A Note on the Theory of Positive Induction, ID_1^*

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Abstract

The article shows a simple way of calibrating the strength of the theory of positive induction, ID_1^*. Crucially the proof exploits the equivalence of Σ_1^1 dependent choice and ω-model reflection for Π_2^1 formulae over $\mathbf{ACA}_0$. Unbeknown to the authors, D. Probst had already determined the proof-theoretic strength of ID_1^* in [13].

1 Introduction

Theories of inductive definitions have always played an important role in mathematical logic. The first order theories of iterated inductive definitions, ID_n, and several of its subsystems have been intensively studied by proof theorists (see [12, 3, 1, 6]). In its much weaker subtheories $\widehat{\mathrm{ID}}_n$, only the fixed-point property of inductive definitions is asserted. The theory ID_1^* lies between $\widehat{\mathrm{ID}}_1$ and ID_1. It is a restricted version of ID_1 in that the scheme for proof by induction on the inductively defined predicates is only permitted for formulae in which the predicates for the inductively defined sets occur positively. Feferman in [6] attributes this theory to H. Friedman [8]. ID_1^* and its evident iterations ID_n^* were considered by Feferman [6] and studied by Cantini [4], giving upper bounds for their proof-theoretic ordinals. However, the upper bound obtained for the strength of ID_n^* exceeded that of $\widehat{\mathrm{ID}}_n$. The exact strength of ID_n^* remained unknown until quite recently. The authors of the present paper determined the proof-theoretic strengths of the theories ID_n^* in 2007. Recently, however, they learned that D. Probst had done this earlier in [13]. The proof given here, though, uses a methodology differing from [13]. It makes use of the concept of ω-model reflection. As this method provides a very transparent proof and can also be used to determine the strength of other subsystems of ID_1 we considered it worthwhile publishing.

2 The theory of positive induction, ID_1^*

For any language $\mathcal{L}$ let $\mathcal{L}^P$ be $\mathcal{L}$ augmented by a unary predicate P. For convenience we shall write $A(u, P^+)$ to emphasise that all occurrences of the predicate P in the formula $A(u, P)$ are positive and that at most the variable u occurs free in it. The language of ID_1^* is the same as that of ID_1, i.e. it comprises the language of primitive recursive arithmetic, $\mathcal{L}_0$, and for each P-positive formula $A(x, P^+)$ of $\mathcal{L}_0^P$, it has a unary predicate symbol I_A. The predicates I_A will be referred to as the *inductive predicates.*

Definition 1. *The axioms of* ID_1^* *are those of* PRA *plus the following:*

$(I_A.1)$ $\forall u[A(u, I_A) \to I_A(u)]$

$(I_A.2)$ $\forall u[A(u, F) \to F(u)] \to \forall u[u \in I_A \to F(u)]$

$(\mathrm{IND}_{\mathbb{N}}^{+})$ $F(0) \wedge \forall u[F(u) \to F(u+1)] \to \forall u F(u)$

for all formulae $A(u, P^{+})$ of $\mathcal{L}_{\mathrm{PRA}}^{P}$ and all formulae $F(u)$ of the language of ID_1^* in which inductive predicates occur only positively.

The general induction scheme for induction on the natural numbers for arbitrary formulae of the language of ID_1^* will be denoted by $\mathrm{IND}_{\mathbf{N}}$.

Caveat: Usually the theory ID_1^* (cf. [6]) is identified with what we denoted by $\mathrm{ID}_1^* + \mathrm{IND}_{\mathbb{N}}$ above.

3 Interpreting ID_1^* in Σ_1^1-DC_0

Definition 2. *The Σ_1^1-DC (Dependent Choice) scheme is*

$$\forall x \forall X \exists Y B(x, X, Y) \to \forall U \exists Z[(Z)_0 = U \wedge \forall x B(x, (Z)_x, (Z)_{x+1})]$$

for Σ_1^1 formulae B. Here by $(Z)_x$ we mean the x-th section of the set Z i.e. $(Z)_x = \{y | \langle x, y\rangle \in Z\}$, where $\langle ., .\rangle$ is some primitive recursive pairing function. The system Σ_1^1-DC_0 is defined to be $\mathrm{ACA}_0 + \Sigma_1^1$-DC. The proof-theoretic ordinal of Σ_1^1-DC_0 was determined to be $\varphi\omega 0$ in [5].

We define an interpretation from the language of ID_1^* into the language of Σ_1^1-DC_0 by letting

$$(I_A(t))^* := \forall X[\forall u(A(u, X) \to u \in X) \to t \in X]$$

and leaving everything else unchanged.

