Reverse Mathematics and Well-ordering Principles: A Pilot Study

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Abstract

The larger project broached here is to look at the generally Π^1_2 sentence *"if X is well ordered then $f(X)$ is well ordered"*, where f is a standard proof theoretic function from ordinals to ordinals. It has turned out that a statement of this form is often equivalent to the existence of countable coded ω-models for a particular theory T_f whose consistency can be proved by means of a cut elimination theorem in infinitary logic which crucially involves the function f. To illustrate this theme, we prove in this paper that the statement *"if X is well ordered then ε_X is well ordered"* is equivalent to $\mathbf{ACA}_0^+$. This was first proved by Marcone and Montalban [7] using recursion-theoretic and combinatorial methods. The proof given here is principally proof-theoretic, the main techniques being Schütte's method of proof search (deduction chains) [11] and cut elimination for a (small) fragment of $\mathcal{L}_{\omega_1,\omega}$.

1 Introduction

This paper will be concerned with a particular Π^1_2 statement of the form

$$\mathbf{WOP}(f): \qquad \forall X\,[\mathbf{WO}(X) \to \mathbf{WO}(f(X))] \tag{1}$$

where f is a standard proof theoretic function from ordinals to ordinals and $\mathbf{WO}(X)$ stands for 'X is a well ordering'. There are by now several examples of functions f where the statement $\mathbf{WOP}(f)$ has turned out to be equivalent to one of the theories of reverse mathematics over a weak base theory (usually $\mathbf{RCA}_0$). The first example is due to Girard [4].

Theorem 1.1. (Girard 1987) *Over $\mathbf{RCA}_0$ the following are equivalent:*

(i) Arithmetic Comprehension

(ii) $\forall \mathfrak{X}\,[\mathbf{WO}(\mathfrak{X}) \to \mathbf{WO}(2^{\mathfrak{X}})]$.

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Recently two new results appeared in preprints [7, 3].

Theorem 1.2. (Marcone,Montalbán 2007) *Over* $\mathbf{RCA}_0$ *the following are equivalent:*

(i) $\mathbf{ACA}_0^+$

(ii) $\forall \mathfrak{X}\, [\mathbf{WO}(\mathfrak{X}) \to \mathbf{WO}(\varepsilon_{\mathfrak{X}})]$.

Theorem 1.3. (Friedman,Montalbán,Weiermann 2007) *Over* $\mathbf{RCA}_0$ *the following are equivalent:*

(i) $\mathbf{ATR}_0$

(ii) $\forall \mathfrak{X}\, [\mathbf{WO}(\mathfrak{X}) \to \mathbf{WO}(\varphi\mathfrak{X}0)]$.

Here $\mathbf{ACA}_0^+$ denotes the theory $\mathbf{ACA}_0$ augmented by an axiom asserting that for any set X the ω-th jump in X exists; $\mathbf{ATR}_0$ asserts the existence of sets constructed by transfinite iterations of arithmetical comprehension; $\alpha \mapsto \varepsilon_\alpha$ denotes the usual ε function while φ stands for the two-place Veblen function familiar from predicative proof theory (cf. [11]). More detailed descriptions of $\mathbf{ACA}_0^+$ and the function $\mathfrak{X} \mapsto \varepsilon_{\mathfrak{X}}$ will be given shortly. Definitions of the familiar subsystems of reverse mathematics can be found in [13].

The proofs of Theorems 1.2 and 1.3 use rather sophisticated recursion-theoretic results about linear orderings and are quite combinatorial. The proof of Theorem 1.2 builds on a theorem by Hirst while the proof of Theorem 1.3 employs a result of Steel's [15] about descending sequences of degrees which states that if $Q \subseteq \mathrm{Pow}(\omega) \times \mathrm{Pow}(\omega)$ is arithmetic, then there is no sequence $\{A_n \mid n \in \omega\}$ such that (a) for every n, A_{n+1} is the unique set such that $Q(A_n, A_{n+1})$, (2) for every n, $A'_{n+1} \leq_T A_n$.

