}, \mathcal{A}) \) be a DL-Lite KB and \( \mathcal{N} \) a DL-Lite ABox consistent with \( \mathcal{T} \). Then the algorithm \( BP_{rec}(\mathcal{K}, \mathcal{N}) \) terminates and returns the finite set which is a prototypical set for \( \mathcal{K} \circ \mathcal{N} \).

We illustrate \( BP_{rec} \) on the following example.

**Example 10.** Consider KB \( \mathcal{K}_2 = (\mathcal{T}_2, \mathcal{A}_2) \) and a new ABox \( \mathcal{N}_2 \) from Example 8. Let us compute \( BP_{rec}(\mathcal{K}_2, \mathcal{N}_2) \). First we run \( BP(\mathcal{K}, \mathcal{N}, \mathcal{J}_0) \) and it returns four prototypes: \( \mathcal{J}_0 \), \( \mathcal{J}_1 \), \( \mathcal{J}_2 \), and \( \mathcal{J}_3 \) (see Example 8). Now we apply the BP procedure to \( \mathcal{J}_1 \), \( \mathcal{J}_2 \), and \( \mathcal{J}_3 \). It is easy to see that \( BP(\mathcal{K}, \mathcal{N} \cup \{ A(x), R(x, e) \}, \mathcal{J}_1) = \emptyset \), since no role atom except for \( R(a, b) \) was affected. Consider \( BP(\mathcal{K}, \mathcal{N} \cup \{ A(y), R(y, d) \}, \mathcal{J}_2) \): it consists of the only prototype \( \mathcal{J}_4 \):

\[
A^{\mathcal{J}_4} = \{ y \}, \quad B^{\mathcal{J}_4} = \{ z \}, \quad C^{\mathcal{J}_4} = \{ b, e \}, \quad R^{\mathcal{J}_4} = \{ (y, d) \}, \quad P^{\mathcal{J}_4} = \{ (z, g) \}.
\]

The uniqueness of the prototype follows from the fact that the role atom that was affected in \( \mathcal{J}_2 \) is \( P(c, d) \) and \( \mathcal{F}A[\mathcal{T}, \mathcal{A}, \mathcal{N} \cup \{ A(y), R(y, d) \}](P) = \{ \exists \! R^{-}(g) \} \). Finally, running \( BP(\mathcal{T}, \mathcal{N} \cup \{ A(y), R(y, d) \}, B(z), P(z, g), \mathcal{J}_4) \) we obtain a prototype \( \mathcal{J}_5 \):

\[
A^{\mathcal{J}_5} = \{ y, v \}, \quad B^{\mathcal{J}_5} = \{ z \}, \quad C^{\mathcal{J}_5} = \{ b \}, \quad R^{\mathcal{J}_5} = \{ (y, d), (v, e) \}, \quad P^{\mathcal{J}_5} = \{ (z, g) \}.
\]

Note that \( BP(\mathcal{T}, \mathcal{N} \cup \{ A(y), R(y, d), B(z), P(z, g), A(v), R(v, e) \}, \mathcal{J}_5) = \emptyset \). Analogously, \( \mathcal{J}_6 \) can be obtained by running \( BP(\mathcal{K}, \mathcal{N} \cup \{ A(x), A(y), R(x, e), R(y, d) \}, \mathcal{J}_5) \):

\[
A^{\mathcal{J}_6} = \{ x, y \}, \quad B^{\mathcal{J}_6} = \{ z \}, \quad C^{\mathcal{J}_6} = \{ b \}, \quad R^{\mathcal{J}_6} = \{ (x, e), (y, d) \}, \quad P^{\mathcal{J}_6} = \{ (z, g) \}.
\]

Thus, the prototypical set \( \mathcal{T}_i \) for \( \mathcal{K} \circ \mathcal{N} \) is \( \{ \mathcal{J}_i \}_{i=0}^6 \).

We conclude with the theorem that \( BP_{rec} \) gives a sound approximation for WS.
Theorem 11. Let $K = (T, A)$ be a DL-Lite$_R$ KB, $\mathcal{N}$ a DL-Lite$_R$ ABox consistent with $T$, and $BP_{rec}(K, \mathcal{N}) = \{J_0, \ldots, J_n\}$ is a prototypal set for $K \circ \mathcal{N}$. Then

$$K \circ \mathcal{N} \subseteq \text{Mod}(T) \cap \text{Mod}(A_0 \lor \cdots \lor A_n) \cap \text{Mod}(\Phi \land \Psi),$$

where $A_i$ is a DL-Lite$_R$ ABox such that $J_i$ is a canonical model for $(T, A_i)$ and $\Phi$ and $\Psi$ are as they defined in Theorem 6.