Definition 3. *We use the usual hierarchy of Π_k^1 and Σ_k^1 for formulae of second order arithmetic. A formula is said to be* essentially Π_1^1 *if it belongs to the smallest collection of formulae which contains all arithmetical formulae and is closed under $\wedge$, $\vee$, $\exists x$, $\forall x$ and $\forall X$.*

Lemma 4. *for formulae $A(x, P^{+}), B(x, P^{+}) \in \mathcal{L}_0^P$ we have*

- *(i) $(I_A(t))^*$ is Π_1^1*
- *(ii) $(A(x, I_B))^*$ is essentially Π_1^1.*

Proof. (i) is obvious by looking at the definition of the translation and (ii) becomes clear by the fact that I_B occurs positively in A. □

Definition 5. *The Σ_1^1-AC (Axiom of Choice) scheme is*

$$\forall u \exists X A(u, X) \to \exists Y \forall u A(u, (Y)_u)$$

for Σ_1^1 formulae A. The system Σ_1^1-AC_0 is defined to be $\mathrm{ACA}_0 + \Sigma_1^1$-AC.

Lemma 6. *For any essentially Π_1^1 formula G we can find a Π_1^1 formula G' with the same free variables such that* *(i)* $\Sigma^1_1\text{-AC}_0 \vdash G' \to G$

(ii) $\text{ACA}_0 \vdash G \to G'$

Proof. G' is defined inductively as follows:

- If G is atomic or negated atomic then we define $G' \equiv G$.
- If G is of the form $A_0 \vee A_1$ and we have that $A'_0 \equiv \forall X_0 B_0(X_0)$ and $A'_1 \equiv \forall X_1 B_1(X_1)$ then we define G' to be $\forall X[B_0((X)_0) \vee B_1((X)_1)]$. Similarly if G is of the form $A_0 \wedge A_1$ define G' to be $\forall X[B_0((X)_0) \wedge B_1((X)_1)]$.
- If G is of the form $\forall u B(u)$ and we have that $B'(u) \equiv \forall X B_0(X, u)$, then we define G' to be $\forall X \forall u B_0(X, u)$.
- If G is of the form $\forall X B(X)$ and we have that $B'(X) \equiv \forall Y B_0(X, Y)$, then we define G' to be $\forall X B_0((X)_0, (X)_1)$.
- If G is of the form $\exists u B(u)$ and we have that $B'(u) \equiv \forall X B_0(u, X)$, then we define G' to be $\forall X \exists u B_0(u, (X)_u)$. In this case $\text{ACA}_0 \vdash \exists u B'(u) \to G'$ and $\Sigma^1_1\text{-AC}_0 \vdash G' \to \exists u B'(u)$.

It is clear from the definition above that G' satisfies our requirements. □

Lemma 7. *For $A(x, P^+) \in \mathcal{L}^P_0$ we have*

$$\text{ACA}_0 \vdash \forall x[A(x, I^*_A) \to I^*_A(x)]\,.$$

Proof. Assume (1) $A(x, I^*_A)$ and (2) $\forall u(A(u, X) \to u \in X)$ for an arbitrary set X. We need to show $x \in X$. Observe that we have $\forall u(I^*_A(u) \to u \in X)$ and therefore since A is positive $A(u, I^*_A) \to A(u, X)$ holds for every u. Now from (1) we have $A(x, X)$ and so with (2) we can derive $x \in X$.
Note that the proof does not use the whole strength of ACA_0; in fact it can be carried out in pure second order logic. □

Lemma 8. $\Sigma^1_1\text{-DC}_0$ *proves induction on* $\mathbf{N}$ *for essentially* Π^1_1 *formulae.*

Proof. It has been shown [15, Theorem VIII.5.12] that Π^1_1 transfinite induction (Π^1_1-TI) and the scheme Σ^1_1-DC are equivalent over ACA_0. Since Π^1_1-TI implies induction along the natural numbers for Π^1_1 formulae we are done. □

Definition 9 (Simpson [15]). *A* countable coded ω-model *is a set M of natural numbers, viewed as encoding a structure for the language of second order arithmetic*

$$(\mathbb{N}, \mathfrak{M}, +, \cdot, 0, 1, <)$$

with the sections of M forming the universe of sets, i.e. $\mathfrak{M} = \{(M)_n : n \in \mathbb{N}\}$. This definition can be made within RCA_0 *(see [15, VII.2.1]).*

We shall write $M \models A$ if $(\mathbb{N}, \mathfrak{M}, +, \cdot, 0, 1, <) \models A$ whenever A is a formula of second order arithmetic with parameters in $\mathfrak{M}$. We also write $X \in M$ instead of $\exists i\, (M)_i = X$.

