Propositions.

<proposition> ::= <predicate> | (<neither nor>)

<neither nor> ::= <proposition> <u>nn</u> <proposition>

A <neither nor> is true if both its constituent propositions are false, and is false if either of them is true. In future, P, Q, and R will stand for propositions.

Negation, – not P:
$$\sim P == P \underline{nn} P$$
Conjunction, – P and Q:
$$P \& Q == \sim P \underline{nn} \sim Q$$
Disjunction, – P or Q:
$$P \underline{v} Q == \sim (P \underline{nn} Q)$$
Implication, – P only if Q:
$$P \Rightarrow Q == \sim P \underline{v} Q$$
Equivalence, – P if and only if Q:
$$P \equiv Q == (P \Rightarrow Q) \& (Q \Rightarrow P)$$

Truth ~~true == T~~

The order of priority is as presented above; and association ~~is to the right~~;

thus: $\sim P \& Q \underline{v} R \Rightarrow S \Rightarrow T \equiv U == ((((\sim P) \& Q) \underline{v} R) \Rightarrow (S \Rightarrow T)) \equiv U$

~ ~~governs~~ the shortest proposition to its right. Other propositional operators ~~bind~~ weaker than

Axioms: any standard axiomatisation of the classical propositional calculus may be used, e.g. Kleene p. 82

Predicates.

<predicate> ::= <atom> | (<generalisation>)

<generalisation> ::= $\underline{A}$ <variable>. <proposition>

<variable> ::= <letter> {<letter> | <digit>}

A generalisation is true if its proposition is true for <u>all</u> values of the variable, and is false otherwise

Existence, – there is an x such that P

$$\underline{E} x . P \qquad == \qquad \sim(\underline{A} x . \sim P)$$

Equality.

<atom> ::= <equation> | <membership>

<equation> ::= <term> = <term>

An equation is true if its two terms denote the same entity, and is false otherwise.

Inequality, — s is not equal to t

$$s \cong t \qquad == \quad \sim (s = t)$$

Descriptions.

<term> ::= <variable> | <description> | <analysis>

<description> ::= ($\underline{I}$ <variable>. <proposition>)

A description denotes the unique value of its variable for which its proposition is true, provided that this exists, and is unique; otherwise, it denotes the empty set.

The undefined element.

$$\underline{O} \quad == \quad \underline{I}x.\ \sim x = x$$

Restriction, - if P then s (otherwise undefined).

$$P \to s \quad == \quad \underline{I}x.\ x = s \ \&\ P$$

Restriction - s such that P

$$s\ \underline{st}\ P \quad == \quad P \to s$$

Alternatives either s or t:

$$s \,|\, t \quad == \quad \underline{I}x.\quad t = \underline{O} \ \&\ x = s \ \underline{v}\ s = \underline{O} \ \&\ x = t \ \underline{v}\ x = s \ \&\ x = t$$

Membership

<membership> ::= <term> $\underline{e}$ <term>

A membership is true if its second term denotes a set and its first term denotes a member of the set; otherwise the membership is false.

?

The empty set

$$\underline{s0} = (\$x.\ x \neq x) \qquad (x\,\$\,x \sim= x)$$

Containment. s is contained in t (s is a subset of t):

$$s \subseteq t \quad == \quad \underline{A}x.\ x\,\underline{e}\,s \Rightarrow x\,\underline{e}\,t \qquad \underline{A}(x\,\$\,x\,\underline{e}\,s \Rightarrow x\,\underline{e}\,t)$$

Set definition, - the set of all x such that P

$$(\$x.P) \quad == \quad \underline{I}y.\ \underline{A}x.\ x\,\underline{e}\,y \equiv P \qquad (x\,\$\,P)$$

Intersection - of s and t

$$s\,\underline{n}\,t \quad == \quad \$x.\ x\,\underline{e}\,s \,\&\, x\,\underline{e}\,t \qquad (x\,\$\,x\,\underline{e}\,s \,\&\, x\,\underline{e}\,t)$$

Union - of s or t

$$s\,\underline{u}\,t \quad == \quad \$x.\ x\,\underline{e}\,s\ \underline{v}\ x\,\underline{e}\,t$$

