Table 1

$$(P;Q);R = P;(Q;R) \qquad P;Q;R$$

$(R;Q)$

$$P/(Q;R) = (P/R)/Q \qquad P;\breve{Q};\breve{R}$$

8

$$P\backslash(Q/R) = (P\backslash Q)/R \qquad \breve{P};Q;\breve{R}$$

$$(P;Q)\backslash R = Q\backslash(P\backslash R) \qquad \breve{Q};\breve{P};R$$

assoc

$$(P;Q)/R = P;(Q/R) \cup P/(R/Q) \qquad \text{lin.} \quad PQR$$

6

$$P\backslash(Q;R) = (P\backslash Q);R \cup (Q\backslash P)\backslash R \qquad \breve{P}Q;R$$

$$(P/Q);R \cup (Q/P)\backslash R = P/(R\backslash Q) \cup P;(Q\backslash R) \qquad P;\breve{Q};R$$

confluence

There are eighteen distinct bracketed terms that can be constructed from three operands, three operators and one pair of brackets. Each of them appears exactly once in the above system of equations.

There are a number of transformations of diagrams that preserve the meaning of the denoted diagram. They involve adding or removing labels, and edges. A label may be formed from any two edges of a triangle and added to a third. Thus the following four triangles are equivalent; that is immediate from the Schroeder property.

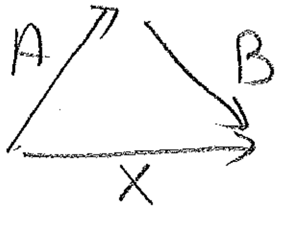
$$\begin{tikzcd} & {} \arrow[dr, "B"] & \\ {} \arrow[ur, "A"] \arrow[rr, "C"'] & & {} \end{tikzcd} \qquad \begin{tikzcd} & {} \arrow[dr, "B"] & \\ {} \arrow[ur, "A"] \arrow[rr, "C \cap (A;B)"'] & & {} \end{tikzcd} \qquad \begin{tikzcd} & {} \arrow[dr, "B"] & \\ {} \arrow[ur, "A \cap (C/B)"] \arrow[rr, "C"'] & & {} \end{tikzcd}$$

$$\begin{tikzcd} & {} \arrow[dr, "B \cap (A\backslash C)"] & \\ {} \arrow[ur, "A"] \arrow[rr, "C"'] & & {} \end{tikzcd}$$

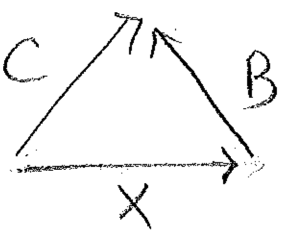
These equivalences hold, no matter how many times the edges are shared, or how many other labels there are on each of the edges.

The following three laws permit the addition or removal of edges, provided that these (deleted edges) are <u>not</u> shared by any other triangle

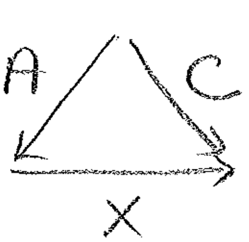
$$\xrightarrow[X \cap (A;B)]{} \quad =$$



$$\xrightarrow[X \cap (C/B)]{} \quad =$$



$$\xrightarrow[X \cap (A \backslash C)]{} \quad =$$



Finally, omission of a label or an edge can only weaken the proposition denoted by a diagram.

Any diagram with an edge labelled by the union of two sets $A \cup B$ may be split into the disjunction of two copies of the diagram, identical except that each omits a different one of the sets "$A \cup$" or "$\cup B$".

An example of reasoning with these rules proves one of the Dedekind (or modular) laws.

Step 0, Assume $c \in C \cap (A;B)$

Step 1. Represent it diagrammatically

$$\overrightarrow{\{c\} \cap C \cap (A;B)}$$

Step 2 Grow the triangle

$$\begin{tikzcd} & \cdot \arrow[dr, "B"] & \\ \cdot \arrow[ur, "A"] \arrow[rr, "\{c\}\cap C"'] & & \cdot \end{tikzcd}$$

Step 3. Add the label

$$\begin{tikzcd} & \cdot \arrow[dr, "B\cap(A\backslash C)"] & \\ \cdot \arrow[ur, "A"] \arrow[rr, "\{c\}\cap C"'] & & \cdot \end{tikzcd}$$

