Shifting the identity of predicate transformers. In the space of predicate transformers, let \( J \) be a preorder, i.e.

\[
\begin{align*}
J & \subseteq I \\
J & \subseteq J ; J \\
J ; \Pi S & = \prod \{ J ; P \mid P \in S \} \\
\end{align*}
\]

It follows that \( J \) is idempotent

\[
J ; J \subseteq I I ; J = J
\]

We define a subset \( J^0 \) of predicate transformers

\[
J^0 \triangleq \{ P \mid J ; P ; J \equiv P \}
\]

We prove that \( J^0 \) is a subspace of predicate transformers, except that it has \( J \) as its identity instead of \( I I \), and \( J ; J \) instead of \( I I \).

\*a complete distributive lattice
1. $J \in J^0$

   $J; J; J = J$ by idempotence

2. $T \in J^0$

   $J; T; J = J; T$  
   \[ T \text{ is a constant.} \]

3. $I \in J^0$  
   \[ I; J; J = J; J \]
   \[ = I \text{ by conjunctivity} \]

4. If $P, Q \in J^0$, so is $P; Q$.

   $J; P; Q; J = J; (J; P; J); (J; Q; J); J$

   $= J; P; J; J; Q; J$ by idempotence

   $= P; Q$ by assumption

5. If $S \in J^0$, then $\forall S \in J^0$

   $J; (\forall S); J = J; \Pi \{P; J \mid P \in S\}$ property of space

   $= \Pi \{J; P; J \mid P \in S\}$ by conjunctive

   $= \Pi \{P \mid P \in S\}$ since $S \in J^0$
6. If $S \subseteq J^0$, then $US \subseteq J^0$.

(E) $J; US; J : \subseteq \Pi; US; \Pi = US$ (since $J \subseteq \Pi$)

(2) $J; P; J = P$

By monotonicity:

For all $P \in S$

.: $J; US; J \supseteq P$

.: $J; US; J \supseteq US$

are there any?

7. If $b$ is a condition in $J^0$ then $J; b; J$ is a condition.

Then $J; b; J$ is a condition by (2)

$T = J; T; J = J; (b \cup b); J$

$= (J; b; J) \cup (J; b; J)$

$= b \cup b$

.: $J; b; J = J; (b \cup b); J$

$= (J; b; J) \cap (J; b; J)$

$= b \cap J; b; J$

By uniqueness of negation, $J; b; J = J; \neg b; J$, which is clearly in $J^0$.

In $J^0$

$K; L \models K$  \quad $H; H \models H$

$L; K \models L$  \quad $H; H \models H$  \quad by disjointness

true \quad true, false  \quad pq
8. In defining fixed points, we confine attention to elements of $J^0$.

$$\mu X. FX = \Pi \{ X \mid FX \in X \land X \in J^0 \}$$

Now

$$\Pi \{ X \mid X \in J \land FX \in X \land X \in J^0 \}$$

$$= \mu X. FX.$$

But we still need to prove this is a fixed point of $F$. This depends on the assumptions

$$F : J^0 \to J^0$$

i.e. $\forall X. X = \Pi \{ X \mid X \in J \land FX \in X \land X \in J^0 \}$.

According to (1) - (7) above, this will be guaranteed if $F$ is an expression in our algebra.

Proof: presumably the same as the standard proof, applied in the domain $J^0$. 

\[ \Delta \]
Shifting the top of predicate transformers.

In the space of predicate transformers, it is universally conjunctive.

Let $K$ be idempotent and conjunctive.

$$K = \lambda x. \text{true} \text{ true} \text{ true}$$

as a predicate transformer.

$$K; K = K; K$$

but

$$K; \text{true} = \text{true}$$

$$K; K \neq K; \text{false}$$

$$K; \text{false} = \text{false}$$

$$K; \text{true} = \text{true}$$

$$K; \text{false} = \text{false}$$

Define $J = \Pi n K$

Now

$$J \in \Pi$$

(properiy of $\Pi$)

$$J \in J; J$$

$$(J; J = (\Pi n K); (\Pi n K)$$

$$= (\Pi n K) n (K; (\Pi n K))$$

$$= \Pi n K n K n (K; K)$$

$$\supseteq \Pi n K$$

Furthermore $J$ is conjunctive, because $\Pi$ and $K$ are.