For a proof theorist, Theorems 1.2 and 1.3 bear a striking resemblance to cut elimination theorems for infinitary logics. This prompted the second author of this paper to look for proof-theoretic ways of obtaining these results. The hope was that this would also unearth a common pattern behind them and possibly lead to more results of this kind. To start this project we shall give a new proof of Theorem 1.2 in this article. It is principally proof-theoretic, the main techniques being Schütte's method of proof search (deduction chains) [11] for proving the completeness theorem and cut elimination for a (small) fragment of $\mathcal{L}_{\omega_1,\omega}$. The general pattern, of which this paper provides a first example, is that a statement $\mathbf{WOP}(f)$ is often equivalent to a familiar cut elimination theorem for an infinitary logic which in turn is equivalent to the assertion that every set is contained in an ω-model of a certain theory T_f. The generality of this theme and the proof technology will be further substantiated in [10]. [10] utilizes well known cut elimination results from predicative proof theory (ramified analysis) to give a proof-theoretic treatment of Theorem 1.3.

2 The ordering $\varepsilon_{\mathfrak{X}}$

Via simple coding procedures, countable well-orderings and functions on them can be expressed in the language of second order arithmetic, L_2. Variables $X, Y, Z, \ldots$ are supposed to range over subsets of $\mathbb{N}$. Using an elementary injective pairing function $\langle , \rangle$ (e.g. $\langle n, m\rangle := (n+m)^2+n+1$), every set X encodes a sequence of sets $(X)_i$, where $(X)_i := \{m \mid \langle i, m\rangle \in X\}$. We also adopt from [13, II.2] the method of encoding finite sequences $(n_0, \ldots, n_{k-1})$ of natural numbers as single numbers $\langle n_0, \ldots, n_{k_1}\rangle$.

Ordinal representation systems for the ordinals below ε_0 (i.e. the first ordinal α such that $\omega^\alpha = \alpha$) as well as stronger ones closed under the function $\alpha \mapsto \varepsilon_\alpha$ can be found in many books on proof theory (cf. [11, 8, 4]). Here we require a generalized version of ε_α where α is replaced by an arbitrary well-ordering.

Definition 2.1. A structure $\mathfrak{X} = (X, <_X)$ is a *well-ordering* (abbrev. $\mathbf{WO}(\mathfrak{X})$) if $<_X$ is a linear ordering of X (the field of $<_X$) and every non-empty subset U of X has a $<_X$-least element.
The ordering $<_{\varepsilon_{\mathfrak{X}}}$ and its field $|\varepsilon_{\mathfrak{X}}|$ are inductively defined as follows:

1. $0 \in |\varepsilon_{\mathfrak{X}}|$.
2. $\varepsilon_u \in |\varepsilon_{\mathfrak{X}}|$ for every $u \in X$, where $\varepsilon_u := \langle 0, u\rangle$.
3. If $\alpha_1, \ldots, \alpha_n \in |\varepsilon_{\mathfrak{X}}|$, $n > 1$ and $\alpha_n \leq_{\varepsilon_{\mathfrak{X}}} \ldots \leq_{\varepsilon_{\mathfrak{X}}} \alpha_1$, then
$$\omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \in |\varepsilon_{\mathfrak{X}}|$$
where $\omega^{\alpha_1} + \ldots + \omega^{\alpha_n} := \langle 1, \langle \alpha_1, \ldots, \alpha_n\rangle\rangle$.
4. If $\alpha \in |\varepsilon_{\mathfrak{X}}|$ and α is not of the form ε_u, then $\omega^\alpha \in |\varepsilon_{\mathfrak{X}}|$, where $\omega^\alpha := \langle 1, \alpha\rangle$.
5. $0 <_{\varepsilon_{\mathfrak{X}}} \varepsilon_u$ for all $u \in X$.
6. $0 <_{\varepsilon_{\mathfrak{X}}} \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$ for all $\omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \in |\varepsilon_{\mathfrak{X}}|$.
7. $\varepsilon_u <_{\varepsilon_{\mathfrak{X}}} \varepsilon_v$ if $u, v \in X$ and $u <_X v$.
8. If $\omega^{\alpha_1}+\ldots+\omega^{\alpha_n} \in |\varepsilon_{\mathfrak{X}}|$, $u \in X$ and $\alpha_1 <_{\varepsilon_{\mathfrak{X}}} \varepsilon_u$ then $\omega^{\alpha_1}+\ldots+\omega^{\alpha_n} <_{\varepsilon_{\mathfrak{X}}} \varepsilon_u$.
9. If $\omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \in |\varepsilon_{\mathfrak{X}}|$, $u \in X$, and $\varepsilon_u <_{\varepsilon_{\mathfrak{X}}} \alpha_1$ or $\varepsilon_u = \alpha_1$, then $\varepsilon_u <_{\varepsilon_{\mathfrak{X}}} \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$.
10. If $\omega^{\alpha_1} + \ldots + \omega^{\alpha_n}, \omega^{\beta_1} + \ldots + \omega^{\beta_m} \in |\varepsilon_{\mathfrak{X}}|$, then
$$\begin{aligned}&\omega^{\alpha_1} + \ldots + \omega^{\alpha_n} <_{\varepsilon_{\mathfrak{X}}} \omega^{\beta_1} + \ldots + \omega^{\beta_m} \text{ iff}\\ &n < m \wedge \forall i \leq n\ \alpha_i = \beta_i \quad \text{or}\\ &\exists i \leq \min(n, m)[\alpha_i <_{\varepsilon_{\mathfrak{X}}} \beta_i \wedge \forall j < i\ \alpha_j = \beta_j].\end{aligned}$$

Definition 2.2. *Let* $\varepsilon_{\mathfrak{X}} = \langle |\varepsilon_{\mathfrak{X}}|, <_{\varepsilon_{\mathfrak{X}}}\rangle$.

It is an empirical fact that ordinal representation systems emerging in proof theory are always elementary recursive and their basic properties are provable in weak fragments of arithmetic like elementary arithmetic. Sommer has investigated the question of complexity of ordinal representation systems in [14]. His case studies revealed that with regard to measures considered in complexity theory ordinal representation systems involved in ordinal analyses emerge at a rather low level. It appears that they are always Δ_0-representable and that computations on ordinals in actual proof-theoretic ordinal analyses can be handled in the theory $I\Delta_0 + \Omega_1$, where Ω_1 is the assertion that the function $x \mapsto x^{\log_2(x)}$ is total.

Lemma 2.3. ($\mathbf{RCA}_0$)

(i) If $\mathfrak{X}$ is a linear ordering then so is $\varepsilon_{\mathfrak{X}}$.

(ii) $\varepsilon_{\mathfrak{X}}$ is elementary recursive in $\mathfrak{X}$.

3 The theory $\mathbf{ACA}_0^+$

Definition 3.1. $\mathbf{ACA}_0^+$ is $\mathbf{ACA}_0$ plus the axiom

$$(\omega\text{-jump}) \qquad \forall X\, \exists Y\, [(Y)_0 = X \;\wedge\; \forall n\, (Y)_{n+1} = \mathrm{TJ}((Y)_n)]$$

where $\mathrm{TJ}(Z)$ denotes the usual Turing jump of Z (see [13]). The Y above is unique and will be denoted by ω-jump(X).

Remark 3.2. $\mathbf{ACA}_0^+$ is sufficiently strong for the development of some interesting countable combinatorics. The Auslander/Ellis theorem of topological dynamics is provable in $\mathbf{ACA}_0^+$ as is Hindman's Theorem (cf. [13, X.3]). The latter says that if the natural numbers are coloured with finitely many colours then there exists an infinite set X such that all finite sums of elements of X have the same colour.

We also note that $\mathbf{ACA}_0^+$ is a theory that bears interesting connections to the bar rule and parameter free bar induction [9].

$\mathbf{ACA}_0^+$ can be characterized in terms of ω-models. This requires a definition.