Step 4. Contract the triangle

$$\overrightarrow{\{c\} \cap C \cap (A;(B \cap (A\backslash C)))}$$

6 Non-emptiness of the label on the final arrow implies

$$c \in C \cap A;(B \cap (A \backslash C)).$$

In summary, this reasoning establishes the algebraic inclusion

$$C \cap (A;B) \subseteq A;(B \cap (A \backslash C))$$

Here are two other laws that may be established in the same way, by starting at a different edge of the triangle:

$$A \cap (C \backslash B) \subseteq (C \cap (A;B))/B$$

$$B \cap (A \backslash C) \subseteq A \backslash (C \cap (A;B))$$

The three laws are derivable from each other by the Schroeder equivalence; but many people find the diagrammatic reasoning simpler.

Each of the six properties of T may be used as an additional law for (diagrammatic) reasoning: it justifies insertion of the missing arrow of the corresponding diagram.

For example, if T has property A1, we can proceed as follows.

Step 0 Assume $d \in (A;B);C$

$$\overrightarrow{\{d\} \cap (A;B);C}$$

Step 1. Grow two triangles

$$\begin{tikzcd}
\bullet \arrow[r, "B"] & \bullet \arrow[d, "C"] \\
\bullet \arrow[u, "A"] \arrow[ur] \arrow[r, "\{d\}"'] & \bullet
\end{tikzcd}$$

Step 3. Insert the diagonal justified by A1, and some further labels

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
{} \arrow[r, "B"] \arrow[dr, "B;C"] & {} \arrow[d, "C"] \\
{} \arrow[u, "A"] \arrow[ur] \arrow[r, "\{d\} \cap A;(B;C)"'] & {}
\end{tikzcd}
$$

This is a proof of one half of an associative law

$$(A;B);C \subseteq A;(B;C).$$

Three other laws may be proved from property A1 by starting with the three other edges of the same diagram. They are

$$(D/C)/B \subseteq D/(B;C)$$

$$A\backslash(D/C) \subseteq (A\backslash D)/C$$

$$(A;B)\backslash D \subseteq B\backslash(A\backslash D)$$

In some cases, the direction of the inserted arrows is not determined by acyclicity; and both possibilities have to be allowed, by drawing both diagrams as a disjunction.

Here is the vital step of a proof that uses B2.

$$
\begin{tikzcd}[row sep=large, column sep=large]
\bullet \arrow[r, "B"] \arrow[d, "A"'] \arrow[dr] & \bullet \arrow[d, "C"] \\
\bullet \arrow[r, "\{d\}"'] & \bullet
\end{tikzcd}
$$

Drawing the two diagonals possible by B2. we get the disjunction

$$
\begin{tikzcd}[row sep=large, column sep=large]
\bullet \arrow[r, "B"] \arrow[d, "A"'] \arrow[dr] & \bullet \arrow[d, "C"] \\
\bullet \arrow[r, "\{d\} \cap (A\backslash B);C"'] \arrow[ur, "A\backslash B"] & \bullet
\end{tikzcd}
\quad \vee \quad
\begin{tikzcd}[row sep=large, column sep=large]
\bullet \arrow[r, dash, "B"] \arrow[d, dash, "A"'] \arrow[dr] & \bullet \arrow[d, dash, "C"] \arrow[dl, "B\backslash A"'] \\
\bullet \arrow[r, dash, "\{d\} \cap (B\backslash A)\backslash C"'] & \bullet
\end{tikzcd}
$$

This gives a proof of

$$A\backslash(B;C) \subseteq (A\backslash B);C \cup (B\backslash A)\backslash C$$

A complete set of algebraic laws derivable from each of the properties of Figure 1 is given in table 1. The laws for A2, B2 and C2 are the exact converses of those of A1, B1 and C1 respectively; they state the reverse inclusion. If both members of a pair are valid, the laws can be stated as equations. /// In the relational calculus, A/B is replaced by $A;\breve{B}$ and $B\backslash A$ by $\breve{B};A$. After this replacement,

~~In this case~~ all the terms (in each law) are equal to the relational term written in the right hand column of the table. This shows the consistency of the laws with the relational calculus.

There are ~~all~~ eighteen (3x3x2) ways of writing a three-operand term with a choice of three operators and one pair of brackets; each of them appears just once in the table.