Define $KJ = J o n \{P | P \in K \}$.

$$(\Pi n K); JS = \Pi \{\Pi n k; x \in S | x \in S\} = \Pi \{\Pi n k; x \in S\}$$

$$\Pi k; \Pi \{x \mid x \in k; J\} = \Pi k; J$$
Now we show that $\mathcal{K} \downarrow$ is an almost complete subspace of predicate transformers, but with $\mathcal{J}$ as its identity and $\mathcal{K}$ as its top.

1) $\mathcal{J} \in \mathcal{K} \downarrow$

$\mathcal{J} \in \mathcal{J}^0$ and $\mathcal{J} = \Pi \cap \mathcal{K} \subseteq \mathcal{K}$.

2) $\perp \in \mathcal{K} \downarrow$

because $\perp \subseteq \mathcal{K}$ and $\perp \in \mathcal{J}^0$.

3) $\mathcal{J} \mathcal{K} \mathcal{J} \in \mathcal{K} \downarrow$. We prove $\mathcal{K} \in \mathcal{J}^0$.

$\mathcal{J} \mathcal{K} \mathcal{J} = (\Pi \cap \mathcal{K}) \mathcal{J} (\Pi \cap \mathcal{K})$

$= \mathcal{K} \cap (\mathcal{K} \mathcal{K}) \cap (\mathcal{K} \mathcal{K} \mathcal{K})$

$= \mathcal{K} \mathcal{K} \mathcal{K}$ (idempotence, transitivity)

4) If $P, Q \in \mathcal{K} \downarrow$, so is $(P; Q)$

$(P; Q) \subseteq \mathcal{K} \mathcal{K}$ by monotonicity.

$= \mathcal{K}$ by idempotence.
(5) If \( S \subseteq K \downarrow \) then \( MS \subseteq K \downarrow \) provided \( S \) is nonempty.

Let \( P \in S \). Then \( P \in K \downarrow \) because \( S \subseteq K \downarrow \).

\[ \therefore P \subseteq (P \cap K) \subseteq K \]

(6) If \( S \subseteq K \downarrow \) then \( US \subseteq K \downarrow \)

for all \( P \in S, \ P \in K \downarrow \)

\[ \therefore \ U P \subseteq K \downarrow \quad (\text{sub}) \]

\( b \) is a condition and \( (K \downarrow, \text{fin} K) \) is a condition.

(7) If \( b \subseteq (K \downarrow) \) then its complement in \( K \downarrow \) is \( (\text{fin} K) \)

\[ \tau = \text{fin} \tau; K \]

\[ K = \tau \cap K = \text{fin} (b \cup b) \cap K \]

\[ = (\text{fin} K) \cup (\text{fin} K) \]

\[ = b \cup (\text{fin} K) \]

\[ \therefore (\text{fin} K) \] is the complement of \( b \) in \( K \downarrow \)

(8) Assume \( F : K \downarrow \rightarrow K \downarrow \). It follows that \( F(K) \subseteq K \downarrow \)

and so \( F(K) \subseteq K \downarrow \). The meet in the following definition is

\[ \mu \chi FX = \mu \{ X \mid FX \subseteq X \& X \in J \} \]

therefore nonempty, and so belongs to \( K \downarrow \)
Dear Hilary,

Thanks for your interested and interesting reply. I think our choice is Heyting algebras, because we find Galois connections so useful. Also, I think it is easier for us to work with morphisms distributing through arbitrary meets (or joins). I should very much like to know more about duality, yours and/or Stone's.

But we are very hooked on monotone functions as morphisms, so that our Homsets are themselves complete distributive lattices. This means that the normal product and coproduct constructions in the category have to be weakened to local adjunctions or worse. Nevertheless, we would like to find some kind of cartesian closedness. The best introduction to the motivation is probably Dijkstra's original paper on Guarded Commands.