Definition 3.3. Let T be a theory in the language of second order arithmetic, L_2. A *countable coded ω-model of T* is a set $W \subseteq \mathbb{N}$, viewed as encoding the L_2-model

$$\mathbb{M} = (\mathbb{N}, \mathcal{S}, \in, +, \cdot, 0, 1, <)$$

with $\mathcal{S} = \{(W)_n \mid n \in \mathbb{N}\}$ such that $\mathbb{M} \models T$.

This definition can be made in $\mathbf{RCA}_0$ (see [13, Definition VII.2]).

We write $X \in W$ if $\exists n\, X = (W)_n$.

Lemma 3.4. ($\mathbf{RCA}_0$) *The axiom ω-jump is equivalent to the statement*

"Every set is contained in a countable coded ω-model of ACA*"*

where ACA *stands for arithmetic comprehension.*

Proof: For "⇐" fix a set X and pick a countable coded ω model $\mathbb{M} = (\mathbb{N}, \mathcal{S}, \in, +, \cdot, 0, 1, <)$ of $\mathbf{ACA}_0$ with $X \in \mathcal{S}$ and $\mathcal{S} = \{(W)_n \mid n \in \mathbb{N}\}$ for some W. Now let

$$Z = \{\langle n, m\rangle \mid \mathbb{M} \models \exists Y\, [(Y)_0 = X \ \wedge\ \forall i < n\ (Y)_{i+1} = \mathrm{TJ}((Y)_i) \ \wedge\ m \in (Y)_n]\}.$$

Z is a set by arithmetic comprehension. Since $\mathbb{M}$ is an ω-model of $\mathbf{ACA}_0$ it follows that $Z = \omega\text{-jump}(X)$.

For "⇒" let X be an arbitrary set. Let $Z = \omega\text{-jump}(X)$ and put

$$\mathcal{S} = \{U \mid \exists n\, U \leq_T (Z)_n\}.$$

Here $U \leq_T V$ expresses that U is Turing reducible to V. One easily shows that $(\mathbb{N}, \mathcal{S}, \in, +, \cdot, 0, 1, <)$ is a model of $\mathbf{ACA}_0$ and, moreover, that $\mathcal{S}$ can be coded as a single set W. □

4 Main Theorem

The main result we want to prove is the following.

Theorem 4.1. $\mathbf{RCA}_0 + \forall \mathfrak{X}\ [\mathbf{WO}(\mathfrak{X}) \to \mathbf{WO}(\varepsilon_{\mathfrak{X}})]$ *proves that every set is contained in a countable coded ω-model of* $\mathbf{ACA}$.

A central ingredient of the proof will be a method of proof search (deduction chains) pioneered by Schütte [11]. This method also been adapted to infinitary logics at various places, e.g. [8, 6].

4.1 Deduction chains in ω-logic

Definition 4.2.

(i) Let $U_0, U_1, U_2, \ldots$ be an enumeration of the free set variables of L_2. For a closed term t, let $t^{\mathbb{N}}$ be its numerical value.

(ii) Henceforth a **sequent** will be a finite set of $\mathcal{L}_2$-formulas *without free number variables.*

(iii) A sequent Γ is **axiomatic** if it satisfies at least one of the following conditions:

1. Γ contains a true **literal**, i.e. a true formula of either form $R(t_1, \ldots, t_n)$ or $\neg R(t_1, \ldots, t_n)$, where R is a predicate symbol for a primitive recursive relation and $t_1, \ldots, t_n$ are closed terms.
2. Γ contains formulas $s \in U$ and $t \notin U$ for some set variable U and terms s, t with $s^{\mathbb{N}} = t^{\mathbb{N}}$.

(iv) A sequent is **reducible** or a **redex** if it is not axiomatic and contains a formula which is not a literal.

Definition 4.3. For $Q \subseteq \mathbb{N}$ define

$$D_Q(n) = \begin{cases} \bar{n} \in U_0 & \text{if } n \in Q \\ \bar{n} \notin U_0 & \text{otherwise} \end{cases}$$

For the proof of Theorem 4.1 it is convenient to have a finite axiomatization of arithmetic comprehension.