Set difference, - s without t

$$s \setminus t \quad == \quad \$x.\ x\,\underline{e}\,s \,\&\, \sim x\,\underline{e}\,t$$

Unit set of s (the set whose only member is s)

$$\underline{i}s \quad == \quad \$x.\ x = s \qquad x\,\$\,x = s$$

The unique member of s $\underline{t}s = \underline{I}x.\ x\,\underline{e}\,s$

The union over x of s set of sets

$$\underline{U}x.s \quad == \quad \$y.\ \underline{E}x.\ y\,\underline{e}\,x \,\&\, x\,\underline{e}\,s \qquad (y\,\$\,\underline{E}x\,\$\,y\,\underline{e}\,x \,\&\, x\,\underline{e}\,s)$$

The intersection over x of s $\underline{u}/\underline{1}'$ $(x \to s)$

$$\underline{N}x.s \quad == \quad \$y.\ \underline{A}x.\ y\,\underline{e}\,x \,\&\, x\,\underline{e}\,s$$

Recursive set definition the smallest set x s.t $x = \underline{U} x : s$

$\underline{R}\, x . s \quad == \quad \underline{N}\, y .\ y \leftarrow \underline{A}\, x .\ x \subseteq y \Rightarrow s \subseteq y \qquad ;\underline{N}\, y \,\$\, A\, x \,\$\, x \subseteq$

The powerset of s, (the set of all subsets of s)

$\underline{P}\, s \quad = \quad \$ x .\ x \subseteq s \qquad (x \,\$\, x \subseteq s)$

$\underline{N}(y \,\$\, \underline{A}(x \,\$\, x \subseteq y \Rightarrow s \subseteq y)))$

$\underline{R}\, s$

$\underline{Y}\, s \qquad (\underline{P}S \Rightarrow \underline{P}S) \Rightarrow \underline{P}S$

$= \quad \underline{N}(x \,\$\, \underline{A}(y \,\$\, y \subseteq x \Rightarrow s(y) \subseteq x))$

$\underline{n}/ \quad \underline{N}\, s \quad = \quad x \,\$ \qquad \underline{P}\underline{P}S \Rightarrow \underline{P}S$

$\underline{u}/ \quad \underline{U}\, s \quad = \quad (x \,\$\, \underline{E}(y \,\$\, x \underline{\in} y \,\&\, y \underline{\in} s))$ $\underline{w}$

$\Sigma\, s \quad = \quad l$

$M\, s \quad = \quad \underline{t}(x \,\$\, x \underline{\in} s \,\&\, \underline{A}(y \,\$\, y \underline{\in} s \Rightarrow x \leq y))$

Analysis.

<analysis> ::= (<term> <analyser> <variable>{, variable}. <term>)

<analyser> ::= @ | @$\underline{1}$ | @$\underline{2}$ | ...

An analyser equates the following variable list with the components of the term on the left; it then denotes the value given by the term on the right.

Pair, – the pair consisting of s and t

$$(s, t) \quad == \quad \underline{T}x.\ (x @ y,z . y) = s \ \& \ (x @ y,z . z) = t$$

Triple, – the triple consisting of s, t, u

(s, t, u) == similar

Tagged value.

$$\underline{a1}\, s \quad = \quad Tx.\ (x @ \underline{1}\, y . y) = s$$

$$\underline{a2}\, s \quad = \quad Tx.\ (x @ \underline{2}\, y . y) = s$$

etc

Direct product.

The direct product of s and t

$$s \,\underline{x}\, t \quad == \quad \$x.\ \underline{E}y.\ \underline{E}z.\ x=(y,z)$$

The first projection (range) of s:

$$\underline{p1}\, s \quad == \quad \$x.\ \underline{E}y.\ s=(x,y)$$

The second projection (domain) of s:

$$\underline{p2}\, s \quad == \quad \$x.\ \underline{E}y.\ s=(y,x)$$

Disjoint union of s and t

$$s \,\underline{\oplus}\, t \quad == \quad \$x.\ \underline{E}y.\ y \underline{e} s \,\&\, x = \underline{a1}\, y$$

$$\underline{v}\ \underline{E}z.\ z \underline{e} t \,\&\, x = \underline{a2}\, z$$

Relations.

