Review by Tony Hoare

ed. John Fauvel, Raymond Flood, Robin Wilson.

Music and Mathematics: from Pythagoras to Fractals

Oxford University Press, 189 pp, £39.50

ISBN 0-19-851187-6

Published September 2003

The close association between music and mathematics can be dated back at least to the time of the ancient Greek philosopher Pythagoras. He or his followers noticed that when the string of a musical instrument is stopped at exactly half its length, the resulting note sounds sweet (consonant) when played at the same time as the original. Similarly when it is stopped at two thirds or three quarters of its length, and at other simple fractions, the resulting harmonies are pleasing. In this way, it is possible to generate all the notes of a major scale, and eventually the sharps and flats as well. This mathematical theory due to Pythagoras gives a method for tuning a musical instrument with a keyboard like a piano, or with fixed frets like a guitar.

But it is not the only way to determine the pitch of the notes, and it suffers from a grave defect: it only permits the instrument to be played in the single key that it was tuned for. In particular, the sharp of a note was very slightly different from the flat of the note above it. Many subsequent mathematicians have made investigated the phenomena, including Plato, Euclid, Kepler, Mersenne, Galileo's father Vincenso, and Newton. The solution eventually adopted is called equal temperament; it has a fixed ratio between the successive semitones of the twelve-semitone chromatic scale. In order to preserve the exactitude of an octave interval, the ratio has to be the twelfth root of two, an irrational number that would have been difficult to accept for Pythagoreans. It so happens that this gives very good approximations to the original Pythagorean harmonies, and the human ear adjusts well to the slight inaccuracies. Fifty three notes in the octave would be even more accurate; such an instrument was built in 1876 by Bosanquet, but the idea never caught on.

This is a summary of the story well told by Neil Bibby in first section of this book. The story is taken further in a fascinating contribution by Ian Stewart, describing a geometrical construction for placing the frets of a guitar that was invented in 1743 by a Swedish craftsman, Daniel Straehle. He was not a mathematician, and when his calculations were checked by Jacob Faggot, one of the founding members of the Swedish Academy, they were found to be in error. But not so: the mistake was actually in the checking by the mathematician., Stewart's account also gives a clear exposition of the underlying geometry, including the method of neusis which was excluded from the classical geometry of constructions by ruler and compass.

The mathematical cosmology of Kepler presents musical harmony as a factor in explaining the structure of the Universe; an idea that persists in the imaginative poetic fallacy of the music of the spheres. Kepler's views and their development by his successors is the subject of a densely written survey by J.V. Field.

Musical sounds and their appreciation by the human ear are a fruitful field for experimental research. The obvious modern approach is to look directly at the musical wave-form as recorded by an oscilloscope; however, this is not readily distinguishable from arbitrary noise, and conveys no insight into the meaning and interpretation of the sound. The essay by Charles Taylor on the science of musical sound concentrates on the ways in which different instruments create notes of different timbre by means of different overtones, and on the ways in which the ear perceives the effects of a chord. This latter topic is further developed in the essay by David Taylor on Helmhotz's experiments with combinational tones and consonances. For example, why is the ear willing to tolerate the inaccuracies of the well-tempered scale, while being very sensitive to the exact octave? And why do many people hear additional combinational tones when two sirens are played at different pitches?

The most direct application of mathematics to music is in the analysis of musical compositions, and occasionally also in their construction. A good example is that of bell-ringing, which was prevalent in England in the seventeenth century. The rules are simple: in each round, all the bells must be rung in some order, and between rounds only two bells may change places (ringing the changes). The objective is to devise ways of ringing all possible rounds exactly once without repetition. For seven bells there are 5040 different orders, a full peal which takes three hours. English bell-ringers solved the problem of playing all the variations two hundred years ago; but it was not until a hundred years ago that the problem was analysed in full generality, using the mathematical techniques of Group Theory. A good introduction (but without too much Group Theory) is given by Dermot Roaf and Arthur White.

Wilfred Hodges contributes an interesting analytic essay entitled "The Geometry of Music" on the musical use of metaphor, inversions, transpositions, dilations, reflections, rotations, and mirror images. It illustrated by short musical scores from composers as varied as Mozart, Elgar, Bartok, Hindemith, and Rimsky-Korsakov.

Modern composers have made explicit use of mathematical constructions to determine structure of their compositions, and sometimes even to generate note sequences. Jonathan Cross in his Chapter on "Composing with Numbers" gives examples from Schoenberg, Berg, Webern, Boulez, Maxwell Davies and Xenakis. Some of these examples seem to owe more to numerology than to mathematics, but the claim is made that musical qualities always take precedence over blind application of a formula. Those with a good appreciation for modern music can judge the merit of this claim.

The book ends with two contributions by modern composers, Carlton Gamer and Robert Sherlaw Johnson. The first gives an example of the use of finite projective planes and their duals to generate sequences in which every interval between two notes of the sequence (not necessarily adjacent) occurs exactly once. The second uses fractal formulae to generate notes, and selects the parameters of the formulae to achieve the desired musical effects. The composition is to be played on an eight-channel MIDI synthesizer. Fragments of the scores are displayed.

It is generally believed that there is a correlation between professional engagement in music and an amateur interest in mathematics; and conversely, many professional mathematicians are also good amateur musicians. This book should appeal equally to both communities. It will also appeal to those whose primary interest is in the history of science. The book is expensively produced on wide and glossy paper, and is illustrated by many black-and-white reproductions of old woodcuts and modern photographs. A few colour illustrations or diagrams would have made the book more attractive for the coffee table; but more to the point would have been inclusion of a compact disc, containing illustrative examples and extracts of the music mentioned in the text. Surely, that could easily have been included in the price of the book.

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Current research and recent publications.

Tony Hoare is a researcher into the theory and practice of computer programming. Recent publications include popular articles and keynote addresses on Grand Challenges for Computing Research. He is neither a mathematician nor a musician, but takes a keen amateur interest in both.

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Fractional shifts produce sweet sound of spheres

Tony Hoare

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Sir Tony Hoare is senior researcher, Microsoft Research.