Lemma 4.4. $\mathbf{ACA}_0$ *can be axiomatized via a single* Π^1_2 *sentence* $\forall X\, C(X)$.

Proof: [13, Lemma VIII.1.5]. □

Definition 4.5. Let $Q \subseteq \mathbb{N}$. A **Q-deduction chain** is a finite string

$$\Gamma_0, \Gamma_1, \ldots, \Gamma_k$$

of sequents Γ_i constructed according to the following rules:

(i) $\Gamma_0 \;=\; \neg D_Q(0), \neg C(U_0)$.

(ii) Γ_i is not axiomatic for $i < k$.

(iii) If $i < k$ and Γ_i is not reducible then

$$\Gamma_{i+1} \;=\; \Gamma_i, \neg D_Q(i+1), \neg C(U_{i+1}).$$

(iv) Every reducible Γ_i with $i < k$ is of the form

$$\Gamma'_i, E, \Gamma''_i$$

where E is not a literal and Γ'_i contains only literals. E is said to be the **redex** of Γ_i.

Let $i < k$ and Γ_i be reducible. Γ_{i+1} is obtained from $\Gamma_i = \Gamma'_i, E, \Gamma''_i$ as follows:

1. If $E \equiv E_0 \vee E_1$ then

$$\Gamma_{i+1} \;=\; \Gamma'_i, E_0, E_1, \Gamma''_i, \neg D_Q(i+1), \neg C(U_{i+1}).$$

2. If $E \equiv E_0 \wedge E_1$ then

$$\Gamma_{i+1} \;=\; \Gamma'_i, E_j, \Gamma''_i, \neg D_Q(i+1), \neg C(U_{i+1})$$

where $j = 0$ or $j = 1$.

3. If $E \equiv \exists x\, F(x)$ then

$$\Gamma_{i+1} \;=\; \Gamma'_i, F(\bar{m}), \Gamma''_i, \neg D_Q(i+1), \neg C(U_{i+1}), E$$

where m is the first number such that $F(\bar{m})$ does not occur in $\Gamma_0, \ldots, \Gamma_i$.

4. If $E \equiv \forall x\, F(x)$ then

$$\Gamma_{i+1} \;=\; \Gamma_i', F(\bar{m}), \Gamma_i'', \neg D_Q(i+1), \neg C(U_{i+1})$$

for some m.

5. If $E \equiv \exists X\, F(X)$ then

$$\Gamma_{i+1} \;=\; \Gamma_i', F(U_m), \Gamma_i'', \neg D_Q(i+1), \neg C(U_{i+1}), E$$

where m is the first number such that $F(U_m)$ does not occur in $\Gamma_0, \dots, \Gamma_i$.

6. If $E \equiv \forall X\, F(X)$ then

$$\Gamma_{i+1} \;=\; \Gamma_i', F(U_m), \Gamma_i'', \neg D_Q(i+1), \neg C(U_{i+1})$$

where m is the first number such that $m \neq i+1$ and U_m does not occur in Γ_i.

The set of Q-deduction chains forms a tree $\mathcal{D}_Q$ labeled with strings of sequents. We will now consider two cases.

Case I: $\mathcal{D}_Q$ is not well-founded. Then $\mathcal{D}_Q$ contains an infinite path $\mathbb{P}$. Now define a set M via

$$(M)_i \;=\; \{t^{\mathbb{N}} \mid t \notin U_i \text{ occurs in } \mathbb{P}\}.$$

Set $\mathbb{M} = (\mathbb{N}; \{(M)_i \mid i \in \mathbb{N}\}, +, \cdot, 0, 1, <)$.