A relation u is identified with the set of ordered pairs (x,y) such that x bears u to y.

The relation x,y such that P

$$\$x,y.P \quad == \quad \$z.\,\underline{E}x.\,\underline{E}y.\; z=(x,y)\,\&\,P$$

The image of t under u (the u's of t's)

$$u\underline{p}t \quad == \quad \$x.\,\underline{E}y.\; y\,\underline{e}\,t\,\&\,(x,y)\,\underline{e}\,u$$

The converse of u:

$$\underline{b}u \quad == \quad \$x,y.\; y,x\,\underline{e}\,u$$

The composition of s and t

$$s\,\underline{o}\,t \quad == \quad \$x,y.\,\underline{E}z.\,(x,z)\,\underline{e}\,s\;\&\,(z,y)\,\underline{e}\,t$$

The ancestral of u, the smallest relation x s.t. $x = II\ \underline{v}\ u\,o\,x$

$$\underline{*}\,u \quad == \quad (\underline{R}x.\; II\ \underline{v}\ x\,\underline{o}\,u)$$

The identity relation

$$II \quad == \quad (\$x,y.\; x=y)$$

Functions.

← A function is identified with a relation in which the first element of each pair is unique

Functional application, - s applied to t

$$s \downarrow t \quad == \quad \underline{T}x\,.\,(x,t)\ \underline{e}\ s \qquad \underline{t}(s\ \underline{e}\ t)$$

Lambda-abstraction, - the function which maps x onto s

$$(\underline{L}x\,.\,s) \quad == \quad \$y,x\,.\ y=s$$

Function space, - the set of total functions from s to t

$$s \Rrightarrow t \quad == \quad \$x\,.\ \underline{A}y.\ y\,\underline{e}\,s \Rightarrow x\downarrow y)\ \underline{e}\ t$$

Correspondence - the set of one-one mappings between s and t

$$s <\!\equiv\!> t \quad == \quad (s \equiv> t)\ \underline{n}\ (t \Rrightarrow s)$$

Updating. s updated by t

$$s\ \underline{g}\ t \quad == \quad Lx.\ x\ \underline{e}\ \underline{p2}\ t \rightarrow t\downarrow x\ [\,]\ \sim x\ \underline{e}\ \underline{p2}\ t \rightarrow s\downarrow t$$

Natural numbers.

The primitive concept of the theory of natural numbers is the successor function, succ.

The set of all natural numbers

$$\mathbb{N}\mathbb{N} \quad == \quad \underline{p2}\ \mathrm{succ}$$

The predecessor function

$$\mathrm{pred} \quad == \quad \underline{b}\ \mathrm{succ}$$

Zero and the numerals.

$$0 \quad == \quad \mathrm{T}x\,.\ x \in \mathbb{N}\mathbb{N}\ \&\ \sim x\ \underline{e}\ \underline{p1}\ \mathrm{succ}$$

$$1 \quad == \quad \mathrm{succ} \downarrow 0$$

etc

Relational powers. s to the power n.

$$s \uparrow n \quad == \quad \left(\underline{R}\, x. \underline{L} n.\, n = 0 \to \mathrm{II}\ \square\ n > 0 \to x \Downarrow (\mathrm{pred} \Downarrow n) \circ s\right) \downarrow n$$

$$\underline{Y}\ (x \to (n \to (\mathrm{II}\ \$\ n = 0\ \cup\ x(n-1)\ \underline{\circ}\ s\ \$\ n > 0)))\ n$$

Arithmetic operators

$$\left[(0 \to \mathrm{II}) \cup (n+1 \to x(n)\ \underline{\circ}\ s)\right] n$$

$$m + n \quad == \quad (\mathrm{succ} \uparrow n) \downarrow m$$

$$m * n \quad == \quad (\mathrm{succ} \uparrow m) \uparrow n \downarrow 0$$

$$m - n \quad == \quad \underline{\mathrm{T}}x\ .\ n + x = m$$

$$m / n \quad == \quad \mathrm{T}x\ .\ n * x = m$$

Remainder. m modulo n

$$m\ \underline{r}\ n \quad == \quad \mathrm{T}x.\ ((m - x)/n) * n = m$$

Relations.