$(A;B);C \subseteq A;(B;C)$	$A;B;C$
$(D/C)/B \subseteq D/(B;C)$	$D;\breve{B};\breve{C}$
$A\backslash(D/C) \subseteq (A\backslash D)/C$	$\breve{A};D;\breve{C}$
$(A;B)\backslash D \subseteq B\backslash(A\backslash D)$	$\breve{B};\breve{A};D$
Laws for A1	
$P;(Q/R) \cup P/(R/Q) \subseteq (P;Q)/R$	$P;Q;\breve{R}$
$(P\backslash Q);R \cup (Q\backslash P)\backslash R \subseteq P\backslash(Q;R)$	$\breve{P};Q;R$
Laws for B1	
$P/(R\backslash Q) \cup P;(Q\backslash R) \subseteq (P/Q);R \cup (Q/P)\backslash R$	
Law for C1	$P;\breve{Q};R$

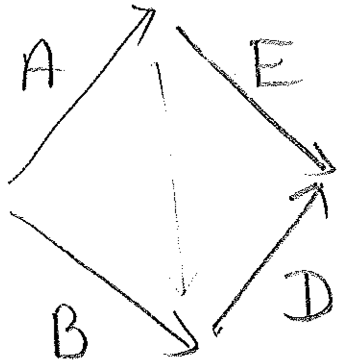
Table 1

The general method of reasoning with diagrams is to start with an edge with a conjunction of labels. Using the expansion laws, each label generates a chain over the initial edge. Further chains can be generated by the crossover theorem; ~~this~~ in general this will introduce a disjunction of diagrams. Finally, a selected chain can be contracted (with further disjunctions perhaps) until the original edge is reached again. The original conjunction of labels is contained in the disjunction of all the final labels.

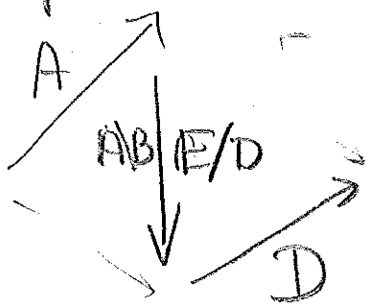
Step 0

$$\overrightarrow{A;E \cap B;D}$$

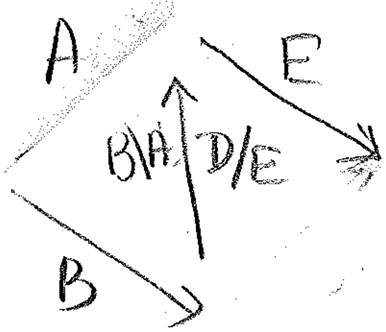
Step 1 growth of chains



Step 2 crossover in either direction



$\vee$



Step 3 contraction to original edge

$$A;(A\backslash B \cap E/D);D \quad \cup \quad B;(B\backslash A \cap D/E);E$$

Related work.

This study shares with arrow logic its starting point, the ternary relation T. Arrow logic also has a primitive binary relation expressing conversity of arrows and a predicate which singles out the self-loops, i.e arrows whose source and target are the same. It has the same overall aim, to explore a range of logics with some but not all of the properties of the relational calculus.

Another generalisation of the relational calculus is provided by quantale theory. Here the division operators are replaced by implications, which can be defined as follows.

$$A \rightarrow_{\ell} C = \overline{A \backslash \overline{C}}$$

$$B \rightarrow_{r} C = \overline{\overline{C} / B}$$

The Schroeder equivalences translate to Galois connections

$$A ; B \subseteq C \text{ iff } A \subseteq B \rightarrow_{r} C$$

$$\text{iff } B \subseteq A \rightarrow_{\ell} C$$

Linear Logic uses the same operators as quantale theory; but the primitive predicate T is assumed to be symmetric in its first two arguments

$$T(x, y, z) = T(y, x, z).$$

Composition therefore commutes, and only one inverse is need. Linear logic also makes a rather strong assumption about the existence of an element D such that

$$D \to (D \to A) = A$$

The relative converse A/B was introduced in [J.H. Conway, Regular Algebra and Finite Machines, Chapman and Hall 1971]. It was used to help solve equations defining regular languages

This study derives its main inspiration from the sequential calculus; the main difference is to start with no assumptions at all about the underlying predicate T, and to delay consideration of units and identities. Of the six theorems proved diagrammatically from properties of T, ~~six theorems~~. The first three are basic axioms of the sequential calculus, and the other three have been found useful in exploration of the foundations of linear temporal logic.

The diagrammatic methods are identical to those proposed by Curtis and Lowe. They have been slightly extended to allow chains with undirected paths as well as directed.