Claim: Under the assignment $U_i \mapsto (M)_i$ we have

$$F \in \mathbb{P} \;\;\Rightarrow\;\; \mathbb{M} \models \neg F \tag{2}$$

The Claim implies that $\mathbb{M}$ is an ω-model of **ACA**. Also note that $(M)_0 = Q$, thus Q is in $\mathbb{M}$. The proof of (2) follows by induction on F using Lemma 4.6 below. The upshot of the foregoing is that we can prove Theorem 4.1 under the assumption that $\mathcal{D}_Q$ is ill-founded for all sets $Q \subseteq \mathbb{N}$.

Lemma 4.6. *Let Q be an arbitrary subset of $\mathbb{N}$ and $\mathcal{D}_Q$ be the corresponding deduction tree. Moreover, suppose $\mathcal{D}_Q$ is not well-founded. Then $\mathcal{D}_Q$ has an infinite path $\mathbb{P}$. $\mathbb{P}$ has the following properties:*

1. *$\mathbb{P}$ does not contain literals which are true in $\mathbb{N}$.*
2. *$\mathbb{P}$ does not contain formulas $s \in U_i$ and $t \notin U_i$ for constant terms s and t such that $s^{\mathbb{N}} = t^{\mathbb{N}}$.*
3. *If $\mathbb{P}$ contains $E_0 \vee E_1$ then $\mathbb{P}$ contains E_0 and E_1.*
4. *If $\mathbb{P}$ contains $E_0 \wedge E_1$ then $\mathbb{P}$ contains E_0 or E_1.*
5. *If $\mathbb{P}$ contains $\exists x F(x)$ then $\mathbb{P}$ contains $F(n)$ for all natural numbers n.*

6. *If $\mathbb{P}$ contains $\forall x F(x)$ then $\mathbb{P}$ contains $F(n)$ for some natural number n.*

7. *If $\mathbb{P}$ contains $\exists X F(X)$ then $\mathbb{P}$ contains $F(U_m)$ for all natural numbers m.*

8. *If $\mathbb{P}$ contains $\forall X F(X)$ then $\mathbb{P}$ contains $F(U_m)$ for some natural number m.*

9. *$\mathbb{P}$ contains $\neg C(U_m)$ for all natural numbers m.*

10. *$\mathbb{P}$ contains $\neg D_Q(m)$ for all natural numbers m.*

Proof: Standard. □

The remainder of the paper will be devoted to ruling out the possibility that $\mathcal{D}_Q$ is well-founded. This is the place were cut elimination for an infinitary calculus $\mathbf{ACA}_\infty$ enters the stage.

4.2 The infinitary calculus $\mathbf{ACA}_\infty$

In the main, the system $\mathbf{ACA}_\infty$ is obtained from $\mathbf{ACA}_0$ by adding the ω-rule. The language of $\mathbf{ACA}_\infty$ is the same as that of $\mathbf{ACA}_0$ but the notion of formula comes enriched with set terms. Formulas and **set terms** are defined simultaneously. Literals are formulas. Every set variable is a set term. If $A(x)$ is a formula without set quantifiers (i.e. arithmetic) then $\{x \mid A(x)\}$ is a set term. If P is a set term and t is a numerical term then $t \in P$ and $t \notin P$ are formulas. The other formation rules pertaining to $\wedge, \vee, \forall x, \exists x, \forall X, \exists X$ are as per usual.

We will be working in a Tait-style formalization of the second order arithmetic. Due to the ω-rule there is no need for formulas with free numerical variables. Thus all sequents below are assumed to consist of formulas without free numerical variables.

Axioms of $\mathbf{ACA}_\infty$

(i) Γ, L where L is a true literal

(ii) $\Gamma, s \in U, t \notin U$ where $s^{\mathbb{N}} = t^{\mathbb{N}}$

Rules of $\mathbf{ACA}_\infty$

Rules for $(\wedge)$, $(\vee)$ and (Cut) as per usual.