6.2 Closure Under Evolution and Approximation

Next theorem allows us to approximate results of evolution under WS, since FO[2] is decidable.

Theorem 12. $K \circ \mathcal{N}$ under WS for KBs in DL-Lite$_R$ can be captured in FO[2].

As a future work we are going to study ways to approximate the resulted FO[2] theories in DL-Lite.

Finally, we discuss cases when the result of Winslett’s evolution is expressible in DL-Lite$_R$. The following formulas appearing in Theorem 6 are not expressible in DL-Lite$_R$: (i) the disjunction of the ABoxes $A_0 \lor \cdots \lor A_n$ and (ii) formula $\Phi \land \Psi$.

The disjunction of ABoxes becomes expressible when it is of the length one, i.e., there is the only prototype: $J_0$. The last statement yields that $F\mathcal{A} = \emptyset$ and therefore $\Phi$ is always true. The formula $\Psi$ becomes trivially true when $AA = \emptyset$, i.e., for every atom $R(a, b) \in fcl_T(A)$ either $\mathcal{N} \models \neg \exists R^- (b)$ or $\text{root}_T(\exists R_i(a_i)) \cap fcl_T(A) = \emptyset$. As one can see, the condition of expressibility of the result in DL-Lite$_R$ (emptiness of $F\mathcal{A}$ and $AA$), depends on a TBox, an ABox, and a new information. Hence, if we do a chain of evolution, at some step the result may be not expressible in DL-Lite$_R$. Since TBox stays unchangeable, to guarantee the expressibility we need to find TBoxes $T$ such that $(T, A) \circ \mathcal{N}$ is expressible in DL-Lite$_R$ for every $A$ and $\mathcal{N}$. A condition that guarantees the emptiness of $F\mathcal{A}$ and $AA$ is: for every role $R \in \Sigma(K \cup \mathcal{N})$ at least one of the following items holds: (1) there is no concept $C$ such that $T \models \exists R^- \subseteq \neg C$, or (2) there is no concept $A$ such that $T \models A \subseteq \exists R$. The former conditions gives that $TR = \emptyset$ since $\mathcal{N} \nvdash \exists R^- (b)$, which leads to $F\mathcal{A} = AA = \emptyset$. The latter one yields that $SC(R) = \emptyset$, therefore $TR$ again is empty.

As a practical summary of this section, given a KB $K$ and a new ABox $\mathcal{N}$, one can check (in polynomial time) whether any dually-affected role is “triggered” by $\mathcal{N}$. If it is not the case, one can compute (in polynomial time) an evolved KB $K'$ that exactly captures $K \circ \mathcal{N}$. Otherwise, it is the case that $K \circ \mathcal{N}$ is inexpressible in DL-Lite$_R$. Thus, one can compute an FO[2] theory that captures $K \circ \mathcal{N}$ and then approximate it in DL-Lite$_R$, by, for example, dropping all the not DL-Lite$_R$ formulas. We will not focus on approximation in this paper.

7 Conclusion

We studied how to capture ABox evolution for DL-Lite$_R$ under WS. In general the result of evolution requires constructs that are not present in DL-Lite$_R$, and even not in DL-Lite, such as disjunction. Moreover, in general the result of evolution, which is a set of models, does not even have a canonical model, which should always exist for
any DL-Lite theory. It turned out that the inexpressibility is caused by a condition on the TBox level, which we called dual-interaction: by pairs of assertions of the form $A \sqsubseteq \exists R$ and $\exists R^{-} \sqsubseteq \neg B$. In order to capture evolution results in the presence of dual-interactions, we introduced prototypes. Our approach is based on the observation that evolution results can be divided into a finite number of subsets and each of them has a canonical model, i.e., a prototype. These subsets can be captured by theories guided by prototypes and the disjunction of these theories, compensated with two formulas, captures evolution results and is in $\text{FO}[2]$. We proved that this technique works for DL-Lite$_R$. We are currently working on efficient approximation of the obtained $\text{FO}[2]$ theory in DL-Lite and on extending results to capture evolution for other DL-Lite languages.

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