$m \leq n \quad == \quad \mathrm{E}x.\ m+x=n$

$m < n \quad == \quad m \leq n \ \&\ \sim m=n$

Maximum and minimum

$m \ \underline{w}\ n \quad == \quad m \leq n \rightarrow n \,[]\, n \leq m \rightarrow m$

$m \ \underline{m}\ n \quad == \quad m \leq n \rightarrow m \,[]\, n \leq m \rightarrow n$

Axioms $\quad pred \underline{o} succ = II$

Induction $\quad NN = Rx(\underline{i}0 \cup succ''x)$

Finite sequences.

The primitive concept of the theory of finite sequences is the append function;
The empty sequence is a sequence; and if x is a sequence,
then ~~appendd~~(x, y) is the sequence obtained by appending y to x
as ~~the~~ its last item.

The set of all sequences

$$QQ \quad = \underline{p1}\ \underline{p2}\ \text{append}$$

Analysis. let x be the initial segment of s and y be the last item in t

$$(s@ x;q y . t) = \underline{I}z.\ \underline{E}x.\ \underline{E}y.\ s = \text{append}(x,y) \& z = t$$

The empty sequence

$$\underline{q0} \quad = Tx.\ x \underline{e} QQ.\ \sim x \underline{e}\ \underline{p1}\text{append}$$

Queueset set of all finite sequences with items from s

$$\underline{*}s \quad = Rx.\ \underline{q0} \cup \underline{U}y.\ \underline{U}z.\ \underline{\iota}\text{append}(y,z) \leftarrow y \underline{e} x \& z \underline{e} s$$
$$= \$x.\ \text{contents}(x \subseteq s)$$

Concatenation of s followed by t.

$$t.\ \underline{\forall}(f \to (u \xrightarrow{u} [(q0 \to s)\ \underline{u}(z;qw \to \dots)]\ (u))\ t$$

$$s;t \quad = \underline{R}f.\ \cancel{\Delta}s.\ \angle t.\ (t = q0 \to s) [] (t@z;qw \to \text{append}(f(s,z),w))\ t$$

Contents of a sequence

$$\text{contents} \quad = (\underline{R}f.\ \Delta s\ .\ (s = q0 \to \underline{0}) [] (s@ z;qw \to f(z) \underline{u} \underline{\iota} w)$$

s begins t

$$s \underline{b} t \quad == \ i.\ Ex\ \ s;x = t$$

$$(Qx < n.\ s)$$

A for all
B
C cardinality
D disjunction of guarded commands domain
E for some
F ? abort: (fail
G
H
I the individual s.t.
J join $\cup$
K constant.
L lambda
M minimum meet $\cap$
N ? intersection
O
P ~~parallel composition~~ product
Q the sequence
R *? repetition range.
S $? ~~set such that~~ sum.
T ? skip theorem the
U union
V program variable.
W maximum
X product
Y recursion
Z universal set

+ sum NO

a ~~achieves~~ (= wp.) alternatives
b begins
c is contained in.
d ~~divided~~ domain
e is a member of.
f False First function fforseq
g
h head
i the unit set of ?
j
k
l ~~empty sequence~~ last length
m min
n intersection
o ~~empty set?~~ ~~empty anything~~ functional composition
p power
q concatenation of sequences? ^
r reverse ~~reversal divider~~ range
s ~~unit~~ set from sequence
t ~~true~~ ~~tail~~ then the.
u union
v or
w max
x ~~multiplication~~ cartesian product
y
z

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t. prefix ,t postfix

~~empty sequence;~~ 'L. infix

not

quantifiers

infix operators.

a 'U' b U'c

+/

t' 't^t' = t

E% V: INTEGER

first and tail
last and head } of a sequence.

x 'X' y

(a v b -> c [] d -> e)

a v b