$$(\omega) \quad \frac{\Gamma, F(\bar{n}) \quad \text{for all } n}{\Gamma, \forall x F(x)}$$

$$(\exists_1) \quad \frac{\Gamma, F(t)}{\Gamma, \exists x F(x)}$$

$$(\forall_2) \quad \frac{\Gamma, F(P) \quad \text{for all set terms } P}{\Gamma, \forall X F(X)}$$

$$(\exists_2) \quad \frac{\Gamma, F(P)}{\Gamma, \exists X F(X)} \quad \text{where } P \text{ is set term.}$$

$$(ST_1) \quad \frac{\Gamma, A(t)}{\Gamma, t \in P} \quad \text{where } P \text{ is the set term } \{x \mid A(x)\}.$$

$$(ST_2) \quad \frac{\Gamma, \neg A(t)}{\Gamma, t \notin P} \quad \text{where } P \text{ is the set term } \{x \mid A(x)\}.$$

Definition 4.7 (Cut rank in $\mathbf{ACA}_\infty$)**.** The **cut-rank** of a formula A, denoted $|A|$ is defined as follows:

1. $|L| = 0$ for literals L. $|s \in P| = |s \notin P| = |A(0)| + 1$ if P is a set term $\{x \mid A(x)\}$.
2. $|B_0 \wedge B_1| = |B_0 \vee B_1| = \max(|B_0|, |B_1|) + 1$
3. $|\forall x B(x)| = |\exists x B(x)| = |B(x)| + 1$
4. $|\forall X A(X)| = |\exists X A(X)| = \omega$ if $A(X)$ is arithmetic.
5. $|\forall X F(X)| = |\exists X F(X)| = |F(X)| + 1$ if $F(X)$ is *not* arithmetic.

Definition 4.8. We use the notation $\mathbf{ACA}_\infty \mid\!\frac{\alpha}{\rho}\, \Gamma$ to convey that the sequent Γ is deducible in $\mathbf{ACA}_\infty$ via a derivation of length $\leq \alpha$ containing only cuts of degree $< \rho$.

One easily shows that $\mathbf{ACA}_\infty$ includes $\mathbf{ACA}$, i.e. $\mathbf{ACA}_0$ plus the full induction scheme.

Lemma 4.9.

$$\mathbf{ACA}_0 \vdash \Gamma \quad \Rightarrow \quad \mathbf{ACA}_\infty \mid\!\frac{\omega+n}{\omega+k}\, \Gamma$$

for some $k, n < \omega$.

4.3 Cut-elimination for $\mathbf{ACA}_\infty$

The following cut elimination theorems can be viewed as folklore. $\mathbf{ACA}_\infty$ is related to the system $\mathbf{EA}^*$ in [11], Part C. The cut elimination theorems below can be gleaned from [11, Theorem 22.13 and Theorem 22.8]. Full details can be found in [1].

Theorem 4.10 (Cut-elimination I)**.** *Let* $\alpha \geq \omega$.

$$\mathbf{ACA}_\infty \mathrel{\substack{\alpha \\ \overline{\omega+n+1}}} \Gamma \quad \Rightarrow \quad \mathbf{ACA}_\infty \mathrel{\substack{\omega^\alpha \\ \overline{\omega+n}}} \Gamma$$

Theorem 4.11 (Cut-elimination II)**.**

$$\mathbf{ACA}_\infty \mathrel{\substack{\alpha \\ \overline{\omega}}} \Gamma \quad \Rightarrow \quad \mathbf{ACA}_\infty \mathrel{\substack{\varepsilon_\alpha \\ \overline{0}}} \Gamma$$

4.4 The variant $\mathbf{ACA}^Q_\infty$

For any fixed $Q \subseteq \mathbb{N}$ we will look at a variant of $\mathbf{ACA}^Q_\infty$ of $\mathbf{ACA}_\infty$ which arises by adding the basic diagram of Q. More precisely, $\mathbf{ACA}^Q_\infty$ results from $\mathbf{ACA}_\infty$ by adding the axioms

1. $\Gamma, s \in U_0$ if $s^{\mathbb{N}} \in Q$.

2. $\Gamma, s \notin U_0$ if $s^{\mathbb{N}} \notin Q$.

Theorems 4.10 and 4.11 hold for $\mathbf{ACA}^Q_\infty$ as well.