Theorem 10 (Simpson [14]). *The following are equivalent over* ACA_0*:*

(i) Σ^1_1-DC

(ii) *ω-model reflection for Π^1_2 formulae, i.e. if $C(X_1, \ldots, X_k)$ is a Π^1_2-formula with all set parameters exhibited and $C(X_1, \ldots, X_k)$ holds then there exists a countable coded ω-model of* ACA_0 *such that $X_1, \ldots, X_k \in M$ and $M \models C(X_1, \ldots, X_k)$.*

Proof. See [14] or [15, Theorem VIII.5.12]. □

Lemma 11. *If $A(x, P^+) \in \mathcal{L}_0^P$ then Σ^1_1-DC_0 proves*

$$\forall x[A(x, F) \to F(x)] \to \forall x[I_A^*(x) \to F(x)]$$

for all essentially Π^1_1 formulae $F(x)$.

Proof. For a contradiction assume

$$\forall x[A(x, F) \to F(x)], \tag{1}$$

$$I_A^*(n_0) \wedge \neg F(n_0) \tag{2}$$

for some n_0. Let $G(x) := A(x, F)$ which is essentially Π^1_1. Let $G'(x)$ and $F'(x)$ be the corresponding formulae provided by Lemma 6. Then we have

$$\forall x[G'(x) \to F'(x)] \text{ and} \tag{3}$$

$$\neg F'(n_0). \tag{4}$$

Using Π^1_2 ω-model reflection (Theorem 10) there exists a countable coded ω-model M of ACA_0 such that

$$M \models \forall x[G'(x) \to F'(x)] \text{ and} \tag{5}$$

$$M \models \neg F'(n_0). \tag{6}$$

Lemma 6(ii) implies

$$M \models \forall x[A(x, F') \to G'(x)]. \tag{7}$$

Using this together with (5) yields

$$M \models \forall x[A(x, F') \to F'(x)]. \tag{8}$$

Now define $Z = \{u : M \models F'(u)\}$. Since the formula $M \models F'(u)$ is arithmetic, Z exists by arithmetical comprehension. As a result of (8) we have

$$\forall x[A(x, Z) \to x \in Z] \tag{9}$$

and hence $I_A^* \subseteq Z$. Thus by (2) $n_0 \in Z$, and so

$$M \models F'(n_0). \tag{10}$$

(6) and (10) are contradictory. □

Theorem 12.

$$\mathrm{ID}_1^* \vdash D \Rightarrow \Sigma^1_1\text{-}\mathrm{DC}_0 \vdash D^*$$

Proof. Lemma 7 and 11 show that $(I_A.1)^*$ and $(I_A.2)^*$ respectively are provable in Σ^1_1-DC_0. Moreover if I_A occurs positively in $A(x, I_B)$ then, by Lemma 4, $(A(x, I_B))^*$ is essentially Π^1_1 and therefore by Lemma 8, Σ^1_1-DC_0 contains the induction scheme for it. $\square$

Corollary 13. *$|ID^*_1| = \varphi\omega 0$ and $|ID^*_1 + \mathrm{IND}_{\mathbb{N}}| = \varphi\varepsilon_0 0$.*

Proof. Since $|\Sigma^1_1\text{-DC}_0| = \varphi\omega 0$ (see [5]) the embedding in Theorem 12 shows that $|\mathrm{ID}^*_1| \leq \varphi\omega 0$. Likewise Theorem 12 shows that $\mathrm{ID}^*_1 + \mathrm{IND}_{\mathbb{N}}$ can be embedded into Σ^1_1-DC_0 plus full induction on natural numbers. As the latter theory is known to have proof-theoretic ordinal $\varphi\varepsilon_0 0$ (cf. [7]), we arrive at $|\mathrm{ID}^*_1 + \mathrm{IND}_{\mathbb{N}}| \leq \varphi\varepsilon_0 0$. $\varphi\omega 0 \leq |\mathrm{ID}^*_1|$ and $\varphi\varepsilon_0 0 \leq |\mathrm{ID}^*_1 + \mathrm{IND}_{\mathbb{N}}|$ are well known results. For the sake of references, the first inequality follows for example from [11] Corollary 8 and the second inequality follows from [4]. $\square$

Prospectus: The method of ω-model reflection has been extended in [2] to give exact bounds for the theories ID^*_n. It can also be employed to analyze stronger fragments of ID_1. For instance the fragment which arises by requiring the formulae $A(u, P^+)$ to be Π^0_1 and the formulae F in $(I_A.2)$ to be Δ^0_0 in inductive predicates has been used in [10] to find an upper bound for a theory of truth E in [9]. Here one utilizes the connection with Π^1_3 ω-model reflection to carry out an argument akin to Lemma 11.

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