4.5 Finalizing the proof of the main Theorem

Recall that we have proved that there exists a countable coded ω-model of $\mathbf{ACA}_0$ containing Q providing $\mathcal{D}_Q$ is ill-founded, i.e. if $\mathcal{D}_Q$ contains an infinite path. To finish the proof of the main Theorem 4.1 let us assume that $\mathcal{D}_Q$ is well-founded. We aim at deriving a contradiction from this. $\mathcal{D}_Q$ can then be viewed as a deduction with **hidden cuts** involving formulas of the shape $\neg C(U_{i+1})$ and $\neg D_Q(i+1)$. Note that $\mathbf{ACA}^Q_\infty \mathrel{\substack{\omega \\ \overline{\omega}}} C(U_i)$ and $\mathbf{ACA}^Q_\infty \mathrel{\substack{0 \\ \overline{0}}} D_Q(i)$. Thus any transition

$$\frac{\Gamma_{i+1}}{\Gamma_i}$$

in $\mathcal{D}_Q$ can be viewed as a combination of three inferences in $\mathbf{ACA}^Q_\infty$, the first one being a logical inferences and the other two being cuts.

By interspersing $\mathcal{D}_Q$ with cuts and adding two cuts with cut formulas $\neg C(U_0)$ and $\neg D_Q(0)$ at the bottom we obtain a derivation $\mathcal{D}^*_Q$ in $\mathbf{ACA}^Q_\infty$ of the empty sequent. The cut formulas of $\mathcal{D}^*_Q$ have at most rank ω. Hence $\mathbf{ACA}^Q_\infty \mathrel{\substack{\alpha \\ \overline{\omega+1}}} \emptyset$ for an ordinal α which corresponds to the Kleene-Brouwer ordering of the tree $\mathcal{D}^*_Q$. Using Cut Elimination I and II for $\mathbf{ACA}^Q_\infty$ we obtain a cut-free proof of length $\varepsilon_{\omega^\alpha}$ of $\emptyset$. But this is impossible because the conclusion of any other rule is always nonempty.

As Cut Elimination I and II (Theorems 4.10, 4.11) can be formalized and proved in $\mathbf{ACA}_0$ and the latter system is contained in $\mathbf{RCA}_0 + \forall \mathfrak{X}\, [\mathbf{WO}(\mathfrak{X}) \to \mathbf{WO}(\varepsilon_{\mathfrak{X}})]$ by Theorem 1.1, this concludes the proof of Theorem 4.1. There are different ways of formalizing infinite deductions in theories like **PA**. We just mention [12] and [2].

5 Finishing the proof of Theorem 1.2

One direction of Theorem 1.2 follows from Theorem 4.1 read in conjunction with Lemma 3.4. It remains to show that $\mathbf{ACA}_0^+$ proves $\forall\mathfrak{X}\ [\mathbf{WO}(\mathfrak{X}) \to \mathbf{WO}(\varepsilon_{\mathfrak{X}})]$. Arguing in $\mathbf{ACA}_0^+$ assume that $\mathbf{WO}(\mathfrak{X})$ holds. For a contradiction suppose there is an infinite $\varepsilon_{\mathfrak{X}}$-descending sequence $\mathfrak{I}$. By Lemma 3.4 pick a countable coded ω-model $\mathbb{M}$ of $\mathbf{ACA}_0$ which contains $\mathfrak{X}$ and $\mathfrak{I}$. However, using transfinite induction on $\mathfrak{X}$ one can show that transfinite induction on $\varepsilon_{\mathfrak{X}}$ holds in $\mathbb{M}$. Basically this can be proved by viewing the proof [11, Theorem 21.4] as taking place in $\mathbb{M}$. Since $\mathfrak{I}$ is in $\mathbb{M}$ we arrive at a contradiction. □

6 Prospectus

In [10] the theme of this paper will be extended to more challenging scenarios. Theorem 1.3 and extensions will be proved by proof-theoretic methods, utilizing cut elimination for systems of ramified analysis.

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